LARGE FINITE STRUCTURES WITH FEW TYPES

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Abstract. We outline the structure theory for infinite structures which are smooth limits of finite structures, or equivalently for sufficiently large finite permutation groups with a bounded number of orbits on 4-tuples. The *primitive* case is treated explicitly in [14] assuming a bound on orbits on 5-tuples, and modifications needed to work with a bound on 4-tuples are indicated in [15]. This theory is an extension of the theory of $\aleph_0$-categorical $\aleph_0$-stable structures. The main technical innovations at this level of generality are due to Hrushovski; some of them are useful in other semistable contexts.

1. Introduction

The class of *smoothly approximable* structures was introduced by Lachlan as a natural generalization of the class of $\aleph_0$-categorical $\aleph_0$-stable structures. These are $\aleph_0$-categorical structures which are well approximated by finite structures in a sense to be given below. One of the achievements of the theory of $\aleph_0$-categorical $\aleph_0$-stable structures was in fact the result that they are smoothly approximable, which is based on ideas of Zilber introduced originally to prove that totally categorical theories are not finitely axiomatizable. Lachlan apparently felt that the natural level of generality for the structure theory which was developed in this case would be the class of smoothly approximable structures. This is in any case an attractive class to the model theorist as their study is essentially equivalent to the study of large finite structures with a restriction on the number of types, which is rather natural from the point of the theory of finite permutation groups as well.

The model theoretic developments tend to reduce all questions to some minimal cases; in permutation group terms, this means in essence that one needs to have good control of the primitive permutation groups in the class, or equivalently: the structures with no nontrivial 0-definable equivalence relation. While not completely helpless in this regard, model theory does not appear to be capable of producing either a classification or the relevant structural properties in the primitive case, but this can be done by purely group theoretic methods, given the classification of the finite simple groups (for which it seems a satisfactorily complete proof [17] is now appearing, as a result of the determination of the late Daniel Gorenstein and the perseverance of his coworkers, Lyons and Solomon). This was carried out in [14], with a further refinement indicated by Macpherson, which is described in [15]. To pass from the primitive case to the general case is a project very much in keeping with the spirit of Shelah's classification theory, and indeed the structures under investigation have much in common with $\aleph_0$-stable structures of finite rank, which serves as a model for much of the development. The main technical innovations that result in a successful theory are due to Hrushovski and are generally of the same type that have proved useful subsequently in other unstable contexts, such as the theory of pseudofinite fields and the theory of difference fields. Naturally the theory incorporates a great deal of material that has appeared in similar forms elsewhere. In particular the key combinatorial property of the coordinatizing geometries under consideration was pointed out first by Ahlbrandt and Ziegler, then reformulated by Hrushovski more flexibly; the generalization of this material to the wider class of geometries needed here – namely those described in [14] – is straightforward.

Other aspects of general stability theory generalize in a less straightforward way to the present context. One needs some notion of rank, a version of definability of types, a notion of local modularity, and a theory of definable groups, and there are in addition some basic features of general stability theory (notably stationarity and canonical bases) that have to be recovered in a rather different form.

My purpose here is to summarize the theory, paying adequate attention to the definitions and basic technical lemmas that make the theory ultimately come out to be a reasonably close parallel to the stable case. A rapid summary of the theory was given in [8]. We will begin by recalling some of the main results as stated there, in a terminology which will be explained in detail subsequently, though it should be noted at the outset that the key term will be "Lie coordinatizability", which replaces smooth approximability in practice. It refers, roughly, to structures built from the primitive pieces identified by [14]. It should be noted that while Hrushovski can legitimately be held responsible for much of what follows, any inaccuracies may reasonably be laid to the account of the present author. It is likely that much can be gained in clarity and simplicity by revisiting the theory from the more general point of view of "simple theories".

The peculiar numbering of the results is intended to preserve compatibility with the full exposition in preparation as [4]; unfortunately this compatibility cannot yet be guaranteed.

Theorem 2

The following conditions on a model $\mathcal{M}$ are equivalent:

1. *smooth approximability;*
2. *Lie coordinatizability;*
3. *4-quasifiniteness.*

Theorem 3

The closure of the class of Lie coordinatizable structures under interpretability is the class of weakly Lie coordinatizable structures.

The fact that this class is not simply closed under interpretability was shown by David Evans, who pointed out what is essentially the only reason that this fails.

Theorem 4

For any k, the theory of finite structures whose automorphism groups have at most k orbits on 4-tuples is decidable, even in an extended language containing certain cardinality comparison quantifiers. Thus one can decide effectively whether a sentence in such a language has a finite model with a given number of orbits on 4-types.

This statement (with 4 in place of 5) incorporates Dugald Macpherson's modifications to [14] as described in [15].

Theorem 6

The weakly Lie coordinatizable structures $\mathcal{M}$ are characterized by the following properties:

1. *$\aleph_0$-categoricity;*
2. *Pseudofiniteness;*
3. *Finite rank;*
4. *Amalgamation of types;*
5. *Modularity;*
6. *Finite basis of definability in definable groups;*
7. *Rank/measure property;*
8. *$\mathcal{M}$ does not interpret the random bipartite graph,*

9. *For every vector space V interpreted in $\mathcal{M}$, the definable dual V^* (the set of all definable elements of the dual) is interpreted in $\mathcal{M}$.*

2. Definitions

Definition 2.1

Let $\mathcal{M}$ be a structure.

1. *$\mathcal{M}$ is* $\aleph_0$-categorical *if for each n, $\mathcal{M}$ has finitely many n-types.*
2. *$\mathcal{M}$ is* pseudofinite *if it is a model of the theory of finite structures.*
3. *$\mathcal{M}$ is* n-quasifinite *if in a nonstandard extension of the universe, it is elementarily equivalent to an internally finite model with finitely many internal n-types.*
5. *A finite substructure $\mathcal{N}$ of $\mathcal{M}$ is* n-homogeneous *in $\mathcal{M}$ if: (i) all 0-definable relations on $\mathcal{N}$ induce 0-definable relations on $\mathcal{N}$, and (ii) any two n-tuples* **a**, **b** *in $\mathcal{N}$ have the same type in $\mathcal{N}$ if and only if they have the same type in $\mathcal{M}$.*
7. *A structure $\mathcal{M}$ is* smoothly approximable *(by finite substructures) if it is $\aleph_0$-categorical, and every finite subset of $\mathcal{M}$ is contained in a finite substructure of $\mathcal{M}$ which is fully homogeneous in $\mathcal{M}$, i.e. $|N|$-homogeneous.*

The notion of Lie coordinatizability rests primarily on an explicit list of the "rank 1" sets allowed as coordinatizing geometries (which precedes any formal notion of rank). There are various technicalities to be dealt with which appear already in the stable case. To a model theorist, it is not completely clear what is meant by a vector space over a finite field F which is not the prime field; this is not clear to a permutation group theorist either, because the question is whether the Galois group of F over the prime field is part of the automorphism group. If it is, then the field F is not 0-definable, but it belongs to the algebraic closure of $\emptyset$. We will neglect these points for the moment but they are omnipresent in the theory. The tradition in algebra is that the galois group is not part of the automorphism group of a vector space, in other words the algebraic closure of $\emptyset$ and its definable closure coincide (in infinite-dimensional models). Such geometries are called *basic*. Leaving these issues aside – which greatly simplifies the notation – we define:

Definition 2.2

A nonquadratic linear geometry *is a structure of one of the following five types:*

1. *A* degenerate space*: a set with equality.*

2. *A* pure vector space *over a finite field.*
3. *A* polar space (V, V^*) *is a pair of vector spaces over a finite field* F *equipped with a nondegenerate bilinear pairing into* F.
4. *An* inner product space *is a vector space equipped with a nondegenerate symplectic or hermitian inner product.*
5. *An* orthogonal space *is a vector space equipped with a nondegenerate quadratic form.*

Frequently orthogonal spaces are viewed as symmetric inner product spaces but to do so requires the assumption that the characteristic is not 2, which is not a reasonable restriction here. We refer to these as *nonquadratic* for emphasis. There is a curious "quadratic" geometry which we will define precisely, and which may possibly be considered as another type of linear geometry. The reader who finds these matters tedious may pass over the quadratic geometry, which has to be dealt with in practice but is generally less trouble than the more familiar affine geometries which will appear momentarily. The more obscure aspects of the following definition will be elucidated immediately below.

Definition 2.2, continued

6. *The* quadratic geometry (V, Q, ω) *associated with a finite field* F *of characteristic 2 is defined as follows. Let* V *be a nondegenerate symplectic space over* F. *Let* Q *be a set of quadratic forms defined on* V *satisfying:*
   i. For $q \in Q$, $q(x+y) = q(x) + q(y) + (x, y)$ *with* (x, y) *the given symplectic form.*
   ii. For $q_1, q_2 \in Q$, *the difference* $q_1 - q_2$ *is the square of a linear form* λ *satisfying:* $\lambda(v) = (v_\circ, v)$ *for some associated* $v_\circ$ *in* V. *Let* $\omega : Q \to \{\pm 1\}$ *be the "Witt defect".*

Some comments on the slightly mysterious Q and the definitely mysterious ω are in order. (See also [15].) As far as Q is concerned, any quadratic form q on V will be associated with a bilinear form $q(x+y) - q(x) - q(y)$ which in odd characteristic is symmetric and determines q, while in even characteristic it is symplectic and does not determine q. The quadratic forms associated with the identically 0 bilinear forms are easily seen to be the squares of linear forms. Thus if we prescribe the associated bilinear form in advance, the associated quadratic forms differ by squares of linear forms. In the finite case the linear forms are all represented by inner products with elements of V and as we are interested in limits of finite structures we will generally work with structures in which all definable linear forms are given by an inner product with some vector. Thus the condition on Q simply reflects what would hold for the space of all appropriate quadratic forms in the finite case.

Similarly, the "Witt defect" is defined in the case of a finite dimensional space of dimension $2n$ (even, as there is a nondegenerate symplectic form) as the difference between n and the dimension of a maximal totally q-isotropic subspace (0 or 1). For infinite dimensional V this presents a serious problem, as there is no such notion; if however V arises as a limit of finite-dimensional approximations, it will inherit a corresponding invariant ω defined on Q. We note that it is possible to define the corresponding equivalence relation on Q with two classes in terms of the structure (V, Q) in a way that we will briefly indicate below, but in any case it is not possible to define each of the two classes separately without parameters. The effect of ω is to name the two classes. This is a significant phenomenon, as ω can be omitted in a reduct and the resulting structure is then not smoothly approximable. This is the example of David Evans. Thus the existence of quadratic geometries does have a direct impact on the content of the theory.

To conclude this discussion, we give – for the record – an explicit definition of the equivalence relation induced by the Witt defect. Let F_{AS} be the image of F under the Artin-Schreier polynomial $x^2 + x$. Two quadratic forms q_1, q_2 will be equivalent if $q_1(\sqrt{q_1 + q_2}) \in F_{AS}$, where $\sqrt{q_1 + q_2} \in V^*$ and V^* is identified with V via the symplectic form. One can check easily that in any model this will give an equivalence relation with two classes, and in finite models this relation is induced by the Witt defect.

We need also the projective and linear versions of these geometries.

Definition 2.4

A projective geometry *is the structure obtained from a linear geometry by removing* $\mathrm{acl}(\emptyset)$ *and factoring out the equivalence relation* $\mathrm{acl}(x) = \mathrm{acl}(y)$.

Definition 2.5

An affine geometry *over a linear geometry* V *(or over one component of a polar geometry) is a pair* (V, A) *in which* V *carries its given structure and* A *is a second copy of* V *carrying no structure beyond the addition map* $+ : V \times A \to A$.

The model theoretic properties of these geometries will be critical. Before dealing with these properties, to complete our terminological discussion we introduce the notion of a Lie coordinatization.

Definition 2.6

Let $\mathcal{M} \subseteq \mathcal{N}$ *be structures with* $\mathcal{M}$ *definable in* $\mathcal{N}$, *and let* $a \in \mathcal{N}^{\mathrm{eq}}$

represent the underlying set of $\mathcal{M}$ (its "canonical parameter").

1. *$\mathcal{M}$ is* canonically embedded *in $\mathcal{N}$ if every a-definable relation on $\mathcal{M}$ in the sense of $\mathcal{N}$ is a 0-definable relation in the sense of $\mathcal{M}$.*
2. *$\mathcal{M}$ is* stably embedded *in $\mathcal{N}$ if every $\mathcal{N}$-definable relation on $\mathcal{M}$ is $\mathcal{M}$-definable, uniformly in the $\mathcal{N}$-definition.*
3. *$\mathcal{M}$ is* fully embedded *in $\mathcal{N}$ if it is canonically and stably embedded.*

Definition 2.7

The structure $\mathcal{M}$ is coordinatized by Lie geometries *if it carries a tree structure of finite height with a unique 0-definable root such that the following* coordinatization *and* orientation *properties hold.*

1. (Coordinatization) *For each $a \in \mathcal{M}$ above the root, either a is algebraic over its immediate tree predecessor, or there is a $b < a$ and a b-definable projective geometry J_b, fully embedded in $\mathcal{M}$, such that either:*
   - (i) *$a \in J_b$; or*
   - (ii) *There is c, $b < c < a$, and a c-definable affine or quadratic geometry whose linear part has projectivization J_b, and with a in the affine or quadratic part.*
2. (Orientation) *If $a, b \in \mathcal{M}$ have the same type and are associated with quadratic geometries J_a, J_b in $\mathcal{M}$, then any definable bijection between them which is an isomorphism up to orientation, also preserves orientation.*

We use the term *Lie coordinatizable* for structures which are biinterpretable with Lie coordinatized structures.

Remark

Lie coordinatized structures are $\aleph_0$-categorical.

We also make use of a weak notion of Lie coordinatization in which the orientation condition on the quadratic geometries is suppressed. This weak notion is the one which will be preserved under interpretation (the difficulty being preservation under reduct).

3. Rank

We use the a notion of rank, which in the $\aleph_0$-categorical case may be phrased as follows.

Definition 3.1; Lemma 3.2

Let $\mathcal{M}$ be $\aleph_0$-categorical, $a, b \in \mathcal{M}$.

1. $\mathrm{rk}\,(a/b) = 0$ *if and only if* $a \in \mathrm{acl}\,(b)$.

2. $\text{rk}\,(a/b) > n$ *if and only if for some* c *there is* $a' \in \text{acl}\,(abc) - \text{acl}\,(bc)$ *with* $\text{rk}\,(a/a'bc) \geq n$.

This is additive:

Lemma 3.3

$$\text{rk}\,(ab/c) = \text{rk}\,(a/bc) + \text{rk}\,(b/c).$$

Using the rank notion one defines independence: a is independent from b over c if $\text{rk}\,(a/bc) = \text{rk}\,(a/c)$.

Via quantifier elimination one shows that the linear geometries have rank 1, and thus that Lie coordinatized structures have finite rank. One also finds that algebraic closure in the nondegenerate cases (when there actually is a base field, in other words) coincides with linear span. This is the first indication of the relative harmlessness of the nonlinear structure which may be present.

