Section 9.1

2. If \( y = \sin x \cos x \), then \( y' = \cos^2 x - \sin^2 x + \sin x \), so \( y' + (\tan x)y = (\cos^2 x - \sin^2 x + \sin x) + (\sin^2 x - \sin x) = \cos^2 x \), and \( y(0) = 0 \). Thus, \( y'' = \cos^2 x - \sin^2 x + \sin x \), so \( y'' + (\tan x)y = (\cos^2 x - \sin^2 x + \sin x) + (\sin^2 x - \sin x) = \cos^2 x \), and \( y(0) = 0 \).

4. If \( y = e^{rt} \), then \( y' = re^{rt} \) and \( y'' = r^2 e^{rt} \). Therefore, \( y'' + y' - 6y = (r^2 + r - 6)e^{rt} \). However, \( e^{rt} \neq 0 \) for all \( r, t \), so if \( y'' + y' - 6y = 0 \), then \( r^2 + r - 6 = (r + 3)(r - 2) = 0 \). Therefore, \( y = e^{rt} \) is a solution for \( r = 2 \) and for \( r = -3 \).

6. (a) If \( y = Ce^{x^2/2} \), then \( y' = Cxe^{x^2/2} = xy \).

(b) 

(c) If \( y = Ce^{x^2/2} \) satisfies the initial condition \( y(0) = 5 \), then plugging in \( x = 0 \), we get \( 5 = Ce^0 = C \). Thus, \( y = 5e^{x^2/2} \).

(d) If \( y = Ce^{x^2/2} \) satisfies the initial condition \( y(1) = 2 \), then plugging in \( x = 1 \), we get \( 2 = Ce^{1/2} \), so \( C = 2e^{-1/2} \). Therefore, \( y = 2e^{(x^2 - 1)/2} \).

8. (a) If \( x \) is close to 0, then the graph of a solution to \( y' = xy^3 \) is nearly horizontal since \( y' \) is close to 0. On the other hand, if \( x \) is large, then the graph is nearly vertical since \( y' \) is large.

(b) If \( y = (c - x^2)^{-1/2} \), then \( y' = -\frac{1}{2}(c - x^2)^{-3/2}(-2x) = x(c - x^2)^{-3/2} = xy^3 \).

(c) 

This graph confirms the predictions in part (a).

(d) If \( y = (c - x^2)^{-1/2} \) satisfies the initial conditions \( y(0) = 2 \), then plugging in \( x = 0 \) gives \( 2 = c^{-1/2} \), so \( c = 2^{-2} = \frac{1}{4} \). Thus, \( y = (\frac{1}{4} - x^2)^{-1/2} = 2(1 - 4x^2)^{-1/2} \).

10. (a) If \( y = C \) is a constant solution, then plugging into the differential equation gives \( 0 = C^4 - 6C^3 + 5C^2 = C^2(C - 5)(C - 1) \); therefore, the constant solutions are \( y \equiv 0 \), \( y \equiv 1 \), and \( y \equiv 5 \).

(b) We have that \( y \) is increasing when \( \frac{dy}{dt} = y^2(y - 1)(y - 5) > 0 \), which is true when \( y < 1 \) and \( y \neq 0 \), or when \( y > 5 \).

(c) We have that \( y \) is decreasing when \( \frac{dy}{dt} = y^2(y - 1)(y - 5) < 0 \), which is true when \( 1 < y < 5 \).
12. At some point \((0, y)\) on the \(y\)-axis, we see that \(y'\) is positive, while \(xy = 0\); this eliminates B as a possibility. On the other hand, there is a point \((x, y)\) in the first quadrant such that \(xy > 0\) and \(y' = 0\), which eliminates A as a possibility. Therefore, by elimination, the differential equation must be C.

**Section 9.2**

2. (a) The answers to (i) through (v) are the five curves in the graph below, from top to bottom.

![Graph of five curves](image)

(b) If \(y = C\) is a constant solution, then plugging into the differential equation gives \(0 = x \sin C\). Since this must be true for all \(x\), this implies \(\sin C = 0\), so \(C = n\pi\) for some integer \(n\). Therefore, the equilibrium solutions are \(y \equiv n\pi\) for \(n\) an integer. (It is clear from the graph that \(y \equiv 0\) is an unstable equilibrium, meaning that solutions which start out near \(y = 0\) move away from it; on the other hand, \(y \equiv \pi\) is a stable equilibrium, meaning that solutions which start out near \(y = \pi\) move towards it.)

