RICCI CURVATURE IN KÄHLER GEOMETRY

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Abstract. These are the notes for lectures given at the Sanya winter school in complex analysis and geometry in January 2016. In the first lecture we review the meaning of Ricci curvature of Kähler metrics and introduce the problem of finding Kähler-Einstein metrics. In the second lecture we describe the formal picture that leads to the notion of K-stability of Fano manifolds, which is an algebro-geometric criterion for the existence of a Kähler-Einstein metric, by the recent result of Chen-Donaldson-Sun. In the third lecture we discuss algebraic structure on Gromov-Hausdorff limits, which is a key ingredient in the proof of the Kähler-Einstein result. In the fourth lecture we give a brief survey of the more recent work on tangent cones of singular Kähler-Einstein metrics arising from Gromov-Haudorff limits, and the connections with algebraic geometry.

1. Introduction

Let (X, g) be a Riemannian manifold of dimension m. Given a point p in X, we can choose local geodesic normal co-ordinates $\{x_1, \cdots, x_m\}$ centered at p. Then we have a Taylor expansion of the metric tensor

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{i,j,k,l} R_{ikjl}(p)x^k x^l + O(|x|^3),$$

where $Rm(p) := \sum_{i,j,k,l} R_{ikjl}(p)dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ is the *Riemann curvature tensor* at p. Roughly speaking, the formula says that *Riemannian curvature is the second derivative of the Riemannian metric*. We also have

$$\sqrt{det(g_{ij}(x))} = 1 - \frac{1}{6}\sum_{i,j} R_{ij}(p)x^i x^j + O(|x|^3),$$

where $Ric(p) := \sum_{i,j} R_{ij}(p)dx^i \otimes dx^j$ is the *Ricci tensor* at p, and $R_{ij}(p) = \sum_k R_{ikjk}$. Thus *Ricci curvature is the second derivative of the volume form.* It follows immediately that the sign of Ricci curvature is closely related to the infinitesimal growth of volume of geodesic balls. Globally we have the classical Bishop-Gromov volume comparison theorem (see for example [45]), a special case of which is the following

Theorem 1. *Suppose X is complete and $Ric(g) \geq 0$, then for all $r > 0$, we have*

$$Vol(B(p, r)) \leq Vol(B_r),$$

where $B(p, r)$ is the geodesic ball of radius r centered at p, and B_r is a ball of radius r in the Euclidean space $\mathbf{R}^m$. Moreover, if $Vol(B(p, s)) = Vol(B_s)$ for some $s > 0$, then $B(p, s)$ is isometric to B_s.

Analytically Ricci curvature often appears in Bochner-type formulae. Given a one-form α, we have

$$\Delta_H \alpha = \nabla^* \nabla \alpha + Ric.\alpha,$$

where $\Delta_H = dd^* + d^*d$ is the *Hodge Laplacian* operator, and $\nabla^*\nabla$ is the *rough Laplacian* operator. Dually given a vector field V, the following holds

$$\mathcal{S}^*\mathcal{S}(V) = \nabla^*\nabla V - Ric.V - \nabla(Tr(\mathcal{S}(V))),$$

where $\mathcal{S}(V)$ is the symmetrization of $\nabla V \in \Omega^1(TX) \simeq TX \otimes TX$. Note $\mathcal{S}(V) = 0$ if and only if V is a Killing vector field, i.e. $\mathcal{L}_V g = 0$. The space of all Killing vector fields on X is the Lie algebra of the isometry group of (X, g). It follows from these the well-known results

Theorem 2. *Suppose X is compact*

- *If $Ric(g) \geq 0$, then any harmonic one-form α on X (i.e. $\Delta_H \alpha = 0$) is parallel, in particular $b_1(X) \leq m$. If furthermore $Ric(g) > 0$ at one point, then there is no nonzero harmonic one-form on X, and $b_1(X) = 0$ (indeed, if $Ric(g) > 0$ everywhere on X then Myers's theorem implies $\pi_1(X)$ is finite);*
- *If $Ric(g) \leq 0$, then any Killing vector field on X is parallel. If furthermore $Ric(g) < 0$ at one point, then there is no non-trivial Killing vector field on X.*

So roughly speaking, *positive Ricci curvature restricts topology, and negative Ricci curvature restricts symmetry.*

A Riemannian metric g is called *Einstein* if it satisfies the equation

$$Ric(g) = \lambda g \tag{1}$$

for some *Einstein* constant λ. If X is compact, then this is the Euler-Lagrange equation of the *Einstein-Hilbert functional*

$$EH : g \mapsto Vol(g)^{-(2-m)/m} \int_X S(g) dVol_g.,$$

where $S(g)$ is the *scalar curvature* function of g, which at a point p is given by $\sum_i R_{ii}(p)$.