4. Elimination of imaginaries

The next issue again involves the coordinatizing geometries.

Definition 4.1

A structure $\mathcal{M}$ *has* weak elimination of imaginaries *if for all* $a \in \mathcal{M}^{\text{eq}}$, $a \in \text{dcl}\,(\text{acl}\,(a) \cap \mathcal{M})$.

Lemma 4.1

The following conditions on a definable subset D *of a structure* $\mathcal{M}$ *are equivalent.*

1. D *is stably embedded in* $\mathcal{M}$ *and admits weak elimination of imaginaries.*
2. *For* $a \in \mathcal{M}^{\text{eq}}$, *the type of* a *over* $D \cap \text{acl}\,(a)$ *determines the type of* a *over* D.

Lemma 4.3

A linear geometry has elimination of imaginaries.

We need to say something similar about the affine geometries. This requires the introduction of the *affine dual.*

Definition 4.2

If A is an affine geometry over a base field F in a structure $\mathcal{M}$, A^* denotes the set of $\mathcal{M}$-definable affine maps $A \to F$.

Affine maps can be described in various ways, but if A is identified with V they are the maps of the form $f + c$ with f linear, c constant. In other words there is an exact sequence:

$$(0) \to F \to A^* \to V^* \to (0)$$

where V^* is the full definable dual. Note that the stable case is degenerate: $V^* = (0)$ and $A^* = F$ in this case.

If J is one of the basic linear nonquadratic geometries and A an affine version of J, the structure (J, A, A^*) has quantifier elimination in its natural language. One can then show:

Lemma 4.6

(J, A, A^*) *has weak elimination of imaginaries.*

5. Orthogonality

A portion of the theory of orthogonality for rank 1 geometries may be presented axiomatically. One should observe that a polar geometry is an example of an interaction of two rank 1 geometries in a rather subtle manner. Since it is beyond the power of a standard theory of orthogonality to deal with this interaction, we take the route of absorbing such interacting pairs into a single geometry. It should be said here that the term orthogonality is currently used in "unstable geometrical stability theory" in two conflicting ways; in some contexts (such as fields with automorphisms) it refers to the absence of interaction at the level of algebraic closure, and in others (such as the present one) it refers to the absence of any significant interaction.

Definition 5.1

1. *A* rigid geometry *is a structure J with the following properties (in every model of its theory)*
   - (i) $\operatorname{acl}(a) = a$ *for* $a \in J$*;*
   - (ii) *Exchange property for* acl *in* J*;*
   - (iii) *If* $a \in J^{\mathrm{eq}}$ *then* $a \in \operatorname{acl}(B)$ *for some* $B \subseteq J$*;*
   - (iv) *For* $J_\circ \subseteq J$ *0-definable and nonempty, any two elements of J realizing the same type over* $J_\circ$ *coincide.*

2. *A rigid geometry is* strongly rigid *if it satisfies:*

   (v) $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ *in* $J^{=}$eq.

One checks that the projective geometries are rigid. The issue of strong rigidity is one that we swept under the rug earlier. From a group theoretic point of view one is reducing the automorphism group to the smallest possible subgroup of finite index.

Lemma 5.2

If J_1, J_2 are rigid geometries, fully embedded in $\mathcal{M}$, then either

1. *J_1 and J_2 are* orthogonal *in the sense that every 0-definable relation on $J_1 \cup J_2$ is a boolean combination of relations of the form $R_1 \times R_2$ with R_i* $\mathrm{acl}(\emptyset)$*-definable on J_i; or*
2. *J_1 and J_2 are* 0-linked*: there is a 0-definable bijection between the geometries.*

If the projectivizations of two linear geometries are 0-linked, one can lift this to a 0-linkage between the linear geometries; if the geometries are parts of polar or quadratic geometries then the linkage can be lifted to the corresponding additional components V^* or Q.

6. Canonical projective geometries

One of the less attractive properties of a geometry carrying an inner product is that there are definable subspaces of arbitrarily large finite codimension, and thus a number of closely related geometries which can be associated with the original one. The following notion distinguishes the "master" geometry from its "offspring",

Definition 6.1

Let $\mathcal{M}$ be a structure, J_b a b-definable projective geometry in $\mathcal{M}$. Then J_b is canonical *if:*

1. *J_b is fully embedded in $\mathcal{M}$ over b;*
2. *if $b' \neq b$ is another realization of the type of b, then J_b and $J_{b'}$ are orthogonal.*

Lemma 6.1

Let $\mathcal{M}$ be Lie coordinatizable, P_b a b-definable projective geometry *in $\mathcal{M}$. Then there is a canonical projective geometry J_b in $\mathcal{M}^{\text{eq}}$ which is nonorthogonal to J_b over a finite set.*

Proof:

One reduces to the case in which $\mathcal{M}$ is not merely coordinatizable, but coordinatized, and P_b is actually part of the coordinatizing tree for $\mathcal{M}$, taken as low in the tree as possible, hence orthogonal to its predecessors. At this stage nonorthogonality of conjugates is an equivalence relation which produces isomorphisms between equivalent pairs, and one checks that this is a *compatible family* of isomorphisms. This allows the equivalence relation to be factored out. ∎

Lemma 6.3

Let J_b and J_c be nonorthogonal projective geometries; we do not require b and c to have the same type. Then $\operatorname{dcl}(b) = \operatorname{dcl}(c)$ *and there is a unique (b, c)-definable bijection between them, preserving everything except possibly the orientation.*

Actually the isomorphism is strictly speaking a weak isomorphism if one allows the galois group of the field to come into play.

This theory is relevant to the theory of "shrinking", which refers to the process of replacing a model in which certain geometries occur by approximations in which those geometries have been replaced by finite dimensional versions. It is natural to specify the dimensions of the canonical projectives.

7. Envelopes

The theory of envelopes originates with Zilber and was elaborated on by Lachlan. In fact the emphasis on smoothly approximable structures is a direct outgrowth of the theory of envelopes. The idea is to replace the coordinatizing geometries of a Lie coordinatized structure by finite approximations.

Definition 6.2

1. *A* standard system *of geometries is a 0-definable function $D \to \mathcal{M}^{\text{eq}}$ from the locus D of a complete type over $\emptyset$ to a family of canonical projective geometries.*
2. *Two standard systems of geometries are* equivalent *if they contain a pair of nonorthogonal geometries; in this case there is a 0-definable identification between the systems, as nonorthogonality gives a 1-1 correspondence between the domains, and the nonorthogonal pairs have canonical identifications.*

Definition 7.1

3. *A* dimension function *is a function μ defined on equivalence classes of standard systems of geometries, taking as its values isomorphism types of approximations to canonical projective geometries of the given type.*
4. *For μ a dimension function, a μ-*envelope *in $\mathcal{M}$ is a subset E satisfying:*
   - (i) *E is algebraically closed in $\mathcal{M}$;*
   - (ii) *For $c \in M - E$ there is a standard system of geometries J with domain D an element $b \in D \cap E$ for which $\operatorname{acl}(E, c) \cap J_b$ properly contains $\operatorname{acl}(E) \cap J_b$;*
   - (iii) *For J a standard system of geometries defined on D and $b \in D \cap E$, $J_b \cap E$ has the isomorphism type specified by $\mu(J)$.*

In working with envelopes one tends to work in a fragment of $\mathcal{M}^e q$ containing both $\mathcal{M}$ and representative canonical projective geometries. In what follows we will refer simply to $\mathcal{M}$ but we have in mind an adequate portion of $\mathcal{M}^{\mathrm{eq}}$.

Lemma 7.2

If $\mathcal{M}$ is Lie coordinatized and $E_\circ \subseteq M$ satisfies:

(i) *$E_\circ$ is algebraically closed;*
(ii) *For each standard system of geometries J with domain D and each $b \in E_\circ \cap D$, $J_b \cap E_\circ$ embeds into a structure of the isomorphism type specified by μ, then $E_\circ$ is contained in a μ-envelope.*

Lemma 7.3

If $\mathcal{M}$ is Lie coordinatized and the dimension function μ is everywhere finite, then every μ-envelope E is finite.

Proof:

As E is algebraically closed it inherits a coordinatizing tree from $\mathcal{M}$. We may suppose every geometry is nonorthogonal to a canonical projective occurring below it. ∎

The final point in the theory of envelopes is their uniqueness and homogeneity. This requires a somewhat closer look at nonorthogonal geometries.

8. Homogeneity of envelopes

Definition 8.1

1. *Let (V, A) be an affine space defined over the set C. A is* free *over C if there is no projective geometry J defined over C for which $A \subseteq \operatorname{acl}(C, J)$.*
2. *Two affine spaces A and A' defined over C are* almost orthogonal *if there is no pair $a \in A$, $a' \in A'$, with $\operatorname{acl}(a, C) = \operatorname{acl}(a', C)$.*

Lemma 8.1

Let the ambient structure $\mathcal{M}$ be Lie coordinatized. Let (V, A) and (V', A') be almost orthogonal affine spaces defined and free over the algebraically closed set C, with PV and PV' loci of complete types over C. Let J be a projective geometry defined over C, not of quadratic type, and stably embedded in $\mathcal{M}$. For $a \in A$, $a' \in A'$, and $b \in J - C$, the triple (a, a', b) is algebraically independent over C.

Lemma 8.2

Let $\mathcal{M}$ be Lie coordinatized, and A an affine space defined and free over the algebraically closed set C. Let $C \subseteq C' = \operatorname{acl}(C')$ and let J be a canonical projective geometry associated with A. Assume:

(i) *$J \cap C' \subseteq C$;*
(ii) *$J \cap C$ is nondegenerate, if there is a form or polarity present;*
(iii) *If J is of quadratic type, then its quadratic part meets C.*

Then either A meets C', or A is free over C'.

In the conclusion, if A does not meet C' then A will not necessarily remain a geometry over C', but will split into a finite number of affine pregeometries over C'; in this case we call A free over C' if the associated geometries are free over C'.

The proof involves an induction over the coordinatization of C', working over C, and depends on the previous lemma.

Lemma 8.4

Let $\mathcal{M}$ be Lie coordinatized, μ a dimension function, and let E and E' be μ-envelopes. If $A \subseteq E$, $A' \subseteq E'$ are finite and $f : A \to A'$ is $\mathcal{M}$-elementary, then f extends to an elementary map carrying E to E'. In *particular, the envelopes are unique up to isomorphism, and homogeneous.*

Proof:

This reduces to the case of finite envelopes by the existence of sufficiently many finite envelopes. We may take A and A' algebraically closed and it suffices to extend f to $\operatorname{acl}(A,b)$ for some $b \notin A$, assuming that $A \neq E$. There are two cases, corresponding essentially to the task of extending the intersection with a canonical projective geometry, or the task of extending to the remainder of the envelope.

In the first case one essentially amalgamates an isomorphism of projective geometries with the given isomorphism over the common part. The compatibility of the two maps is given by elimination of imaginaries and stable embedding.

In the second case the maps already handle the canonical projective geometries completely. The task then reduces to the affine free case, where the previous lemma is useful; one also needs an understanding of the affine dual and elimination of imaginaries in the affine context. So this requires some attention. ∎

It follows that Lie coordinatized structures are smoothly approximated by finite substructures, namely, appropriate envelopes. For the converse, in addition to the results of [14, 15] one needs to take up the issue of orthogonality from the point of view of permutation group theory.

9. Finite structures

We begin with the relevant facts from permutation group theory.

Definition 9.1

A simple *Lie geometry is either a linear geometry of any type other than polar or quadratic, the projectivization of such a geometry, or the affine or quadratic part of a geometry.*

Here we definitely allow some or all of the Galois group of the base field to act on the geometry. The notion of a Lie geometry as such is a rather rudimentary notion from a model theoretic point of view, in the sense that is not well adapted to orthogonality theory and ignores the issue of stable embedding, but it represents the form in which such geometries are first encountered.

Definition 9.2

1. *A* coordinatizing structure *is a structure $\mathcal{C}$ with transitive automorphism group, carrying a 0-definable equivalence relation E with finitely*

many classes, such that each class is a simple Lie geometry over a finite field.

2. *If $\mathcal{C}$ is a coordinatizing structure and τ is the type over the empty set of some finite algebraically closed subset of $\mathcal{C}$, the* Grassmannian structure $\Gamma(\mathcal{C},\tau)$ *is the locus of τ in $\mathcal{C}$, with its inherited structure.*
3. *A coordinatizing structure is* proper *if the structure induced on each equivalence class is the geometrical structure, and* semiproper *if the two structures have essentially the same automorphism group in the sense that the automorphism groups have the same socle (a single simple group).*

Fact 9.1 [14]

For each k there is an n_k such that for every finite primitive structure $\mathcal{M}$ of order at least n_k, if $\mathcal{M}$ has at most k 5-types then $\mathcal{M}$ is isomorphic to a semiproper grassmannian with the size of the set whose type is τ, the size of the base field, and the number of equivalence classes in the coordinatizing structure bounded by k.

One needs to know also that the automorphism groups of the simple Lie geometries are almost simple in a strong sense, notably that the simple part has the same action on k-tuples as the full automorphism group, as the dimension goes to ∞. In particular one has to avoid various low-dimensional pathologies, occurring as high as dimension 8 for orthogonal groups of positive Witt defect.

As a special case of the results of [2] one can read off:

Fact 9.2

Let G be a subgroup of a classical group acting naturally on a finite simple classical projective geometry P without galois action, and suppose that G has the same orbits on P^3 as $\mathrm{Aut}\,P$. Then G contains $[\mathrm{Aut}\,P]^{(\infty)}$.

The following is relevant to the theory of orthogonality for structures whose automorphism group is nearly simple.

Lemma 9.1

Let H be a normal subgroup of a product $G=\prod_i G_i$ which projects surjectively onto each product of two terms $G_i\times G_j$. Then G/H is nilpotent of class at most $n-2$. In particular, if $G=G'$ then $H=G$.

10. Orthogonality for finite structures

In order to work with large finite structures we will work with infinite internally finite structures in the sense of nonstandard analysis for the present.

Definition 10.1

Let $\mathcal{M}$ be an internally finite structure in a nonstandard universe. Then $\mathcal{M}^*$ is the standard structure with the same universe whose relation symbols consist of names for all the relations in finitely many variables which are defined in $\mathcal{M}$.

Thus $\mathcal{M}^*$ is richer than than $\mathcal{M}$ in the standard part of its language, but is a reduct of $\mathcal{M}$ from the nonstandard language. As an example, a nonstandard finite linear order will carry the binary predicates $D_n(x, y)$ signifying that the distance from x to y is n, for every n, finite or infinite.

Lemma 10.1

Let $\mathcal{M}$ be an internally finite structure, J a finite disjoint union of 0-*definable projective simple Lie geometries with no additional structure. Let G be* Aut J *and let G_1 be $G^{(\infty)}$ with both groups understood internally. Let H be the group of automorphisms of J induced by internal automorphisms of $\mathcal{M}$. Then J is canonically embedded in $\mathcal{M}^*$ if and only if H contains G_1.*

For the case in which J has a single component this is more or less stated in the preceding section. The case of several components is derived from the same facts but requires more attention.