3. The direction field must be horizontal at every point where \(y = 1\), since then \(y' = y - 1 = 0\). The only given direction field which matches this condition is IV.

6. This direction field is horizontal exactly when \(y^3 - x^3 = 0\), or \(y = x\); only direction fields I and II satisfy this condition (in III, the direction field is also horizontal along the line \(y = -x\)). It appears we can distinguish I and II, for example, by calculating the slope of the direction field at \((2, 1)\). Here the slope is \(1^3 - 2^3 = -7\), which more closely matches I.

8. Again, the answers to (a) through (c) are the three curves in the graph from top to bottom.

![Graph of three curves](image)
From this diagram, we see:

- If \( y(0) > 2 \), then \( \lim_{x \to -\infty} y(x) = 2 \) and \( \lim_{x \to \infty} y(x) = \infty \).
- If \( 1 < y(0) < 2 \), then \( \lim_{x \to -\infty} y(x) = 2 \) and \( \lim_{x \to \infty} y(x) = 1 \).
- If \( -1 < y(0) < 1 \), then \( \lim_{x \to -\infty} y(x) = -1 \) and \( \lim_{x \to \infty} y(x) = 1 \).
- If \( -2 < y(0) < -1 \), then \( \lim_{x \to -\infty} y(x) = -1 \) and \( \lim_{x \to \infty} y(x) = -2 \).
- If \( y(0) < -2 \), then \( \lim_{x \to -\infty} y(x) = -\infty \) and \( \lim_{x \to \infty} y(x) = -2 \).

Section 9.3

2. The given equation implies \( 4y^3 \, dy = e^{2x} \, dx \); integrating gives \( y^4 = \frac{1}{2} e^{2x} + C \), so \( y = \pm \left( \frac{1}{2} e^{2x} + C \right)^{1/4} \).

4. We see that \( y \equiv 0 \) is a solution. Otherwise, if \( y \neq 0 \), then dividing both sides by \( y^2 \) gives \( \frac{y'}{y^2} = \sin x \).

Integrating both sides with respect to \( x \) gives \(-\frac{1}{y} = -\cos x + C_2\), so \( y = \frac{1}{\cos x + C} \), where \( C = -C_2 \).

6. The given equation implies \( (1 + \sqrt{u}) \, du = (1 + \sqrt{r}) \, dr \); integrating gives \( u + \frac{2}{3} u^{3/2} = r + \frac{2}{3} r^{3/2} + C_2 \), or \( 3u + 2u^{3/2} = 3r + 2r^{3/2} + C \), where \( C = 3C_2 \). (Solving for \( u \) in terms of \( r \) would require solving a cubic equation.)

8. If \( y' = \frac{x y}{2 \ln y} \), then we cannot have \( y = 0 \), or \( \ln y \) would be undefined. Therefore, we may multiply both sides by \( \frac{\ln y}{y} \) to get \( \frac{\ln y \, y'}{y} = \frac{1}{2} x \). Integrating both sides with respect to \( x \) gives \( \frac{1}{2} (\ln y)^2 = \frac{1}{4} x^2 + C_2 \). Solving for \( y \) gives \( y = \exp \left( \pm \sqrt{\frac{x^2}{2} + C} \right) \), where \( C = 2C_2 \) is an arbitrary constant.

10. From the given equation we get \( \frac{dz}{dt} = -e^z e^x \), so \( e^{-z} \, dz = -e^x \, dt \). Integrating, we get \( -e^{-z} = -e^x + C_2 \); solving for \( z \) gives \( z = -\ln(e^x + C) \), where \( C = -C_2 \).
12. Since \( y(0) = 1 \), we see that \( y \neq 0 \) for \( x \) near 0. Therefore, we may multiply both sides by \( \frac{1+x^2}{y} \) and by \( dx \) to give 
\[
\frac{dx}{y} = \frac{1+x^2}{y} \, dy = \cos x \, dx = \left( \frac{1}{y} + y \right) \, dy.
\]
Integrating, we get \( \ln |y| + \frac{y^2}{2} = \sin x + C \); however, since \( y(0) = 1 \), \( y \) is positive for \( x \) near zero, so this reduces to \( \ln y + \frac{y^2}{2} = \sin x + C \). Plugging in \( y = 1 \) and \( x = 0 \), we get \( \frac{1}{2} = C \), so the solution to the initial value problem is \( \ln y + \frac{y^2}{2} = \sin x + \frac{1}{2} \).