Now we assume (X, g) is *Kähler*. This means that the holonomy group of g is contained in $U(m/2)$ (in particular m is even), or in other words, there is a parallel almost complex structure J, i.e.,

$$\nabla_g J = 0. \tag{2}$$

We will denote by $n = m/2$ the *complex dimension* of X.

Equation (2) implies two facts

- J is integrable. This means locally one can choose holomorphic coordinates $\{z_1, \cdots, z_n\}$ so that X is naturally a *complex manifold.* There are natural ∂ and $\bar{\partial}$ operators on differential forms on X.

- The *Kähler form* $\omega = g(J\cdot,\cdot)$ is closed. Locally in holomorphic co-ordinates we may write

$$\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{3}$$

where $g_{\alpha\bar{\beta}} = g(\partial_{z^\alpha}, \partial_{\bar{z}^\beta})$ in terms of the natural complexification of the Riemannian metric g. On the complex manifold (X, J), ω is a *positive* $(1,1)$ *form*, i.e., $(g_{\alpha\bar{\beta}})$ is a positive definite Hermitian matrix. For simplicity we may also refer to ω as the Kähler metric if the underlying complex structure J is fixed in the context.

From now on we assume X is compact. Then ω defines a de-Rham cohomology class $[\omega] \in H^2(X; \mathbf{R})$. Since the volume form of g is given by

$$\mathrm{dVol}_g = \omega^n/n!,$$

the volume of (X, ω) depends only on $[\omega]$.

Now we shift point of view, and fix the underlying complex structure J of X. Let $\mathcal{K}_\omega$ be the space of all Kähler forms on (X, J) that is cohomologous to ω in $H^2(X; \mathbf{R})$. By the $\partial\bar{\partial}$-lemma, we have

$$\mathcal{K}_\omega = \{\omega + i\partial\bar{\partial}\phi | \phi \in C^\infty(X; \mathbf{R}), \omega + i\partial\bar{\partial}\phi > 0\}.$$

This is called the *Kähler class* of ω.

Fix a point $p \in X$, using the fact that ω is closed, by a local transformation of holomorphic co-ordinates we may assume

$$g_{\alpha\bar{\beta}}(z) = \delta_{\alpha\beta} - \frac{1}{2} \sum_{\gamma,\bar{\delta}} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p) z^a \bar{z}^b + O(|z|^3),$$

where $R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p) := Rm(p)(\partial_{z^\alpha}, \partial_{\bar{z}^\beta}, \partial_{z^\gamma}, \partial_{\bar{z}^\delta})$ is the complexified Riemannian sectional curvature. Then we have

$$\det(g_{\alpha\bar{\beta}}) = 1 - \frac{1}{2} R_{\alpha\bar{\beta}}(p) z^\alpha \bar{z}^\beta + O(|z|^3),$$

where $R_{\alpha\beta}(p) := Ric(p)(\partial_{z^\alpha}, \partial_{\bar{z}^\beta})$. We define the *Ricci form* as a real-valued $(1,1)$ form given by

$$Ric(\omega) := Ric(g)(J\cdot,\cdot) = \frac{\sqrt{-1}}{2} R_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Then we have

$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}). \tag{4}$$

It follows immediately that $Ric(\omega)$ is closed, so it defines a cohomology class in $H^2(X; \mathbf{R}) \cap H^{1,1}(X; \mathbf{C})$.

From another perspective, the volume form $\omega^n/n!$ can be viewed as a hermitian metric h on the anti-canonical line bundle K_X^{-1}, using the fact that a $2n$ form on X is equivalently a section of $K_X \otimes \overline{K_X}$. Then (4) simply means that the Ricci form is the curvature form of h. It also follows that $Ric(\omega) \in 2\pi c_1(X) \in H^2(X; \mathbf{R}) \cap H^{1,1}(X; \mathbf{C})$, where $c_1(X) := c_1(K_X^{-1}) \in H^2(X; \mathbf{Z})$ is the *first Chern class* of the complex manifold X.

We say a cohomology class $\gamma \in H^2(X;\mathbf{R}) \cap H^{1,1}(X;\mathbf{C})$ is positive (negative, zero, correspondingly) if there is a representative real-valued $(1,1)$ form $\eta \in [\gamma]$ that is positive (negative, zero, respectively) everywhere, i.e., locally the corresponding matrix of functions $(\eta_{\alpha\bar{\beta}})$ is positive definite. Using the $\partial\bar{\partial}$-Lemma and maximum principle it is easy to see that these three notions are mutually exclusive. When $\gamma = 2\pi c_1(L)$ for some holomorphic line bundle L, the positivity of γ is equivalent to algebro-geometrically that L being *ample*; namely, for sufficiently large integer m, holomorphic sections of L^m define an embedding of X into a complex projective space.

In particular, it makes sense to talk about the "sign" of $c_1(X)$ if it has one. When $\dim_{\mathbf{C}} X = 1$, the sign of $c_1(X)$ coincides with the sign of the Euler characteristic of X.