Lemma 10.4

Let $\mathcal{M}$ be an internally finite structure. Let J_i for $i \in I$ be canonically embedded projective Lie geometries in $\mathcal{M}^*$, orthogonal in pairs over the set A in $\mathcal{M}^*$. Then they are jointly orthogonal over A in $\mathcal{M}^*$.

This mainly reflects the lemma of the previous section.

Lemma 10.5

Let $\mathcal{M}$ be an internally finite structure. Let J_1 and J_2 be 0-definable basic simple projective Lie geometries canonically embedded in $\mathcal{M}^*$. Then in $\mathcal{M}^*$ we have one of the following:

1. *J_1 and J_2 are orthogonal;*
2. *There is a 0-definable bijection between J_1 and J_2;*
3. *J_1 and J_2 are of pure projective type, that is with no forms, and there is a 0-definable duality between making the pair (J_1, J_2) a polar space.*

This is based on an understanding of the outer automorphisms of the socle of the relevant automorphism group and the subgroups of a product of two simple groups which project onto both factors.

Lemma 10.6

Let $\mathcal{M}$ be an internally finite structure, A a 0-definable basic affine space, with corresponding linear and projective geometries V and J. Suppose that J is canonically embedded in $\mathcal{M}^*$. Then one of the following holds in $\mathcal{M}^*$:

1. *A is canonically embedded in $\mathcal{M}^*$;*
2. *There is a 0-definable point of A in $\mathcal{M}^*$;*
3. *J is of quadratic type and there is a 0-definable bijection of A with some multiple αQ of Q, for a unique α.*

This depends on the cohomological information in [10].

11. Coordinatization

Definition 11.1

An internally finite structure $\mathcal{M}$ in some nonstandard universe is locally Lie coordinatized *if it has finitely many 1-types, and has a coordinatizing tree of finite height whose unique root is 0-definable, and whose successors at b are either a finite set algebraic over b or a b-definable geometry J_b, basic projective, linear, or affine, with the projective and linear geometries canonically embedded in $\mathcal{M}$, and with affine spaces preceded by their linear versions.*

Lemma 11.4

Let $\mathcal{M}$ be an infinite, internally finite structure such that $\mathcal{M}^*$ has a finite number of 5*-types. Then $\mathcal{M}^*$ is biinterpretable with a locally Lie coordinatized structure.*

This is a translation of [14]. One checks that this holds for grassmannians and that $\mathcal{M}^*$ is coordinatized by grassmannians and finite structures.

Lemma 11.6

Let $\mathcal{M}$ be an internally finite locally Lie coordinatized structure with respect to the coordinate geometries in $\mathcal{J}$, and suppose:

i. Whenever $J_b \in \mathcal{J}$ is pure projective, with linear model V, the definable dual V^* is trivial.

ii Whenever $J_b \in \mathcal{J}$ is symplectic of characteristic 2, there are no definable quadratic forms on J_b compatible with the given form.

Then for any finite subset C of $\mathcal{M}$ closed downward with respect to the coordinatizing tree we have:

1. *For $b \in C$, if J_b is not affine the for some finite subset C_b of J_b, the structure (J_b, C_b) is fully embedded in $\mathcal{M}^*$ over C.*
2. *For $J_1, J_2 \in \mathcal{J}$ not affine, with defining parameters in C, if $C_i = \mathrm{acl}\,(C) \cap J_i$ then either $(J_1; C_1)$ and $(J_2; C_2)$ are orthogonal over C, or else there is a C-definable bijection between the localizations J_i/C_i.*

This requires a detailed analysis involving the general structure of the automorphism groups in both the affine and projective cases.

Proposition 11.1

Let $\mathcal{M}$ be an infinite, internally finite locally Lie coordinatized structure. Then $\mathcal{M}$ is Lie coordinatizable.

One uses geometries as rich as possible; thus if a vector space can be viewed as part of a polar pair or a quadratic geometry, we do so. From the previous Lemma one gets stably embedded nonaffine geometries, and this implies that the affine geometries are also stably embedded.

12. Geometrical finiteness

The key combinatorial property of these geometries was found by Ahlbrandt and Ziegler and reworked by Hrushovski in [7].

Definition 12.1

A countable structure $\mathcal{M}$ is geometrically finite *with respect to an ordering $<$ of type ω, if for any n and any sequence of n-tuples $\mathbf{a}_i$ in $\mathcal{M}$ there is an order-preserving elementary embedding $\alpha : \mathcal{M} \to \mathcal{M}$ taking $\mathbf{a}_i$ to $\mathbf{a}_j$ for some $i < j$.*

Lemma 12.3

The countably infinite versions of the linear geometries are geometrically finite.

This is proved as in the pure vector space case by Ahlbrandt-Ziegler/ Hrushovski based on a combinatorial result of Higman, a precursor of Kruskal's tree theorem:

Fact 12.1

The set of words in a fixed finite alphabet contains no infinite sequence *of incomparable words; words are comparable if one is a subword of the other (the letters should occur in the proper order, but not necessarily consecutively).*

The idea of the proof in the vector space case, over a finite field, is that relative to a basis, vectors are coded by finite strings of field elements. If the vector space is decorated by forms one can use appropriate orthogonal bases; in the symplectic case the 'basis' elements should be nondegenerate 2-dimensional subspaces, also known as hyperbolic planes. A certain amount of linear algebra is needed as well to keep track of more than one vector at a time.

The geometrical finiteness will lead to a finite language as well as finite axiomatizability modulo appropriate axioms of infinity.

After treating the linear case, one can handle the variations by direct reduction to the linear case, so all of the coordinatizing geometries are geometrically finite.

13. Sections

Once one has the coordinatizing geometries geometrically finite with respect to appropriate orderings (i.e., enumerations) the next task is to enumerate the coordinatizing tree in a corresponding manner, and to check the geometrical finiteness of a Lie coordinatized structure (not necessarily in order type ω). Initial segments of $\mathcal{M}$ with respect to an appropriate ordering are called sections of $\mathcal{M}$. We use a breadth-first enumeration of the tree. Each section is determined by a finite set of data, called its support, specifying how much of the tree structure has been completely enumerated (which is given by a bounded amount of data) as well as how things stand with respect to the part of the enumeration currently 'active'; the latter involves a finite set of data of unbounded size.

As background for all of this it is also useful to introduce the notion of a skeletal language and a skeletal type, which amounts to a description of the structure of a coordinatizing tree, specifying the types of the geometries involved, without going into the details of any particular structure $\mathcal{M}$. The basic problem is to axiomatize the theory of $\mathcal{M}$ modulo the skeletal data, essentially by describing as far as possible how the structure "evolves" as one follows the induced enumeration of the coordinatizing tree. The first step is to finitize the language.

14. Finite language

We referred in passing to the notion of a skeleton above, as a rough description of a coordinatizing tree. We will refer to a structure $\mathcal{M}$ as a proper skeletal expansion of a skeletal type if it actually has the properties the skeleton is intended to describe. For example, the skeleton will specify which geometries are supposed to be orthogonal and which are not, and in a proper skeletal expansion this will actually be the case.

We now capture the geometrical finiteness of $\mathcal{M}$. The basic relation of interest is a relation on triples (E, X, a) with E an envelope for $\mathcal{M}$, X a subset of E – either a section or the support of a section – and a a finite sequence of specified length. We let $\mathcal{U}_n$ and $\mathcal{S}_n$ be the sets of triples where a has length n and where X is a section ($\mathcal{U}$) or section support ($\mathcal{S}$) respectively.

Lemma 14.2

Let $\mathcal{M}$ be a proper countable skeletal expansion. Then the quasi-ordered sets $\mathcal{U}_n$ and $\mathcal{S}_n$ contain no infinite antichains.

This is bootstrapped up from the case of geometries but depends on the theory of orthogonality. A version of this is found in [7, Lemma 2.10] in a rather abstract notation.

This leads to:

Lemma 14.4

Let $\mathcal{M}$ be a Lie coordinatized structure. Then there is an integer k such that:

1. *For any envelope E, any section U of E, and any $a \in E$, if $a \in \operatorname{acl}(U)$ there for some subset C of U of size at most k, a is algebraic over C and its multiplicity over U and over C coincide.*
2. *For any envelope E, any section support S in E, and any $a \in E$, if $a \in \operatorname{acl}(S)$ then for some subset C of S of size at most k, a is algebraic over C and its multiplicity over S and over C coincide.*

Taking as our language for $\mathcal{M}$ the restriction of the canonical language of $\mathcal{M}$ to types in at most $k+1$ variables, which we call the standard language, we get:

Proposition 14.1, Lemma 14.5

If $\mathcal{M}$ is Lie coordinatized then $\mathcal{M}$ admits a finite language, with respect to which the theory of any envelope is model complete.

Proof:

The first claim can be reduced to the following: For any section U of an envelope E of $\mathcal{M}$, and any $a \in E$, the type of a over U in the standard language determines its type over U. The algebraic case is built in to the definitions, via the choice of k. The nonalgebraic case can be driven down to a property of the geometry J in which a lies: $\operatorname{acl}(U) \cap J \subseteq U$. In the main case this reduces to $\mathrm{acl}(U \cap J) \cap J \subseteq U \cap J$ (when the next element is nonalgebraic) which is a property of the standard enumerations used in these geometries.

The proof of the second point involves a similar analysis. ∎

Once one has a finite language, it makes sense to consider questions of finite axiomatizability relative to the axioms of infinity specifying that certain coordinatizing geometries are infinite.

15. Quasifinite axiomatizability

Let $\mathcal{M}$ be Lie coordinatized and L its standard language.

Definition 15.1

A characteristic sentence *for $\mathcal{M}$ is an L-sentence whose proper countable models are the envelopes of $\mathcal{M}$, up to isomorphism.*

Proposition 15.1

With the skeletal data and the language L fixed, there is a recursive class Ξ of potential characteristic sentences (which may include spurious candidates) which contains characteristic sentences for every Lie coordinatized structure in the language L; any member of Ξ which actually has a proper model will be a characteristic sentence for that model.

We allow spurious characteristic sentences as the problem of consistency is a fundamental problem of effectivity which is approached by other methods.

The proof of this is rather delicate, as are its precursors in [1] and [7]. The model completeness of sections noted above is one of the ingredients. In addition the global geometric finiteness is invoked again to finitize auxiliary parameters needed for the axiomatization. In essence one simply wants to axiomatize the way the type of the next element in an enumeration will depend on the preceding section. After writing down what appears to be an appropriate axiomatization, one carries out a "forth" argument – a 1-way back-and-forth – establishing isomorphism of any model with an envelope in the standard model. This creates the odd impression that they are only isomorphic in one direction. However this feature has been present in the theory since [1]. The details of this final step are considerably closer to [7] than to [1] but one gets less help from stability theory.

The next point is that the set Ξ can itself be taken to be finite, once one has fixed both the skeleton and the language L. One can however enrich a fixed finite cover of a projective geometry in infinitely many different ways, letting the arity of the language L go to infinity. The structures involved have less and less structure as the complexity of the language increases; they will agree with the unadorned cover up to some fixed dimension. One might possibly expect the compactness theorem to bound the complexity of the finite language L, but these examples show that the language can degenerate in the limit.

16. Ziegler's Finiteness Conjecture

Proposition 16.1

Let a skeletal type and corresponding skeletal language L_{sk} be fixed, and let L be a fixed finite language containing L_{sk}. Then there are only finitely many Lie coordinatized structures in the language L having a given skeleton M_{sk}, up to isomorphism.

Proof:

It suffices to combine Proposition 15.3 with the Compactness Theorem. For this one must check that the class of Lie coordinatized structures in the language L with the specified skeleton is an elementary class. ∎

Definition 16.1

Let $\mathcal{M}$ be a structure.

1. *A* cover *of $\mathcal{M}$ is a structure $\mathcal{N}$ and a map $\pi : N \to M$ such that the equivalence relation E_π given by "$\pi x = \pi y$" is 0-definable in $\mathcal{N}$, and the set of E_π-invariant 0-definable relations on $\mathcal{N}$ coincides with the set of pullbacks along π of the 0-definable relations in $\mathcal{M}$.*
2. *Two covers $\pi_1 : \mathcal{N}_1 \to \mathcal{M}$, $\pi_2 : \mathcal{N}_2 \to \mathcal{M}$ are* equivalent *if there is a bijection $\iota : N_1 \leftrightarrow N_2$ compatible with π_1, π_2 which carries the 0-definable relations of $\mathcal{N}_1$ onto those of $\mathcal{N}_2$.*
3. *If $\pi : \mathcal{N} \to \mathcal{M}$ is a cover, then $\mathrm{Aut}\,(\mathcal{N}/\mathcal{M})$ is the group of automorphisms of $\mathcal{N}$ which act trivially on the quotient $\mathcal{M}$. Thus $\mathrm{Aut}\,(\mathcal{N}/\mathcal{M}) \leq \prod_{a\in M} \mathrm{Aut}_{\mathcal{N}}(C_a)$ where $C_a = \pi^{-1}(a)$ and $\mathrm{Aut}_{\mathcal{N}}(C_a)$ is the permutation group induced by the setwise stabilizer of C_a in $\mathrm{Aut}\,\mathcal{N}$.*

The problem of the theory of covers is to classify or at least restrict the possible covers with given quotient and specified fiber.

Proposition 16.3

Let $\mathcal{M}$ be a fixed Lie coordinatized structure and let J be a fixed geometry or a finite structure. Then there are only finitely many covers $\pi : \mathcal{N} \to \mathcal{M}$ up to equivalence which have fiber J and a given relative *automorphism group* $\mathrm{Aut}\,(\mathcal{N}/\mathcal{M}) \le \prod_{N/E} \mathrm{Aut}\, J$.

Proof:

We apply Proposition 16.2. The skeleton N_{sk} of $\mathcal{N}$ is determined by the given data and thus it suffices to find a single finite language L adequate for all such covers $\mathcal{N}$. Thus it suffices to bound the arity k of L and the number of k-types occurring in $\mathcal{N}$.

We will discuss the arity, using the language of permutation groups. We must find k so that $\mathrm{Aut}\,(\mathcal{N})$ is a k-closed group, for all suitable covers $\mathcal{N}$. $\mathrm{Aut}\,(\mathcal{M})$ is $k_\circ$-closed for some $k_\circ$. If we restrict attention to $k \ge k_\circ$, then $\mathrm{Aut}\,(\mathcal{N})$ is k-closed if and only if $\mathrm{Aut}\,(\mathcal{N}/\mathcal{M})$ is k-closed, as is easily checked. Thus for $k \ge k_\circ$ the choice of k is independent of the cover, as long as the relative automorphism group is fixed in advance.

The number of types can be estimated more directly in the two cases (finite or affine fibers). ∎

This completes the first layer of the theory. In the $\aleph_0$-categorical, $\aleph_0$-stable setting, this is reasonably satisfactory, though it leaves open questions of effectivity. In any case this class is certainly closed under interpretability. In the more general context considered here we have proceeded with the rudiments of stability theory, using little more than algebraicity in rank 1 sets and orthogonality as a means of achieving global results. For a deeper analysis one must now return to the foundations of the subject and build up appropriate parallels to modern stability theory.