14. Since \( P(1) = 2 \), we see that \( \sqrt{P} \neq 0 \) for \( t \) near 1; thus, we may divide both sides by \( \sqrt{P} \) and multiply by \( dt \) to get 
\[
\frac{dy}{\sqrt{P}} = \sqrt{t} \, dt.
\]
Integrating, we get \( 2\sqrt{P} = \frac{2}{3} t^{3/2} + C \). Plugging in \( P = 2 \) and \( t = 1 \), we get \( 2\sqrt{2} = \frac{2}{3} + C \), so \( C = 2\sqrt{2} - \frac{2}{3} \). Therefore, \( 2\sqrt{P} = \frac{2}{3} t^{3/2} + 2\sqrt{2} - \frac{2}{3} \). Solving for \( P \) gives \( P = \left( \frac{1}{2} t^{3/2} + \sqrt{2} - \frac{1}{3} \right)^2 \).

16. Dividing both sides by \( e^y \) and multiplying by \( dt \), we get 
\[
e^y \, dy = \frac{t}{2} \, dt.
\]
Plugging in \( y = 0 \) and \( t = 1 \) gives \( -1 = \frac{1}{2} + C \), so \( C = -\frac{3}{2} \), so \( -e^{-y} = \frac{t^2}{2} - 1 \). Solving for \( y \) gives \( \ln \left( \frac{t^2}{2} - 1 \right) = \ln(2) \). Integrating both sides, we see that \( \ln(1) = 1 \), and the solution has no singularities, but for \( -1 < C \leq 1 \), the solution does have singularities. (And there are no solutions with \( C \leq -1 \).)

20. If the curve passes through \((1, 1)\), then \( y(1) = 1 \), and we are also given that the slope \( y' \) is equal to \( \frac{y^3}{x^2} \). Now since \( y(1) = 1, y \neq 0 \) for \( x \) close to 1, so we may divide both sides by \( y^2 \) to get \( \frac{y'}{y^2} = \frac{y}{x} \). Integrating, we see 
\[
\frac{1}{y} = -\frac{1}{2x} + C.
\]
Plugging in \( y = 1 \) and \( x = 1 \) gives \( -1 = -\frac{1}{2} + C \), so \( C = -\frac{1}{2} \), and \( \frac{1}{y} = -\frac{1}{2x} - \frac{1}{2} = -\frac{x^2 + 1}{2x^2} \). Solving for \( y \) gives \( y = \frac{2x^2}{x^2 + 1} \).

22. Since \( -y' = -\cos x \), integrating both sides gives \( -e^{-y} = -\sin x + C_2 \), so \( y = -\ln(C + \sin x) \), where \( C = -C_2 \). From the graph below, we see that for \( C > 1 \), the solution has no singularities, but for \( -1 < C \leq 1 \), the solution does have singularities. (And there are no solutions with \( C \leq -1 \).)

28. Taking the derivative of both sides, we get \( 2x - 2y y' = 0 \), so \( y' = \frac{x}{y} \). Therefore, in the orthogonal trajectories, we have \( y' = -\frac{y}{x} \). Here \( y = 0 \) is one solution; otherwise, dividing by \( y \) gives \( \frac{y'}{y} = -\frac{1}{x} \). Integrating, we get 
\[
\ln |y| = -\ln |x| + C_2 \text{, so } y = C \frac{x}{x^2} \text{, where } C = \pm e^{C_2} \text{ is a nonzero constant. In summary, the curves are } y = \frac{C}{x} \text{ for any constant } C.
\]
In the graph on the next page, the solid curves are the curves \( x^2 - y^2 = k \), and the dashed curves are the curves \( y = \frac{C}{x} \).
30. The given equation implies $e^2y = k$; taking the derivative gives $e^2y + e^2y' = 0$, so $y' = -y$. Therefore, in the orthogonal trajectories, we have $y' = \frac{1}{y}$. This implies that $yy' = 1$; integrating, we get $\frac{y^2}{2} = x + C_2$, so $y = \pm \sqrt{2x + C}$, where $C = 2C_2$. Again, in the above graph, the solid curves are the curves $y = ke^{-x}$, and the dashed curves are the curves $y = \pm \sqrt{2x + C}$.