The first of these is Hrushovski's so-called "independence theorem", for which I currently prefer the term "type amalgamation".

17. Type amalgamation

Definition 17.1

Let $\mathcal{M}$ be a structure.

1. *An* amalgamation problem *of length n is given by the following data:*

(1) *A base set A*

(2) *Types $p_i(x_i)$ over A for $1 \le i \le n$*

(3) *Types $r_{ij}(x_1, x_j)$ over A for $1 \le i < j \le n$*

subject to:

$$r_{ij} \text{ contains } p_i(x_i) \cup p_j(x_j)$$

(4) *$r_{ij}(x_i, x_j)$ implies the independence of x_i from x_j*

2. *A* solution *to an amalgamation problem is a type r of an independent n-tuple $x_1, \ldots, x_n$ such that the restrictions of r coincide with the given types.*

Our goal here is to prove that amalgamation problems of this type over an algebraically closed subset of $\mathcal{M}^{\mathrm{eq}}$ with $\mathcal{M}$ Lie coordinatized always have solutions.

One establishes this result first in the individual geometries, and then one builds up to the general result via a series of special cases. At key points we work directly in the geometries, using the fact that we know them concretely. In the process one also encounters “generic equivalence relations”, which are useful elsewhere, so we will begin with that topic.

Definition 17.3.1

Let $\mathcal{M}$ be a structure, E a definable binary relation, D a definable set, a, b elements of $\mathcal{M}$. E is a generic equivalence relation *on D if it is generically symmetric and transitive: for any independent triple a, b, c in its domain, $E(a, b)$ and $E(b, c)$ imply $E(b, a)$ and $E(a, c)$.*

Lemma 17.5.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and E a generic equivalence relation defined on the locus of a complete type p over $\operatorname{acl}(\emptyset)$. Then E agrees with a definable equivalence relation E^* on independent pairs from p.

Proof:

Define $E^*(x, y)$ by: $p(x)$ and $p(y)$ and either $x = y$ or there is a z realizing p independent from x, y so that $E(x, z)$ and $E(y, z)$ hold. ∎

We will indicate briefly how this used in the present context.

Definition 17.3.2-3

An indiscernible sequence I is 2-independent *if $\operatorname{acl}(a) \cap \operatorname{acl}(b) = \operatorname{acl}(\emptyset)$ for $a, b \in I$ distinct. E_2 is the smallest equivalence relation containing all pairs which belong to infinite 2-independent indiscernible sequences.*

Lemma 17.5.2

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Then any two elements with the same type over $\operatorname{acl}(\emptyset)$ are E_2-equivalent.

The key special case of type amalgamation, and the only one we will actually discuss, is the following.

Lemma 17.6

Let $\mathcal{M}$ be a Lie coordinatized structure. Let $(p_i; r_{ij})$ be an amalgamation problem of length 3 over $\operatorname{acl}(\emptyset)$ *with p_1 the type of a pair (ab) with a in a rank 1 geometry J and b algebraic over a. If $r_{12} = r_{13}$ up to the necessary change of variable, then the amalgamation problem has a solution.*

Proof:

As a matter of notation, use variables x, y for realizations of p_1 and z_2, z_3 for realizations of p_2, p_3. Let C be the set defined by p_2; it is also defined by p_3. After some preliminary adjustment we may suppose that for $c \in C$, $r_{12}(xy, c)$ isolates a type over $\operatorname{acl}(c)$.

Now for $a \in J$ satisfying p_1, $c, c' \in C$ we consider the set $B(a, c) = \{y : r_{12}(ay, c)\}$ and the sets $J(c) = \{a \in J : B(a, c) \neq \emptyset\}$, $J(c, c') = \{a \in J : B(a, c) = B(a, c') \neq \emptyset\}$. In particular $J(c, c') \subseteq J(c) \cap J(c')$. We define a relation E on C as follows: $E(c, c')$ if and only if $J(c, c')$ is infinite. Using our detailed understanding of J we can show that E is a generic equivalence relation extending E_2 – this is the heart of the analysis. Then by the preceding lemmas, $E(c_2, c_3)$ holds for any independent pair c_2, c_3 in C, in particular for a realization of r_{23}. This then allows us to solve the amalgamation problem directly. ∎

We will not go through the various reductions to the case treated above. The following corollary to type amalgamation is very useful.

Corollary

Let $\mathcal{M}$ be a Lie coordinatized structure, I an independent set, $p(x)$ a complete type over $\operatorname{acl}(\emptyset)$, *and $\phi_a(a, x)$ $(a \in I)$ a collection of formulas for which $\phi_a \& p$ is consistent of rank* $\operatorname{rk} p$. *Then $\bigwedge_I \phi_a \& p$ is consistent of rank* $\operatorname{rk} p$.

Proof:

We may assume first that I is finite and then that $|I| = 2$ as the statement is iterable. So we are considering $\phi_1(a_1, a_3) \& \phi_2(a_2, a_3) \& p(a_3)$, with a_1, a_2 independent. This can be converted into an amalgamation problem of the type covered by the preceding proposition. ∎

18. The sizes of envelopes

For technical reasons it is useful to have some information on the sizes of envelopes. This gives a little more control over the approximations by finite structures. We wish to express the sizes of envelopes as polynomial functions of the relevant data, and to do so it will be convenient to work with square roots of the sizes of the associated fields.

Notation

Let $\mathcal{M}$ be Lie coordinatized and p a canonical projective geometry. For an envelope E we let $d(p)$ be the corresponding dimension (or cardinality in the degenerate case) and we let $d^*(p) = (-\sqrt{q})^{d(p)}$ where q is the size of the base field; in the degenerate case we set $d^*(p) = \sqrt{d(p)}$.

Proposition 18.1

Let $\mathcal{E}$ be a family of envelopes for the Lie coordinatized structure $\mathcal{M}$ such that for each dimension p corresponding to an orthogonal space, the signature and the parity of the dimension is constant on the family. Then there is a polynomial ρ such that for every E in $\mathcal{E}$, $|E| = \rho(d^*(E))$. The total degree of ρ is $2\mathrm{rk}\,\mathcal{M}$ and all leading coefficients are positive. If $\mathcal{M}$ is the locus of a single type, then ρ is a product of polynomials in one variable.

This comes down to a computation in the basic geometries. It then leads to:

Lemma 18.2

Let $\mathcal{M}$ be a Lie coordinatized structure and D a definable subset. Then the following are equivalent:

1. $\mathrm{rk}\, D < \mathrm{rk}\, \mathcal{M}$;
2. $\lim_{E\to\mathcal{M}} |D[E]|/|E|) = 0$.
   Here the limit is taken over envelopes whose dimensions all go to infinity.

There is also a finitary Lowenheim-Skolem principle.

Lemma 18.3

Let $\mathcal{M}$ be pseudofinite. For any subset X of $\mathcal{M}$ there is an envelope E of $\mathcal{M}$ containing X, in which each dimension is at most $2\mathrm{rk}\,(X) \leq 2\mathrm{rk}\,\mathcal{M}\cdot|X|$.

19. Nonmultidimensional expansions

We show next that Lie coordinatizable structures have nonmultidimensional expansions, lifting a result of [7, §3] to the present context. While not essential, this does have a simplifying effect on the analysis of situations where an expansion is permissible – a rather common occurrence.

Definition 19.1

A Lie coordinatized structure is non-multidimensional if it has only finitely many dimensions, or equivalently if all canonical projectives are definable over $\operatorname{acl}(\emptyset)$.

Proposition 19.1

Every Lie coordinatized structure expands to a non-multidimensional Lie coordinatized structure.

We will not say much about the argument. One works inductively up the coordinatizing tree, and by induction it suffices to deal with the first level at which geometries are encountered which are orthogonal to the preceding ones. These must be "glued together" by imposing additional structure without decreasing the automorphism group on the original structure.

20. Canonical bases

We do not in fact have a theory of canonical bases as such, but the following result may serve as a very useful substitute.

Proposition 20.1

Let $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank. Suppose* a_1, a_2, a_3 *is a triple of elements which are independent over* a_1*, over* a_2*, and over* a_3*. Then* a_1, a_2, a_3 *are independent over the intersection of* $\operatorname{acl}(a_i)$, $i = 1, 2, 3$, *in* $\mathcal{M}^{\mathrm{eq}}$.

Our first lemma is a variation on the theme of generic equivalence relations.

Lemma 20.1

Let $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank and let* R *be a 0-definable symmetric binary relation satisfying:*

> *Whenever* $R(a,b)$, $R(b,c)$ *hold with* a, c *independent over* b
> *then* $R(a,c)$ *holds and* b, c *are independent over* a.

Then there is a 0-definable equivalence relation E *such that:*

> $R(a,b)$ *implies:* $E(a,b)$*, and* a, b *are independent over* $a/E = b/E$.

Proof:

We define $E(a,b)$ as follows: For some c independent from a over b and from b over a, $R(a,c)$ and $R(b,c)$ holds. There are then quite a number of points to be checked. ∎

Definition 20.1

Let $a_1,\ldots,a_n$ be a sequence of elements in a structure of finite rank.

1. *The sequence is said to be* locally independent *if it is indpendent over any of its elements.*
2. *We set $\delta(a_1,\ldots,a_n) = \sum_i \mathrm{rk}\, a_i - \mathrm{rk}\,(a_1\ldots a_n)$.*

The next lemma can be verified fairly directly by computation. It is quite useful.

Lemma 20.2

Let $\mathcal{M}$ be a structure of finite rank, $\mathbf{a} = a_1,\ldots,a_n$ a sequence of elements. Then the sequence $\mathbf{a}$ is locally independent *if and only if:*

$$\delta = \delta(a_ia_j) \text{ is independent of } i,j \text{ (distinct); and } \delta(\mathbf{a}) = (n-1)\delta.$$

This reduces the following result to a computation.

Lemma 20.3

Let $\mathcal{M}$ be a structure of finite rank.

1. *Suppose that $\mathbf{a} = a_1,a_2,a_3,a_4$ is a sequence with a_1,a_2,a_3 and a_2,a_3,a_4 locally independent. If a_1 and a_4 are independent over a_2, a_3 then $\mathbf{a}$ is locally independent.*
2. *If $\mathbf{a} = a_1a_2b_1b_2c_1c_2$ is a sequence whose first four and last four terms are locally independent, and a_1a_2 is independent from c_1c_2 over b_1b_2, then $\mathbf{a}$ is locally independent.*

Proof of Proposition 20.1:

We have a_1,a_2,a_3 locally independent. Let X be the set of pairs $x = (x_1,x_2)$ such that each coordinate x_i realizes the type of one of the three elements a_i and define a relation R on X by: $R(x,y)$ if and only if with x_1,x_2,y_1,y_2 is a locally independent quadruple. We will apply Lemma 20.1 to R. Note first that if $R(x,y)$ and $R(y,z)$ hold with x and z independent over y then the 6-tuple (x,y,z) satisfies the conditions of case 2 of the previous lemma, and thus the six coordinates form a locally independent sequence. Thus Lemma 1 applies and there is a 0-definable equivalence relation E such that:

$$R(x,y) \text{ implies: } E(x,y), \text{ and } x,y \text{ are independent over } x/E$$

Now consider the locally independent triple (a_1, a_2, a_3) we extend it by two further elements a_4, a_5 satisfying: $\text{tp}(a_i/a_2a_3) = \text{tp}(a_1/a_2a_3)$, a_i independent from a_1 over a_2a_3, for $i = 4, 5$. The sequences $a_1a_2a_3a_4$ and$a_1a_2a_3a_5$ are covered by case 1 of the previous lemma and thus are locally independent. Using a_5 we can show that any two pairs with coordinates among $a_1a_2a_3a_4$ are E-equivalent. This is the case by definition if the pairs partition the sequence, and to link for example a_1a_2 with a_1a_4 we use: $E(a_1a_2, a_3a_5)$; $E(a_3a_5, a_2a_3)$; $E(a_2a_3, a_1a_4)$. Let e be the E-class of any such pair. Then a_1a_2 is independent from a_3a_4 over e and a_1a_3 is independent from a_2a_4 over e. Thus in particular over e we get: a_3 is independent from a_1a_2 and a_1 is independent from a_2, so $a_1a_2a_3$ is an independent set over e. It remains only to be checked that e is algebraic over each a_i. Certainly $e \in \text{acl}(a_1a_2)$ and $\text{acl}(a_3a_4)$ and as these pairs are independent over any a_i, $e \in \text{acl}(a_i)$ for all i. ∎

21. Modularity

Definition 21.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. $\mathcal{M}$ is modular *if whenever A_1, A_2 are algebraically closed sets in $\mathcal{M}^{\text{eq}}$, they are independent over their intersection.*

This is traditionally called "local modularity", which corresponds to a characterization of the property in terms of the structure of coordinatizing geometries geometries.

Proposition 21.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Then the following are equivalent.

1. *$\mathcal{M}$ is modular.*
2. *For all finite A_1, A_2 in $\mathcal{M}$, A_1 and A_2 are independent over the intersection of their algebraic closures.*
3. *For all finite A_1, A_2 in $\mathcal{M}$, there is a finite C independent from A_1, A_2 such that A_1, A_2 are independent over the intersection of the algebraic closures of $A_1 \cup C$ and $A_2 \cup C$.*
4. *The lattice of algebraically closed subset of $\mathcal{M}^{\text{eq}}$ is a modular lattice.*

Though this requires some argument, the present situation is not terribly different from the stable case. However Proposition 20.1 comes in repeatedly. This is worth illustrating. Suppose for example that the modular law holds for algebraically closed subsets of $\mathcal{M}^{\text{eq}}$ and we wish to verify the modularity of $\mathcal{M}$ according to the definition. Thus we have A, B given and we claim $A \perp B$ over the intersection of their algebraic closures. We proceed by induction on $r = \mathrm{rk}\,(A/B)$, and for fixed r, on $\mathrm{rk}\,A$. Adding constants, we may suppose $\mathrm{acl}\,(A) \cap \mathrm{acl}\,(B) = \mathrm{acl}\,(\emptyset)$. It is convenient to denote the latter by: 0. Making use of the modular law and the induction hypothesis one may reduce with some argument to the case in which A is an atom in the lattice of algebraically closed sets.

Now consider a conjugate B' of B over A which is independent from B over A. Then $\mathrm{acl}\,(AB) \cap B' \subseteq \mathrm{acl}\,(A) \cap B' = 0$. If the triple A, B, B' is locally independent then by Proposition 20.1 it is independent over the intersection of the algebraic closures, which is 0, and we are done. Otherwise one has dependence over B or B' and in either case $\mathrm{rk}\,(A/BB') < \mathrm{rk}\,(A/B)$, and now induction applies to give A independent from BB' over $\mathrm{acl}\,(A) \cap \mathrm{acl}\,(BB')$. Now the fact that A is an atom comes into play. The main case then is: $A \subseteq \mathrm{acl}\,(BB')$, which by the modular law, applied to $\mathrm{acl}\,(A, B)$ B, and B', will quickly produce $A \subseteq B$, a real *reductio ad absurdum*.

The next step, following normal lines of development, is:

Proposition 21.2 [Rank inequality]

Let $\mathcal{M}$ be $\aleph_0$-categorical, of finite rank, modular, and with the type amalgamation property. Let D, D' be 0-definable sets with D' parametrizing a family of definable subsets D_b of D of constant rank r for $b \in D'$. Suppose that E is a 0-definable equivalence relation on D' such that for inequivalent $b, b' \in D'$ we have $\mathrm{rk}\,(D_b) \cap \mathrm{rk}\,(D_{b'}) < r$. Then $\mathrm{rk}\,(D'/E) + r \leq \mathrm{rk}\,D$.

Since this involves the type amalgamation property, which is very heavily used in this theory, we give the proof, which otherwise consists of normal arguments.

Proof:

We may assume that both D and D' each realize a unique type over the empty set. Take $b \in D'$ and $a \in D_b$ with $\mathrm{rk}\,(a/b) = r$. Let $C = \mathrm{acl}\,(a) \cap \mathrm{acl}\,(b)$. Thus $a \perp b$ over C by modularity, and $\mathrm{rk}\,(a/C) = \mathrm{rk}\,(a/b) = r$. We will show

$$(*) \qquad b/E \in C$$

Thus $\mathrm{rk}\,(D'/E) \leq \mathrm{rk}\,C = \mathrm{rk}\,(aC) - \mathrm{rk}\,(a/C) = \mathrm{rk}\,(a) - r$ as claimed. So we turn to $(*)$.

Let b'/E be a conjugate of b/E over C distinct from b/E, with b' independent from b over C. We seek an element b'' of D' satisfying:

$$\mathrm{tp}(b''b/C) = \mathrm{tp}(b'b/C);\ \mathrm{tp}(b'', a/C) = \mathrm{tp}(b, a/C)$$

with a, b, b'' independent over C. which amounts to an amalgamation problem for the three compatible 2-types $\mathrm{tp}(ba/C)$, $\mathrm{tp}(b'b/C)$, $\mathrm{tp}(ba/C)$. By the type amalgamation property, this can be done.

In particular $a \in D_b \cap D_{b'}$ and thus $\mathrm{rk}\,(a/bb') < r$, so $\mathrm{rk}\,(a/C) < r$, a contradiction. Thus there is no such conjugate b' and $b \in \mathrm{dcl}\,(C) = C$. ∎

As an application one can show that there is no pseudoplane interpreted in $\mathcal{M}$, answering a question raised in [14].

There are two other basic issues to be dealt with. One is the characterization of modular structures in terms of the properties of coordinatizing geometries, which runs along standard lines. The other is the behavior under reducts, since we do not have the luxury of working in a class which is itself closed under reducts.

22. Reducts of modular structures

Proposition 22.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, and modular. Then every reduct $\mathcal{M}'$ of $\mathcal{M}$ inherits these properties.

We will just summarize the approach taken, which has one useful side effect apart from the Proposition stated.

Definition 22.1

Let a, b be elements of a structure of finite rank. Then b is filtered *over a if there is a sequence $\mathbf{b} = b_1, \ldots, b_n$ with $\mathrm{rk}\,(b_i/ab_1 \ldots b_{i-1}) = 1$ and $\mathrm{acl}\,(a\mathbf{b}) = \mathrm{acl}\,(ab)$.*

Lemma 22.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular, $\mathcal{M}'$ a reduct in $\mathcal{M}$, and $a, b_1, b_2, \ldots, b_n$ elements of $\mathcal{M}'$ with b_i not $\mathcal{M}$-algebraic over $a, b_1, \ldots, b_{i-1}$ for all i. Then there are $b'_1, \ldots, b'_n$ with $\mathrm{tp}(\mathbf{b}'/a) = \mathrm{tp}(\mathbf{b}/a)$, b'_i not $\mathcal{M}$-algebraic over $ab'_1 \ldots b'_{i-1}$ for any i, and $a \notin \mathrm{acl}_{\mathcal{M}}(\mathbf{b}')$.

Lemma 22.2

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular, $\mathcal{M}'$ a reduct in $\mathcal{M}$, and a, b elements of $\mathcal{M}'$ with b filtered over a. Then a is independent from b over $\mathrm{acl}\,(a) \cap \mathrm{acl}\,(b)$.

We emphasize that the model theoretic notions used are those of $\mathcal{M}'$ rather than $\mathcal{M}$. The proof is an induction on $\mathrm{rk}\,(a)$ which naturally aims at driving the situation back to modularity in $\mathcal{M}$, after replacing b by a more suitably placed element of $\mathcal{M}$ which is conjugate to b in $\mathcal{M}'$. The Proposition 20.1 again comes into play.

The next lemma remains useful even when applied to the original structure $\mathcal{M}$. The previous lemma reduces it to a straightforward induction.

Lemma 22.3

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and modular and let $\mathcal{M}'$ be a reduct in $\mathcal{M}$. Then for any a, b in $\mathcal{M}'$, b is filtered over a in $\mathcal{M}'^{\mathrm{eq}}$.

These results immediately yield Proposition 22.1.

23. Local characterization of modularity

Definition 23.1

Let $\mathcal{M}$ be a structure.

1. *A definable subset D of $\mathcal{M}$ is* modular *if for every finite subset A of $\mathcal{M}$, the structure with universe D and relations the A-definable relations of $\mathcal{M}$ restricted to D, is modular.*
2. *Let $\mathcal{F}$ be a collection of definable subsets of $\mathcal{M}$. Then $\mathcal{M}$ is* eventually coordinatized *by $\mathcal{F}$ if for any $a \in M$ and finite $B \subseteq M$, with $a \notin \operatorname{acl}(B)$, there is $B' \supseteq B$ independent from a over B and a B'-definable member D of $\mathcal{F}$ for which $D \cap \operatorname{acl}(aB')$ contains an element not algebraic over B'.*

Lemma 23.1

If $\mathcal{M}$ is eventually coordinatized by a family of modular definable sets, then it is eventually coordinatized by a family of modular definable sets of rank 1.

The main point perhaps is that this depends on the results of the previous section, both the preservation under reducts (applied directly to definable rank 1 sets) and the filtration result of Lemma 22.3. Modulo this, it is a direct and brief induction argument.

Proposition 23.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. If $\mathcal{M}$ is eventually coordinatized by modular definable sets, then $\mathcal{M}$ is modular.

After applying the previous lemma, the idea of the argument is to show that a suitably minimized counterexample to modularity necessarily takes place in one of the coordinatizing geometries (initially, it may seem more likely that it would take place in two of them).

Corollary

If $\mathcal{M}$ is Lie coordinatized then $\mathcal{M}$ is modular.

This completes the general theory of modularity in this context. The next topic, closely related to modularity, is the behavior of definable groups.

24. Generation and stabilizers

We consider definable groups in Lie coordinatized structures. While modularity is the main ingredient at the beginning, it does not suffice for the full theory, so at a certain point in the development it is again necessary to take into account the Lie coordinatization again.

The main results are fairly standard, though the definitions and proofs vary noticeably from narrower versions of the theory.

We work with $\mathcal{M}^{eq}$ and consider certain subsets that may meet infinitely many sorts of $\mathcal{M}^{eq}$. However in such cases we adopt the following terminology, reflecting the greater generality of this situation relative to the usual context of model theory.

Definition 24.1

Let $\mathcal{M}$ be a many-sorted structure. A subset S of $\mathcal{M}$ is locally definable *if its restriction to any sort (equivalently, any finite set of sorts) is definable. In particular a group is locally definable in $\mathcal{M}$ if its underlying set and its operations are locally definable. When the sorts of $\mathcal{M}$ all have finite rank, a locally definable subset is said to have* finite rank *if its restrictions to each sort have bounded rank; the maximum such rank is then the rank of S.*

We record a number of results on generation, mainly without proofs, that do not present much in the way of novelty.

Lemma 24.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G be a locally definable group in $\mathcal{M}^{eq}$, S a definable subset closed under inversion and generic multiplication: for a, b in S independent, $ab \in S$. Then $H = S \cdot S$ is the subgroup of G generated by S; and $\operatorname{rk}(H - S) < \operatorname{rk} S$

The next lemma reduces to the previous one. Since it illustrates the use of type amalgamation, we give the proof.

Lemma 24.2

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property. Let G be a locally 0-definable group of finite rank k in $\mathcal{M}^{eq}$ and

$S \subseteq G$ the locus of a complete type over $\operatorname{acl}(\emptyset)$*, of rank k. Then $S \cdot S^{-1}$ generates a definable subgroup of G.*

Proof:

Let $X = \{ab^{-1} : a, b \in S; \operatorname{rk}(a, b) = 2k\}$. We claim that the previous lemma applies to X, and that the groups generated by $S \cdot S^{-1}$ and by X coincide. In any case X is closed under inversion. We show now that X is closed generically under the operation ab^{-1}, and hence under multiplication.

Let $c_1, c_2 \in X$ be independent, $c_i = a_i b_i^{-1}$ with $a_i, b_i \in S$, $\operatorname{rk}(a_i, b_i) = 2k$. We may suppose that (a_1, b_1) is independent from (a_2, b_2) and hence that a_1, a_2, b_1, b_2 is an independent quadruple. We seek d independent from this quadruple satisfying:

$$\operatorname{tp}(d/c_1) = \operatorname{tp}(b_1/c_1); \quad \operatorname{tp}(d/c_2) = \operatorname{tp}(b_2/c_2)$$

As S is a complete type over $\operatorname{acl}(\emptyset)$ and b_i is independent from c_i, this is a type amalgamation problem of the sort that can be solved. The type of d now ensures the solvability of the equations

$$c_1 = a_1' d^{-1}; \quad c_2 = a_2' d^{-1}$$

with a_1', a_2' in S. Thus $c_1 c_2^{-1} = a_1' a_2'^{-1}$ We claim that this forces $c_1 c_2^{-1}$ into S, with a_1', a_2' as witnesses. Since $a_i' \in \operatorname{dcl}(a_i, b_i, d)$, we have a_1' and a_2' independent over d. Also $\operatorname{rk}(a_i', b_i, d) = \operatorname{rk}(a_i, b_i, d) = 3k$, so a_i and e are independent. Thus a_1' and a_2' are independent. Thus $c_1 c_2^{-1} \in X$.

Now suppose $a, b \in S$. Take $d \in S$ independent from a. Then $ab^{-1} = (ad) \cdot (bd)^{-1} \in X \cdot X$. Thus $S \cdot S^{-1}$ and X generate the same subgroup. ∎

Lemma 24.3

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G be a locally definable group in $\mathcal{M}^{\mathrm{eq}}$, and S a definable subset generically closed under the ternary operation $ab^{-1}c$ (an affine group law). Then S lies in a coset C of a definable subgroup H of G, with $\operatorname{rk}(C - S) < \operatorname{rk} S$.

Definition 24.2

Let $h : G_1 \to G_2$ be a map between groups. Then h is an affine homomorphism *if it respects the operation $ab^{-1}c$.*

Reworking the previous lemmas in terms of graphs of homomorphisms we get:

Lemma 24.4

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank. Let G, H be locally 0-definable *groups in $\mathcal{M}^{\rm eq}$, S a 0-definable subset of G, and $h : S \to H$ a 0-definable function.*

1. *If S is generically closed under the affine group operation $ab^{-1}c$ and h respects this operation generically, then h extends to an affine group homomorphism with domain the coset of a definable subgroup generated by S (under the affine group operation).*
2. *If S is generically closed under the operation ab^{-1} and h respects this operation generically, then h extends to a group homomorphism defined on the subgroup of G generated by S.*

In the next lemma, as we deal with locally definable groups, the hypothesis of bounded exponent is essential.

Lemma 24.5

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank, with the type amalgamation property. Let G be a locally definable group in $\mathcal{M}^{\rm eq}$ of bounded rank which is abelian of bounded exponent. Then for any definable subset S of G, the subgroup generated by S is definable.

We now turn to the notion of the stabilizer of a definable set S. This is a fairly delicate notion in our context.

Definition 24.3

Let $\mathcal{M}$ have finite rank, G a definable group in $\mathcal{M}$, and let D, D' be complete types over $\operatorname{acl}(\emptyset)$, *contained in G, with $\operatorname{rk} D = \operatorname{rk} D' = r$. Then*

1. $\operatorname{Stab}_\circ(D, D') = \{g \in G : \operatorname{rk}(Dg \cap D') = r\}$.
2. $\operatorname{Stab}_\circ(D) = \operatorname{Stab}_\circ(D, D)$ *and* $\operatorname{Stab}(D)$ *is the subgroup of G generated by* $\operatorname{Stab}_\circ(D)$.

Though we claim that $\operatorname{Stab}(D)$ is generically closed under multiplication, it will not in general actually be a subgroup.

Example

Let (V, Q) be an infinite dimensional orthogonal space over a finite field of characteristic 2, with the associated symplectic form degenerate, with a 1-dimensional radical K on which Q is nonzero. Let $D = \{x \neq 0 : Q(x) = 0\}$. Then $\operatorname{Stab}_\circ(D) = V - (K - (0))$ *is not a subgroup.*

Lemma 24.6

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, G a 0-definable group in $\mathcal{M}^{\rm eq}$. Let D, D', D'' be complete types *over* acl $(\emptyset)$ *of rank* r *contained in* G. *If* $a \in \mathrm{Stab}_\circ(D, D')$ *and* $b \in \mathrm{Stab}_\circ(D', D'')$ *are independent, then* $ab \in \mathrm{Stab}_\circ(D, D'')$.

Proof:

$\mathrm{rk}\,(Da) \cap D' = r = \mathrm{rk}\,(D''b^{-1} \cap D')$ so by the Corollary to type amalgamation we have also $\mathrm{rk}\,(Da \cap D' \cap D''b^{-1}) = r$ and after multiplication on the right by b we have $\mathrm{rk}\,(Dab \cap D'') = r$. ∎

Lemma 24.7

Let $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$*, and* D *a complete type over* acl $(\emptyset)$. *Then* $\mathrm{Stab}\,(D) = \mathrm{Stab}_\circ(D)\mathrm{Stab}_\circ(D)$ *and* $\mathrm{rk}\,(\mathrm{Stab}\,(D) - \mathrm{Stab}_\circ(D)) < \mathrm{rk}\,(\mathrm{Stab}_\circ(D))$;

Proof:

Lemmas 1 and 6. ∎

Lemma 24.8

Let $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property,* G *a 0-definable group in* $\mathcal{M}^{\mathrm{eq}}$*, and* D *a complete type over* acl $(\emptyset)$ *with* $\mathrm{rk}\, D = \mathrm{rk}\, G$. *Then* $[G : \mathrm{Stab}\,(D)] < \infty$.