38. (a) Assume $t$ is given in days, and let $\Delta t$ be some small amount of time. Then between time $t$ and $\Delta t$, 

$$(5 \cdot 10^7)\Delta t$$

dollars of currency enter the bank, of which a proportion of approximately $x(t)/10^{10}$ is new currency. Therefore, $1 - x(t)/10^{10}$ of the bills are old currency, so $\Delta x = x(t + \Delta t) - x(t) = (5 \cdot 10^7)\Delta t(1 - x(t)/10^{10}) = (5 \cdot 10^7 - 5 \cdot 10^{-3}x(t))\Delta t$. Dividing by $\Delta t$, we see $\frac{dx}{dt} \approx \frac{x(t + \Delta t) - x(t)}{\Delta t} = 5 \cdot 10^7 - 5 \cdot 10^{-3}x(t)$. Therefore, our initial value problem will be $\frac{dx}{dt} = 5 \cdot 10^7 - 5 \cdot 10^{-3}x$, with $x(0) = 0$.

(b) Dividing both sides by $5 \cdot 10^7 - 5 \cdot 10^{-3}x$ and multiplying by $dt$ gives $\frac{dx}{5 \cdot 10^7 - 5 \cdot 10^{-3}x} = dt$. Integrating, 

$$-200 \ln|5 \cdot 10^7 - 5 \cdot 10^{-3}x| = t + C;$$

however, since $5 \cdot 10^7 - 5 \cdot 10^{-3}x$ is positive at $t = 0$, it is positive for all $t$, so $-200 \ln(5 \cdot 10^7 - 5 \cdot 10^{-3}x) = t + C$. Plugging in $t = 0$ and $x = 0$ gives $-200 \ln(5 \cdot 10^7) = C$. Therefore, $\ln(5 \cdot 10^7 - 5 \cdot 10^{-3}x) = -\frac{1}{200} + \ln(5 \cdot 10^7)$. Exponentiating gives $5 \cdot 10^7 - 5 \cdot 10^{-3}x = 5 \cdot 10^7 e^{-t/200}$, and solving for $x$ gives $x = 10^{10}(1 - e^{-t/200})$.

(c) Since the total supply is $10^{10}$, we need $10^{10}(1 - e^{-t/200}) = 10^{10} \cdot \frac{9}{10}$, for $e^{-t/200} = \frac{1}{10}$. Solving for $t$ gives $t = 200 \ln(10) \approx 460$ days.

40. (a) Let $x(t)$ be the amount of salt in the tank after time $t$, in kilograms. Since the net flow is zero, the volume of water stays constant at 1000 liters, so the concentration at time $t$ is $\frac{x}{1000}$ kilograms per liter. Now at time $t$, the first inflow adds salt at a rate of $5(0.05) = 0.25$ kilograms per minute, and the second inflow adds salt at a rate of $10(0.04) = 0.4$ kilograms per minute. On the other hand, the outflow removes salt at a rate of $15 \cdot \frac{x}{1000} = 0.015x$ kilograms of salt per minute. Therefore, $\frac{dx}{dt} = 0.65 - 0.015x = 0.015(\frac{130}{3} - x)$. Also, since the tank starts out with pure water, $x(0) = 0$.

Dividing both sides of the equation by $\frac{130}{3} - x$ and multiplying by $dt$ gives $\frac{dx}{\frac{130}{3} - x} = 0.015 dt$; integrating, we get $-\ln(\frac{130}{3} - x) = 0.015t + C$. However, $\frac{130}{3} - x$ is positive when $t = 0$, so $-\ln(\frac{130}{3} - x) = (0.015)t + C$. Substituting $x = 0$ and $t = 0$, we get $-\ln(\frac{130}{3}) = C$, so $\ln(\frac{130}{3} - x) = 0.015t - \ln(\frac{130}{3})$. Solving for $x$, we get $x = \frac{130}{3}(1 - e^{-0.015t})$.

(b) Plugging in $t = 60$, we get $x = \frac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kilograms.