Proof:

It suffices to show that $\mathrm{rk}\,\mathrm{Stab}_\circ(D) = \mathrm{rk}\, G$. Let a, b be independent elements of D of rank $r = \mathrm{rk}\, G$ and $c = a^{-1}b$. Then $\mathrm{rk}\,(b, c) = 2r$ so $\mathrm{rk}\,(b/c) = r$, and $b \in D \cap Dc$. Thus $c \in \mathrm{Stab}_\circ D$. As c has rank r, we are done. ∎

25. Modular groups

Definition 25.1

Two subgroups H_1, H_2 *of a group* G *are* commensurable *if their intersection has finite index in each. This is an equivalence relation. When* G *has finite rank this is equivalent to:* $\mathrm{rk}\,(H_1) = \mathrm{rk}\,(H_2) = \mathrm{rk}\,(H_1 \cap H_2)$.

Lemma 25.1

Let $\mathcal{M}$ *be* $\aleph_0$*-categorical of finite rank with the type amalgamation property, and modular. Let* G *be a definable group in* $\mathcal{M}$*, and* H_d *a subgroup defined uniformly from the parameter* d *for* d *varying over a definable set* D. *Let* $E(d, d')$ *hold if and only if* H_d *and* $H_{d'}$ *are commensurable. Then* *the relation E has finitely many equivalence classes.*

Proof:

Choose $d \in D$ of maximal rank, $a \in G$ of maximal rank over d, and b in $H_a b$ of maximal rank over a, b. Let $B = \operatorname{acl}(b) \cap \operatorname{acl}(d, a)$. Let d', a' be conjugate to d, a over b and independent from d, a over b. Then b, d, a, and d', a' are independent over B by modularity and the choice of d', a'. Thus $\operatorname{rk}(b/aa'dd') = \operatorname{rk}(b/B) = \operatorname{rk}(b/ad)$ and $\operatorname{rk}(H_d a \cap H_{d'} a') = \operatorname{rk}(H_d a)$. Therefore $\operatorname{rk}(H_d \cap H_{d'}) = \operatorname{rk}(H_d)$, in other words $E(d, d')$ holds. Thus $d/E \in B$.

Furthermore as $H_d \cap H)d'a'a^{-1}$ is nonempty, $a'a^{-1}$ lies in $H_d H_{d'} = X_a X_{a'}(H_d \cap H_{d'})$ for sets $X_d, X_{d'}$ of coset representatives of the intersection in H_d, $H_{d'}$ respectively. Thus $\operatorname{rk}(a'/a, d, d') \leq \operatorname{rk} H_d$ and hence $\operatorname{rk}(a/B) \leq \operatorname{rk} H_d$. Now we compute $\operatorname{rk}(d/E)$:

$$\begin{aligned} \operatorname{rk}(d,a,b) &= \operatorname{rk}(d) + \operatorname{rk}(a) + \operatorname{rk}(b/a,d) = \operatorname{rk}(a) + \operatorname{rk} G + \operatorname{rk} H_d \\ &= \operatorname{rk}(b) + \operatorname{rk}(a/b) + \operatorname{rk}(d/a,b) \leq \operatorname{rk} G + \operatorname{rk} H_d + \operatorname{rk}(d/(d/E)) \end{aligned}$$

showing $\operatorname{rk}(d/(d/E)) = \operatorname{rk}(d)$ and $\operatorname{rk}((d/E)) = 0$, $d/E \in \operatorname{acl}(\emptyset)$. ∎

Proposition 25.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let G be a 0-definable group in $\mathcal{M}$, and H a definable subgroup. Then H is commensurable with a group defined over $\operatorname{acl}(\emptyset)$.

Proof:

Let $H = H_d$ have defining parameter $d \in D$, with D a complete type over $\operatorname{acl}(\emptyset)$. Let $E(d, d')$ be the equivalence relation: H_d, $H_{d'}$ are commensurable. As this has finitely many classes and D realizes a unique type over $\operatorname{acl}(\emptyset)$, all groups H_d $(d \in D)$ are commensurable.

Define $B = \{g \in G : \text{ For some } d \in D \text{ independent from } g,\ g \in H_d\}$. By the corollary to Proposition 17.1:

$$\text{For } b_1,\ b_2 \text{ in } B \text{ independent, } b_1 b_2^{-1} \in B$$

Thus by Lemma 24.1, $H = \langle B \rangle$ is a definable subgroup of G with $\operatorname{rk}(H - B) < \operatorname{rk} H$. Let $h \in H$ be an element of maximal rank. Then $h \in B$. Take $d \in D$ independent from h with $h \in H_d$. Then $\operatorname{rk}(h) \leq \operatorname{rk} H_d$ and thus $\operatorname{rk} H \leq \operatorname{rk} H_d$. On the other hand any element of H_d independent from d is in B, so $\operatorname{rk}(H \cap H_d) \geq \operatorname{rk} H_d$. This shows that H and H_d are commensurable. ∎

Proposition 25.2

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation *property, and modular. Let G be a 0-definable group in $\mathcal{M}$. Then G has a finite normal subgroup N such that G/N contains an abelian subgroup of finite index.*

Proof:

Let $Z^* = \{g \in G : [G : C(g)] < \infty\}$. We work mainly in $G^2 = G \times G$. For $a \in G$ let H_a be the subgroup $\{(x, x^a) : x \in G\}$ of G^2. Define $E(a, a')$ by: H_a and $H_{a'}$ are commensurable. This is an equivalence relation with finitely many classes. Notice that $E(a, a')$ holds if and only if $Z^*a = Z^*a'$: $E(a, a')$ holds if and only if on a subgroup G_1 of G of finite index we have $x^a = x^{a'}$, that is: $G_1 \leq C(a'a^{-1})$, $a'a^{-1} \in Z^*$.

Thus we have proved that Z^* is of finite index in G and we may replace G by Z^*. Then any element of G has finitely many conjugates and thus for $x, y \in G$ $[x, y]$ is algebraic over x and over y. In particular for $x, y \in G$ independent, the commutator $[x, y]$ is algebraic over $\emptyset$. On the other hand ever commutator $[x, y]$ can be written as $[x, y']$ with y' independent from x, since $C(x)$ has finite index in G. Thus $N = G'$ is finite, and G/N is abelian. ∎

Frank Wagner points out that this result is contained in a purely algebraic theorem of Bergman and Lenstra ("*Subgroups close to normal subgroups*", J. Alg **127** (1989), 80-97.

This result tends to reduce the study of definable groups to the abelian case. The next result incorporates information coming from the rank inequality, Proposition 21.2.

Lemma 25.2

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let A be a 0-definable abelian group in $\mathcal{M}$, and $D \subseteq A$ the locus of a complete type over $\operatorname{acl}(\emptyset)$, S the stabilizer of D in A. Then:

1. $\operatorname{rk} S = \operatorname{rk} D$;
2. *D is contained in a single coset of S;*
3. *If D' is the locus of another complete type over $\operatorname{acl}(\emptyset)$ of the same rank, and if $\operatorname{Stab}_\circ(D, D')$ is nonempty, then $\operatorname{Stab}_\circ(D, D')$ agrees with a coset of S up to sets of smaller rank, and $\operatorname{Stab}(D') = S$.*
4. *If $a, b \in S$ are independent with the same type over $\operatorname{acl}(\emptyset)$, then $a - b \in \operatorname{Stab}_\circ(D)$.*

The previous lemma will in particular deliver enough rank 1 subgroups to prove:

Lemma 25.3

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation *property, and modular. Let A be a definable group in $\mathcal{M}^{\rm eq}$ of rank n. Then there is a sequence of subgroups $(0) = A_\circ \lhd A_1 \lhd \ldots \lhd A_n = A$ with* $\mathrm{rk}\,(A_i/A_{i-1})$=*1.*

We record a useful technical result which comes in to the theory of reducts.

Lemma 25.5

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let A_1, A_2 be 0-definable abelian groups in $\mathcal{M}^{\rm eq}$. Suppose that any $\mathrm{acl}\,(\emptyset)$-definable subgroup of $A_1 \times A_2$ is 0-definable, and that $\mathrm{acl}\,(\emptyset) \cap A_1 = (0)$. Let C be a finite set with $\mathrm{acl}\,(C \cap A_1) \subseteq C$, and let $a_2 \in A_2$ have maximal rank over C. Then

1. $\mathrm{acl}\,(a_2, C) \cap A_1 \subseteq \mathrm{dcl}\,(a_2, \mathrm{acl}\,(C))$*;*
2. *If no proper definable subgroup of A_2 of finite index is definable over $\mathrm{acl}\,(\emptyset)$, then $\mathrm{acl}\,(a_2, C) \cap A_1 = \mathrm{dcl}\,(a_2) \cap A_1 + C \cap A_1$.*

Proposition 25.3

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank with the type amalgamation property, and modular. Let A be a 0-definable rank 1 abelian group in $\mathcal{M}^{\rm eq}$. Assume that $\mathrm{acl}\,(\emptyset) \cap A = (0)$ and that A has no proper $\mathrm{acl}\,(\emptyset)$-definable subgroup of finite index. Then there is a finite field F such that A has a definable vector space structure over F for which linear dependence coincides with algebraic closure.

Proof:

Let F be the ring of $\mathrm{acl}\,(\emptyset)$-definable endomorphisms of A. Our assumptions on A imply that F is a division ring and by $\aleph_0$-categoricity of $\mathcal{M}$, F is finite; thus it is a finite field. Taking A as a vector space over F, one shows by induction on n that any n algebraically dependent elements $a_1, \ldots, a_n$ of A will be linearly dependent. ∎

This provides the basis for the following, whose proof we omit.

Lemma 25.6

Let $\mathcal{M}$ be Lie coordinatized, and A an abelian group interpreted in $\mathcal{M}$. Suppose that A has no nontrivial $\mathrm{acl}\,(\emptyset)$-definable proper subgroup, and that $\mathrm{acl}\,(\emptyset) = \mathrm{dcl}\,(\emptyset)$. Then A is part of a basic linear geometry in $\mathcal{M}$.

26. Duality

Definition 26.1

If $\mathcal{M}$ is a structure, A a group of prime exponent p interpreted in $\mathcal{M}$, then A^* denotes the group of $\mathcal{M}^{\text{eq}}$-definable homomorphisms from A to a cyclic group of order p (equivalently the set of definable F-linear maps from A to the field F of order p).

Note that the elements of A^* are almost determined by their kernels, which are definable subgroups of A. However we do not necessarily have $A^* \subseteq A^{\text{eq}}$ since for example A may be one side of a polar geometry.

Proposition 26.1

Let $\mathcal{M}$ be a Lie coordinatized structure, A a 0-definable group in $\mathcal{M}^{\text{eq}}$ of prime exponent p. Then A^* and the evaluation map $A \times A^* \to F$ are 0-definable in A^{eq}. If A has no nontrivial proper 0-definable subgroups then either $A^* = (0)$ or the pairing $A \times A^* \to F$ is a perfect pairing.

Proof:

A^* is a piecewise definable group. Arrange the sorts of $\mathcal{M}^{\text{eq}}$ in some order and let D_n be the definable subset of A^* consisting of elements which lie in the first n sorts.

Our first claim is that $\text{rk}\, A^*$ is finite, bounded by $\text{rk}\, A$. Fix a definable subset D of A^*, and suppose $\text{rk}\, D > \text{rk}\, A$. We apply Proposition 18.1 concerning the sizes of envelopes. Accordingly the number of elements of D is a polynomial of degree $2\text{rk}\, D$ in the variables used there, and similarly for A. Taking envelopes of large and constant dimension, we deduce that $D \cap E$ eventually is larger than $A \cap E$, while (again for large enough envelopes) $D \cap E \subseteq (A \cap E)^*$; this is a contradiction.

We apply Lemma 24.5 and deduce that for any n the subgroup A_n^* generated by D_n is definable. Let A_n be the annihilator in A of A_n^*. The decreasing chain A_n of 0-definable groups must stabilize with $K_n = K$ constant from some point on. We may factor out K and suppose $K = (0)$ (note in passing that the last part of the Proposition will be covered by the argument from this point on).

After these preliminaries we see that $A \times A_n^* \to F$ is a perfect pairing for all large n. Therefore with n, n' fixed, looking at the same situation in large finite envelopes, we find $A_n^* \cap E = A_{n'}^*$ in such envelopes. Thus A_n^* is independent of n for n large, and $A_n^* = A^*$. ∎

Lemma 26.2

Let $\mathcal{M}$ be a Lie coordinatized structure, A a 0-definable vector space in

$\mathcal{M}^{\mathrm{eq}}$ relative to a finite field K. A^* the definable $Zz/n\mathbb{Z}$-dual of A, and Tr *the trace from K to the prime field. Then A^* can also be given a K-space structure, and there is then a definable K-bilinear map $\mu : A \times A^* \to K$ so that* $\mathrm{Tr}\,\mu(a, f) = f(a)$ *for $(a, f) \in A \times A^*$. This pairing makes A^* the full definable K-linear dual of A.*

Proof:

Let A' be the space of all definable K-linear maps of A to K. Let $\mathrm{Tr} : A' \to A^*$ be defined by $\mathrm{Tr}\,(f)(a) = \mathrm{Tr}\,(f(a))$. If $\mathrm{Tr}\,(f) = 0$ then for $a \in A$ and $\alpha \in K$ we have $\mathrm{Tr}\,(\alpha f(a)) = \mathrm{Tr}\,(f)(\alpha a) = 0$, and thus $f(a) = 0$ by the nondegeneracy of the bilinear form $\mathrm{Tr}\,(xy)$. Thus Tr embeds A' into A^*. Conversely, if $g \in A^*$ then for $a \in A$ the linear map $g_a : K \to F$ defined by $g_a(\alpha) = g(\alpha a)$ must have the form $\mathrm{Tr}\,(\gamma_a \alpha) = g(\alpha a)$ for a unique $\gamma_a \in K$. Letting $f(a) = \gamma_a$ we get $\mathrm{Tr}\,(f) = g$, and f is K-linear since $f(\alpha\beta a) = \mathrm{Tr}\,(\beta\gamma_a\alpha)$. Thus Tr identifies the K-linear dual with the F-linear dual. Let μ be the transport to A^* of the natural pairing on $A \times A'$.
∎

Definition 26.2

Let $\mathcal{M}$ be a structure of finite rank, A a group interpretable in $\mathcal{M}$.

1. *Let S, T be definable sets. We write $S \subseteq^* T$ if $\mathrm{rk}\,(S - T) < \mathrm{rk}\,S$. For corresponding definable formulas σ, τ we use the notation $\sigma \Longrightarrow^* \tau$.*
2. *If B is a subgroup of A^*, and $a \in A$, then $\mathrm{gtp}(a/B)$ denotes the atomic type of a over B in the language containing only the bilinear map $A \times A^* \to \mathbb{Z}/n\mathbb{Z}$, with n the exact exponent of A.*
3. *The group A is* settled *if for every algebraically closed parameter set C and $a \in A$ of maximal rank over C, we have* $\mathrm{tp}(a) \cup \mathrm{gtp}(a/A^* \cap C) \Longrightarrow^* \mathrm{tp}(a/C)$.
4. *The group A is* 2-ary *if for any algebraically closed parameter set C and any set $\mathbf{b} = b_1, \ldots, b_n$ in A of elements which are independent over C of maximal rank, we have* $\cup_i\ \mathrm{tp}(b_i/C) \cup \cup_{ij}\ \mathrm{tp}(b_i b_j/\ \mathrm{acl}\,\emptyset) \Longrightarrow^*\ \mathrm{tp}(\mathbf{b}/C)$.

Our primary objective in the long run is to show that every group becomes both settled and 2-ary after introducing finitely many constants. The linear part of a quadratic geometry is an example of an unsettled group.

27. Rank and measure

We can attempt to derive a measure on subsets of a group A by taking cosets of a subgroup of index n to have measure $1/n$. Thus we may assign to a set S the infimimum of the sums $\sum_i 1/n_i$ corresponding to coverings of S by finitely many such cosets. The objective is to show that the measure zero sets are those of less than full rank.

Lemma 27.1

Let $\mathcal{M}$ be a Lie coordinatizable structure and A an abelian group of exponent p, 0-definably interpretable in $\mathcal{M}$. Let D be a 0-definable subset of A of full rank, and $a_1^*, \ldots, a_n^* \in A^*$ independent generics. Let $\alpha_1, \ldots, \alpha_n$ be elements of the prime field F. Then $\{d \in D : (d, a_i^*) = \alpha_i\}$ has full rank.

The proof is based on the formulas for the sizes of envelopes.

Lemma 27.2

Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted in $\mathcal{M}$, and let $D \subseteq A$ be definable with $\operatorname{rk} D = \operatorname{rk} A$. Then finitely many translates of D cover A. More specifically, if D is c-definable then one may find $\mathbf{b} = b_1, \ldots, b_n$ in A with $A = \cup_i (D + b_i)$ and $\mathbf{b}$ independent from c.

Proof:

We proceed by induction on the maximal length of a chain of $\operatorname{acl}(\emptyset)$-definable subgroups of A.

One checks first that this holds when A is part of a basic linear geometry for $\mathcal{M}$.

Now suppose that A has a nontrivial $\operatorname{acl}(\emptyset)$-definable finite subgroup B. Then $\bar{D} = (D+B)/B$ has full rank in A/B and induction applies to $\bar{D}$, A/B. As B is finite this yields the claim in A.

Assume now that A has no nontrivial $\operatorname{acl}(\emptyset)$-definable finite subgroup, and is not part of a basic linear geometry. There is an $\operatorname{acl}(\emptyset)$-definable subgroup A_1 of A which is part of a stably embedded basic linear geometry of $\mathcal{M}$. Let D be c-definable of full rank in A. Pick $b \in A$ of maximal rank over c such that $b + A_q \cap D$ is infinite. Then $D - b$ meets A_1 in an infinite set and thus there is a finite subset $F \subseteq A_1$ such that $A_1 \subseteq F + D - b$, and we may take the elements of F to be independent from b, c. Let B be the locus of b over $F \cup \{c\}$. Then B has full rank and for $b' \in B$, $A_1 \subseteq F + D - b'$. Now by induction in A/A_1, for some finite set F', $F' + B + A_1$ covers A. We claim that $F + F' + D = A$.

Let $a \in A$. Then for some $b' \in B$, we have $a \in F' + b' + A_1 \subseteq F' + b' + (F + D - b') = F' + F + D$, as claimed. ∎

Lemma 27.3

Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, and suppose A has no proper 0-definable subgroups of finite index. Let $h_i : A \to B_i$ for $i = 1, 2$ be homomorphisms onto finite *groups B_1, B_2 and let $h = (h_1, h_2) : A \to B_1 \times B_2$ be the induced map. If h_1, h_2 are independent then h is surjective.*

Proof:

Let the range of h be $C \leq B_1 \times B_2$ and let $C_1 = C \cap B_1 \times (0)$, $C_2 = C \cap (0) \times B_2$. C can be interpreted as the graph of an isomorphism between B_1/C_1 and B_2/C_2. Let $g_i : A \to B_i/C_i$ be the map induced by h_i. Then $g_i \in \operatorname{acl}(h_i)$ and g_1 and g_2 differ only by an automorphism of the range. Thus $g_i \in \operatorname{acl}(h_1) \cap \operatorname{acl}(h_2) = \operatorname{acl}(\emptyset)$ and thus by assumption $B_1 = C_1$, $B_2 = C_2$, and h is surjective. ∎

Lemma 27.4

Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, let A^0 be the smallest 0-definable subgroup of finite index, and let $D \subseteq A$ be 0-definable with $\operatorname{rk} D = \operatorname{rk} A$. Assume that D lies in a single coset C of A^0 and let $h : A \to B$ be a definable homomorphism into a finite group B. Then for any $b \in h[C]$, D meets $h^{-1}[b]$ in a set of full rank.

Proof:

If h is algebraic over $\operatorname{acl}(\emptyset)$ then h is constant on C and there is nothing to prove. Suppose therefore that $h \notin \operatorname{acl}(\emptyset)$.

Using the previous lemma, the proof of Lemma 27.1 can be repeated (for the case $n = 1$), using independent conjugates of h. ∎

Lemma 27.5

Let $\mathcal{M}$ be Lie coordinatizable, let A be an abelian group interpreted 0-definably in $\mathcal{M}$, and let D be the locus of a complete type over $\operatorname{acl}(\emptyset)$ of maximal rank. Then there are independent $a, a' \in D$ such that $a - a'$ lies in every a-definable subgroup of A of finite index.

Proof:

Take $a \in D$. Let A^a be the smallest a-definable subgroup of A of finite index. We consider the canonical homomorphism $h : A \to A/A^a$. The previous lemma applies and shows that $(A^a + a) \cap D$ has full rank. It suffices to take a' in the intersection of maximal rank. ∎

28. The semi-dual cover

Duality can be used to reduce the treatment of affine covers to the treatment of finite covers. This will be needed for the sharpest result on definability in groups interpreted in Lie coordinatized structures, the Finite Basis Theorem of the next section. The definition is rather technical.

Definition 28.1

Let A_1, A_2 be groups. A bilinear cover *of A_1, A_2 is a surjective map $\pi = (\pi_1, \pi_2) : L \to A_1 \times A_2$ where L is a structure with two partial binary operations $q_1, q_2 : L \times L \to L$, with the following properties:*

(BL1) *q_i is defined on $\cup_{a \in A_i} \pi_i^{-1}[a]$ and gives an abelian group operation on each subset $L[a] = \pi_i^{-1}[a]$.*

(BL2) *For $i, i' = 1, 2$ in either order, $\pi_{i'}$ is a group homomorphism on each group $(L[a]; q_i)$ for $i \in A_i$.*

(BL3) *Given elements $a_{ij} \in A_i$ for $i = 1, 2$, $j = 1, 2$, and elements $c_{ij} \in \pi^{-1}(a_{1i}, a_{2j})$:*

$$q_2(q_1(c_{11}, c_{12}), q_1(c_{21}, c_{22})) = q_1(q_2(c_{11}, c_{21}), q_2(c_{12}, c_{22}))$$

Generally q_1 and q_2 will be given the more suggestive notation $+^1$, $+^2$ or just $+$ if no ambiguity results. The same applies to iterated sums $\sum^1$, $\sum^2$ or $\sum$. We will also write $L(a_1, a_2)$ for $\pi^{-1}[(a_1, a_2)]$.

Lemma 28.1

Let $\pi : L \to A_1 \times A_2$ be a bilinear cover relative to the operations q_1 and q_2. Then:

1. *q_1 and q_2 agree on $L(0, 0)$. Let this group be denoted $(A, +)$.*
2. *If $0_1, 0_2$ are the identity elements of A_1 and A_2 respectively, then there canonical identifications $L(0_1) \simeq A \times A_2$ and $L(0_2) \simeq A_1 \times A$.*
3. *Each set $L(a_1, a_2)$ is naturally an affine space over $L(0, a_2)$ and $L(a_1, 0)$, giving two A-affine structures on $L(a_1, a_2)$ which coincide.*

These are direct verifications from the axioms.

Lemma 28.2

Let L be a bilinear cover of $A_1 \times A_2$. Let $a_i \in A_1$, $a'_j \in A_2$, and let $x_{ij} \in L(a_i, a'_j)$, r_i, s_j integer coefficients. Then $\sum_i^2 r_i \sum_j^1 s_j x_{ij} = \sum_j^1 s_j \sum_i^2 r_i x_{ij}$ and in particular if $r_i = s_j = 1$ then the order of summmation can be reversed.

This is proved by induction, first with positive coefficients and then in general. The base case is $i = j = 2$ which is actually the main axiom.

Lemma 28.3

Let $\mathcal{M}$ be a structure, and

$$0 \to A_1 \to B \to A_2 \to 0$$

be an exact sequence of abelian groups with A_1*,* A_2 *of prime exponent* p*, and assume this sequence is interpreted in* $\mathcal{M}$*. For* $a \in A_2$ *let* B_a *be the preimage in* B *of* a*, a coset of* A_1*, and let* B_a^* *be the set of definable affine homomorphisms from* B_a *to the field* F *of* p *elements. Let* $L = \{(a, f) : a \in A_2, f \in B_{a_2}^*\}$*, take* $\pi_1 : L \to A_2$ *natural and let* $\pi_2 : L \to A_1^*$ *be defined by* $\pi_2 f \in A_1^*$ *the linear map associated to* f*, i.e.* $f(x+y) - f(y)$ *as a function of* x*. Then* L *is a cover of* $A_2 \times A_1^*$ *with respect to the following operations* q_1, q_2*. The operation* q_1 *acts by addition in the second coordinate. The operation* q_2 *also acts by addition but in a somewhat more delicate sense: if* $\pi_2(a, f) = \pi_2(a', f')$ *then* f *and* f' *are affine translates of the same linear map* $f_\circ$*, and we set* $q_2((a, f), (a', f')) = (a + a', f + f')$ *where* $f + f'$ *is the function* g *on* $B_{a+a'}$ *defined by* $g(b+b') = f(b) + f'(b')$ *for* $b \in B_a$*,* $b' \in B_{a'}$*.*

The cover associated to an exact sequence as described above will be called a *semi-dual* cover since it involves two groups, one of which is a dual group. Notice that the "structure group" $L(0, 0)$ for the semi-dual cover associated with such an exact sequence is the set of constant maps from A_1 to F, which we identify with F. If $\mathcal{M}$ is Lie coordinatized then the cover obtained is definable since the dual group is definable.

Now we present a construction in the reverse direction.

Lemma 28.4

Let $\mathcal{M}$ *be a structure,* A_1 *and* A_2 *groups interpreted in* $\mathcal{M}$*, and* L *a bilinear cover of* $A_2 \times A_1$ *interpreted in in* $\mathcal{M}$*. Let*

$$\begin{aligned} B \;=\; & \{(a, f) : a \in A_2,\ f : L(a) \longrightarrow F,\ f \text{ is the identity on} \\ & L(a, 0) \text{ identified with } L(0, 0)\}\,. \end{aligned}$$

Then B *is a group with respect to the operation* $(a, f) + (a', f') = (a + a', f'')$ *with* $f''(q_2(x, x')) = f(x) + f(x')$ *for* $x \in L(a)$*,* $x' \in L(a')$*, and setting* $F = L(0, 0)$*, there is an exact sequence* $0 \to \mathrm{Hom}(A_1, F) \to B \to A_2 \to 0$ *where* Hom *is the group of definable homomorphisms.*

Definition 28.2

A group A *of prime exponent interpreted in a Lie coordinatized structure will be called* reflexive *if the natural map* $A \to A^{**}$ *is an isomorphism.*

Lemma 28.5

Let $\mathcal{M}$ *be a Lie coordinatizable structure,* A *a group interpreted in* $\mathcal{M}$*. Then the following are equivalent.*

(1) A is reflexive.
(2) The natural map $A \to A^{**}$ is injective.
(3) A is definably isomorphic to a dual group B^*.

Lemma 28.6

Let $\mathcal{M}$ be a Lie coordinatized structure, and A_1, A_2 groups interpreted in $\mathcal{M}$ of prime exponent p, with A_1 reflexive. Let F be the field of order p. Then there is a natural correspondence between interpretable exact sequences $0 \to A_1 \to B \to A_2 \to 0$ and definable bilinear covers L of $A_2 \times A_1^*$ with structure group $L(0,0) = F$, up to the natural notions of isomorphism.

Notation

1. *For $D \subseteq A \times B$, $s : A \times B \to C$, and $a \in A$, we write D_a for $\{b \in B : (a,b) \in D\}$ and $s_a : D_a \to C$ for the map induced by s.*
2. *For A an $\aleph_0$-categorical group, c a parameter or finite set of parameters, let A^c be the smallest c-definable subgroup of A of finite index. This will be called the* principal component *of A over c. Notice the law $(A_1 \times A_2)^c = A_1^c \times A_2^c$ and hence $(A^n)^c = (A^c)^n$.*

The utility of these semidual covers lies in the following result, whose proof we omit.

Lemma 28.7

Let $\mathcal{M}$ be Lie coordinatizable, A and B groups and $\pi : L \to A \times B$ a bilinear cover, all 0-definably interpreted in $\mathcal{M}$, with structure group $F = L(0,0)$. Let $f : A' \to A$ be a generically surjective 0-definable map, $D \subseteq A' \times B$ the locus of a complete type over $\operatorname{acl}(\emptyset)$ of maximal rank, and $s : D \to L$ a 0-definable section relative to f, i.e. $s(a',b) \in L(fa',b)$ on D. Assume:

(1) The group B is settled.
(2) A and B have no 0-definable proper subgroups of finite index.
(3) $\operatorname{acl}(a') \cap B^* = \operatorname{dcl}(a') \cap B^*$ for $a' \in A'$.
(4) For $(a',b) \in D$, b lies in $B^{a'}$, the principal component of B over a'.

Then for any $a' \in A'$, the map $s_{a'} : D_{a'} \to L(fa')$ is affine, that is, is induced by an affine map.

The next proposition is the preceding lemma with its fourth hypothesis deleted. It is proved by reduction to the previous case.

Proposition 28.1

Let $\mathcal{M}$ be Lie coordinatizable, A and B groups and $\pi : L \to A \times B$ a bilinear cover, all 0-definably interpreted in $\mathcal{M}$, with structure group $F = L(0,0)$. Let $f : A' \to A$ be a generically surjective 0-definable map, $D \subseteq A' \times B$ the locus of a complete type over $\operatorname{acl}(\emptyset)$ of maximal rank, and $s : D \to L$ a 0-definable section relative to f, i.e. $s(a', b) \in L(fa', b)$ on D. Assume:

(1) The group B is settled.
(2) A and B have no 0-definable proper subgroups of finite index.
(3) $\operatorname{acl}(a') \cap B^* = \operatorname{dcl}(a') \cap B^*$ for $a' \in A'$.

Then for any $a' \in A'$, the map $s_{a'} : D_{a'} \to L(fa')$ is affine, that is, is induced by an affine map.

29. The finite basis property

Our objective in the present section is to pin down definability in groups rather thoroughly, as follows.

Proposition 29.1 - Finite Basis Property

Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. Then there is a finite collection of definable subsets D_i of A such that every definable subset of A is a boolean combination of the sets D_i, cosets of definable subgroups of A of finite index, and sets of rank less than $\operatorname{rk}(A)$.

We record the steps of the argument. Using Lemma 27.4 one may show:

Lemma 29.1

Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. The following are equivalent:

(1) A is settled over $\emptyset$, i.e., we have

$$(*) \qquad \operatorname{tp}(a/\emptyset) \cup \operatorname{gtp}(a/C \cap A^*) \Longrightarrow^* \operatorname{tp}(a/C)$$

for a of maximal rank over the algebraically closed set C.

(2) For every finite set $C_\circ$ there is an algebraically closed set C containing $C_\circ$ such that for $a \in A$ of maximal rank over C the relation $(*)$ holds.

(3) Every definable subset of A is a boolean combination of 0-definable sets, cosets of definable subgroups of finite index, and sets of rank less than $\operatorname{rk} A$.

Thus Proposition 29.1 is equivalent to the statement that every group becomes settled over some finite set.

Lemma 29.2

Let $\mathcal{M}$ be a Lie coordinatizable structure, and let $A_1, \ldots, A_n$ be settled groups 0-definably interpreted in $\mathcal{M}$, with no proper 0-definable subgroups of finite index. Then the product $A = \prod_i A_i$ is settled over $\operatorname{acl}(\emptyset)$.

Definition 29.1

Let A be an abelian group interpreted in a Lie coordinatizable structure $\mathcal{M}$. A definable subset Q of A will be called tame *if every definable subset of Q is the intersection with Q of a boolean combination of cosets of definable subgroups of finite index, and sets of lower rank. This notion is of interest only when $\operatorname{rk} Q = \operatorname{rk} A$.*

Lemma 29.3

Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$.

1. *If A contains a definable tame subset of full rank, then A is settled over some finite set.*
2. *If A contains a settled definable subgroup B of finite index then A is settled over some finite set.*

This depends on Lemma 27.2: A is covered by finitely many translates of Q.

Lemma 29.4

Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$. If A contains a finite subgroup $A_\circ$ for which the quotient $A/A_\circ$ is settled over a finite set, then A is settled over a finite set.

The next step constitutes a significant reduction of the problem.

Lemma 29.5

Let $\mathcal{M}$ be a Lie coordinatizable structure, and let A be an abelian group interpreted in $\mathcal{M}$, A_1 a rank 1 $\operatorname{acl}(\emptyset)$-definable subgroup of A, with $\operatorname{acl}(\emptyset) \cap A^* = (0)$, $\operatorname{acl}(\emptyset) \cap A_1 = (0)$. Suppose a is an element of A of full rank over $\emptyset$, with $a \in \operatorname{acl}(a/A_1, c)$ for some c independent from a/A_1 (an *element of the quotient group). Then there is an* $\operatorname{acl}(\emptyset)$*-definable subgroup* A_2 *with* $A = A_1 \oplus A_2$*.*

Proof:

Let Q be the locus of a over $\operatorname{acl}(c)$. With $n = \operatorname{rk} A$, the hypotheses give $\operatorname{rk}(a/c) = n-1$. Let $S = \operatorname{Stab}(Q)$. Then S is a subgroup of A of rank $n-1$, and Q lies in a single coset of S. We claim that $S \cap A_1$ is finite.

If $S \cap A_1$ is infinite, let $b \in S \cap A_1$ have rank 1. By Lemma 25.2, part (4), we may take $b \in \operatorname{Stab}_\circ Q$. Then there is $a' \in Q$ of rank $n-1$ over b, c such that $a'' = a' - b \in Q$. Thus $\operatorname{tp}(a''/c) = \operatorname{tp}(a/c)$ and $a'' \in \operatorname{acl}(a''/A_1, c)$, that is $a' - b \in \operatorname{acl}(a'/A_1, c)$ and hence $b \in \operatorname{acl}(a', c)$. This contradicts the independence of a', b over c.

Now by Proposition 25.1 there is an $\operatorname{acl}(\emptyset)$-definable subgroup A_2 commensurable with S. It follows easily that $A_1 \oplus A_2$ is a definable subgroup of A of finite index defined over $\operatorname{acl}(\emptyset)$, and thus $A_1 \oplus A_2 = A$.

Lemma 29.6

Let $\mathcal{M}$ *be a Lie coordinatizable structure, and let* $(0) \to A_1 \to B \to A_2 \to (0)$ *be an exact sequence interpreted in* $\mathcal{M}$*, and let* $\pi : L \to A_2 \times A_1^*$ *be the corresponding bilinear cover. Assume* $\operatorname{acl}(\emptyset) \cap A_1 = (0)$ *and* $\operatorname{acl}(\emptyset) \cap A_2^* = (0)$*. Let* C *be algebraically closed, and let* D *be a complete type over* C *in* A_2 *of maximal rank. Let* $a^* \in C \cap A_1^*$*, and suppose* $g : D \to L(a^*)$ *is a* C*-definable section, that is:* $g(a) \in L(a, g_2(a))$ *for some function* g_2*; here we use the standard representation of the bilinear cover* L*, and in particular* $g_2(a)$ *induces* a^* *on* A_1*.*

Then there is a C*-definable homomorphism* j *from* A_2 *to a finite group, so that for any* $b \in B$ *with* $b/A_1 \in D$*, the quantity*

$$[g_2(a/A_1)](a)$$

is determined by $j(a)$*.*

Lemma 29.7

Let $\mathcal{M}$ *be a Lie coordinatizable structure, let* A *be 0-definably interpretable in* $\mathcal{M}$*,* A_1 *a definable subgroup, and suppose that* A_1 *is settled. Suppose there is a 0-definable type of full rank in* A *with locus* Q *such that for any* C *and any* $a \in Q$ *with* a/A_1 *of maximal rank over* C*,*

$$(*) \qquad \operatorname{tp}(a/(a/A_1)) \cup \operatorname{gtp}(a/\operatorname{acl}(C) \cap A^*) \Longrightarrow \operatorname{tp}(a/(a/A_1), C)$$

Then Q *is tame in* A*, and hence* A *is settled over some finite set.*

The following lemma is critical.

Lemma 29.8

Let $\mathcal{M}$ be a Lie coordinatizable structure, let A be 0-definably interpretable in $\mathcal{M}$, with $\operatorname{acl}(\emptyset) \cap A^* = (0)$, and let A_1 be a 0-definable subgroup of $\mathcal{M}$ which is part of a stably embedded linear geometry J in $\mathcal{M}$, not of quadratic type. Assume that A/A_1 is settled and that there is no $\operatorname{acl}(\emptyset)$-definable complement to A_1 in A. Then A is settled over some finite set.

Proof:

We will arrive at the situation of the previous lemma, relative to some finite set of auxiliary parameters $C_\circ$ (so the sets C of the previous lemma should contain $C_\circ$). We work over $\operatorname{acl}(\emptyset)$.

Let $\bar{A} = A/A_1$. Fix an element $a \in A$ of maximal rank, and let $\bar{a} = a/A_1$. Let $S = a + a_1$ viewed as an affine space over A_1. Let $S^{*\circ}$ be the prime field affine dual defined in §4. Call a set C *basal* if C is algebraically closed and independent from a. Then we claim:

$$\text{For } C \text{ basal, } a \text{ is not in } \operatorname{acl}(b, C, J)$$

Otherwise, take $a \in \operatorname{acl}(b, C, d_1, \ldots, d_k$ with $d_i \in J$ and k minimal. Then the sequence $b, C, d_1, \ldots, d_k$ is independent. We apply Lemma 5 noting that $\operatorname{acl}(\emptyset) \cap A_1 = (0)$ by our hypothesis. Then Lemma 5 produces a complement to A_1 in A, a contradiction. Also, by Lemma 25.5 $\operatorname{acl}(b, C) \cap J = \operatorname{dcl}(b, C) \cap J$. Now Lemma 4.10 applies, giving:

$$\operatorname{tp}(a/b, \operatorname{dcl}(b, C) \cap S^{*\circ} \Longrightarrow \operatorname{tp}(a/b, C)$$

Let $T(C)$ be $\operatorname{dcl}(C) \cap S^{*\circ}$. We need to examine $T(C)$ more closely for basal C. For $f \in A_1^*$ let $S^{*\circ}(f)$ be the set of elements of $S^{*\circ}$ lying above f; this is an affine space over the prime field $F_\circ$, of dimension 1. Let $A_1^*(C) = \operatorname{acl}(C) \cap A_1^*$. Let $T_1(C) = \operatorname{dcl}(C, b) \cap \cup\{S^{*\circ}(f) : f \in A_1^*(C)\}$. We claim that for some basal C, for all C' containing C, we have

$$(*) \qquad T(C') = T(C) + T_1(C')$$

and hence $T(C') \subseteq \operatorname{dcl}(b, T(C), T_1(C'))$.

Let $\beta(C) = \{x \in A_1^*(b) : \text{for some } y \in A_1^*(C),\ S^{*\circ}(x+y) \cap T(C) \neq \emptyset\}$. Chose C basal with $\beta(C)$ maximal. Let $C' \supseteq C$ be basal, $t \in T(C')$. Then $t \in S^{*\circ}(x+y)$ for some $x \in A_1^*(b)$, $y \in A_1^*(C')$. So $t \in \beta(C') - \beta(C)$. Thus there is $y' \in A_1^*(C)$ and $t' \in T(C) \cap S^{*\circ}(x+y')$. Then $t - t' \in T(C') \cap S^{*\circ}(y - y') \subseteq T_1(C')$ and as $t = t' + (t - t')$, our claim is proved.

Using quantifier elimination in $(J, S, S^{*\circ})$, the claim gives:

$$\operatorname{tp}(a/b, T(C)) \cup \operatorname{tp}(a/b, T_1(C')) \Longrightarrow \operatorname{tp}(a/b, T(C'))$$

Now in order to show

$$\operatorname{tp}(a/C') \cup \operatorname{gtp}(a/\operatorname{acl}(C') \cap A^*) \Longrightarrow^* \operatorname{tp}(a/C')$$

it will suffice to check:

$$(**) \qquad \operatorname{tp}(a/b) \cup \operatorname{gtp}(a/C' \cap A^*) \Longrightarrow \operatorname{tp}(a/b, T_1(C'))$$

We fix C' and let $\pi : L \to \bar{A} \times A_1^*$ be the semi-dual cover corresponding to $(0) \to A_1 \to A \to B \to (0)$. Let D' be the locus of b over C'. If $t \in T_1(C')$ then $(b, t) \in L$; let $a^* = \pi_2(b, t)$ be the induced element of A^*. Then $a^* \in C' \cap A_1^*$. As $t \in \operatorname{dcl}(b, C')$ we may write $(b, t) = g(b) = (b, g_2(b))$ where $g : D' \to L(a^*)$ is a C'-definable section. By Lemma 6 there is a C'-definable homomorphism j onto a finite group whose values determine $g_2(\bar{u})(u)$ for $u \in A$, $\bar{u} \in D'$. By definition $\operatorname{gtp}(a/C' \cap A^*)$ determines the value of $j(a)$ and hence of $t(a)$. Claim $(**)$ follows. ∎

For the proof of Proposition 29.1, proceeding by induction on the length of a maximal chain of $\operatorname{acl}(\emptyset)$-definable subgroups, taking A_1 to be a 0-definable subgroup of rank 1, by induction A/A_1 is settled over some set C and after taking into account the various special cases dealt with above and in particular assuming that a 0-definable rank 1 subgroup A_1 is not complemented in A, one arrives at a situation where the previous lemma applies; the quadratic case can be avoided by naming a quadratic form, if necessary. ∎

The following is an equivalent version of the finite basis property.

Proposition 29.2

Let $\mathcal{M}$ be Lie coordinatizable and A an abelian group interpreted in $\mathcal{M}$. Then there is a finite collection D_i of definable subsets of A, such that every definable subset of A is a boolean combination of translates of the D_i together with cosets of definable subgroups.

This completes the general theory of definable groups. The more specialized developments that follow are aimed at controlling reducts of Lie coordinatized structures.

We take note of a few further results which may be viewed as belonging to the general theory.

30. Recognizing geometries

Proposition 30.1

Let $\mathcal{M}$ be $\aleph_0$-categorical of finite rank and let A, A^* be rank 1 groups equipped with vector space structures over a finite field F, and a definable

F-bilinear pairing into F, with everything 0-definably interpreted in $\mathcal{M}$. Assume the following properties:

1. *Every $\mathcal{M}$-definable F-linear map $A \to F$ is represented by some element of A^*, and dually.*
2. *Algebraic closure and linear dependence coincide on A and on A^*.*
3. *A and A^* have no nontrivial proper 0-definable subspaces.*
4. *Every definable subset of A or of A^* is a boolean combination of translates of 0-definable subsets and cosets of definable subgroups.*
5. *If D is the locus of a complete type in A over $\mathrm{acl}(\emptyset)$ and $a'_1, \ldots, a'_n$ are F-linearly independent, then there is an element d of D with (d, a'_i) prescribed arbitrarily.*

Then the pair (A, A^*) is a linear Lie geometry, possibly weak, which is stably embedded in $\mathcal{M}$.

33. Reducts with groups

After a fairly lengthy development, which we omit here, the main result is the following, or its corollary.

Proposition 33.3

Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$, with $\mathrm{acl}_{\mathcal{M}}(\emptyset) \cap (\mathcal{M}^-)^{\mathrm{eq}} = \mathrm{dcl}_{\mathcal{M}^-}(\emptyset)$. If A is settled over $\emptyset$ in $\mathcal{M}$ then it is settled over $\emptyset$ in $\mathcal{M}^-$ and thus every definable subset in $\mathcal{M}^-$ is a boolean combination of 0-definable subsets, a finite set, and cosets of definable subgroups.

Corollary

Let $\mathcal{M}^-$ be a reduct of a Lie coordinatizable structure $\mathcal{M}$, A a rank 1 0-definable group in $\mathcal{M}^-$. If A is settled over $\emptyset$ in $\mathcal{M}$ then it is settled over a finite set of algebraic constants.

Proof:

By the preceding result A becomes settled over $\mathrm{acl}(\emptyset)$ and hence over the subsets of A which belong to $\mathrm{acl}(\emptyset)$; there are finitely many such. ∎

34. Reducts

Proposition 28.1

Let $\mathcal{M}$ be a weakly Lie coordinatized structure, $\mathcal{M}^-$ a reduct of $\mathcal{M}$, and D a primitive, rank 1, definable subset of $\mathcal{M}^-$. Then D is a Lie geometry *forming part of a Lie geometry stably embedded in $\mathcal{M}^-$; this geometry may be unoriented, and may be affine.*

35. Effectivity

One shows primarily that the "characteristic sentences" described in connection with the discussion of quasifinite axiomatizability can be effectively recognized – that is, the "bogus" ones can be deleted. This argument has the very curious feature that if one begins with a stable structure the argument will pass through an unstable expansion. From this point of view a polar space is easier to understand than a pure vector space. There is a further discussion of these matters in [8] for which the theory as summarized here provides the background.

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