When X is projective, i.e., when X can be embedded as a complex submanifold of $\mathbf{P}^N$, the positivity of $c_1(X)$ can be numerically checked via the *Nakai-Moishezon criterion.* This says that, for example, $c_1(X)$ is positive if and only if $\int_Y c_1(X)^{\dim_{\mathbf{C}} Y} > 0$ for all non-trivial complex subvarieties Y.

In a sense, the sign of $c_1(X)$, as a purely complex geometric invariant, is a *numerical analogue* of the sign of Ricci curvature. A much deeper relationship between the two is given by Yau's resolution of the Calabi conjecture.

Theorem 3 ([67])**.** *Given any compact Kähler manifold X, the natural map $Ric : \omega \mapsto Ric(\omega)$ from $\mathcal{K}_\omega$ to the space of all closed real-valued $(1,1)$ forms in the cohomology class of $2\pi c_1(X)$ is bijective.*

This implies that if $c_1(X)$ has a sign, say $c_1(X) = \lambda[\omega]$ for some $\lambda \in \mathbf{R}$ and Kähler form ω, then we can find a Kähler form $\omega' \in \mathcal{K}_\omega$ such that $Ric(\omega')$ has the same sign as λ.

We have similar results to Theorem 2, with somewhat stronger conclusions

Theorem 4. *Suppose X is compact*

- *If $c_1(X) > 0$ (X is called* Fano *in this case), then X is simply-connected.*
- *If $c_1(X) < 0$ (X is called a* smooth canonical model *in this case), then X does not admit any non-trivial holomorphic vector field.*

Proof. (1) If $c_1(X) > 0$ then by Theorem 3 we know X admits a Kähler metric ω with positive Ricci curvature, so by Myers's theorem $\pi_1(X)$ is finite. Denote by $\chi(X,\mathbf{C}) = \sum_{q\geq 0}(-1)^q h^{0,q}(X)$ the holomorphic Euler characteristic of X, then by Hirzerbruch-Riemann-Roch theorem we have

$$\chi(X,\mathbf{C}) = \int_X Td(\omega),$$

where $Td(\omega)$ is the *Todd form* of ω. On the other hand, since $c_1(X) > 0$, it follows easily from Kodaira vanishing theorem (whose proof involves a generalized Bochner formula) and Serre duality that $h^{0,q}(X) = 0$ for all $q > 0$. Hence $\chi(X,\mathbf{C}) = 1$.

Now let $\pi : \hat{X} \to X$ be a finite cover of degree d, then $\hat{X}$ admits a natural complex structure so that π is holomorphic, and $\pi^*\omega$ is a Kähler metric on $\hat{X}$ with positive Ricci curvature. So $c_1(\hat{X})$ is also positive, hence as above we know $\chi(\hat{X}, \mathbf{C}) = 1$. Since $Td(\omega)$ is determined by the curvature form of ω, we have $Td(\pi^*\omega) = \pi^* Td(\omega)$, so by Hirzebruch-Riemann-Roch again it follows that $\chi(\hat{X}, \mathbf{C}) = \chi(X, \mathbf{C})d$. Therefore $d = 1$.

(2) If $c_1(X) < 0$, then it is a direct consequence of Kodaira vanishing theorem that $H^0(X, TX) \simeq H^{1,n}(X, K_X) = 0$.

□

When $c_1(X)$ is positive, there are indeed many more constraints, for example, there are only finite many deformation families in each dimension [40]. The philosophy here is that *positive first Chern class restricts the complex structure moduli, and negative first Chern class restricts the holomorphic symmetry.*

For testing examples, let X be a smooth hypersurface of degree d in $\mathbf{CP}^{n+1}$ ($n \geq 2$), then $c_1(X) > 0/ = 0/ < 0$ if and only if $d - (n+1) < 0/ = 0/ > 0$. There are only finitely many d such that $d < n+1$, and for $d \geq 3$ there is no non-zero holomorphic vector field on X [41].

A Kähler metric g is called *Kähler-Einstein* if g is also Einstein, i.e., $Ric(g) = \lambda g$ for some $\lambda \in \mathbf{R}$; or equivalently,

$$Ric(\omega) = \lambda\omega \tag{5}$$

Clearly if such a metric exists on X then $c_1(X)$ must have a sign. The converse is the famous

Conjecture 5 (Calabi 1954 [10])**.** *Let X be a compact Kähler manifold. If $c_1(X)$ has a sign then X admits a unique Kähler-Einstein metric.*

The problem can be formulated in terms of solving a complex Monge-Ampère equation. Suppose $c_1(X)$ has a sign, by scaling the Kähler class we may assume $2\pi c_1(X) = \lambda[\omega]$ where $\lambda \in \{+1, 0, -1\}$. So ω determines a hermitian metric on K_X^{-1}, and thus a volume form Ω. If $\omega' = \omega + i\partial\bar{\partial}\phi$, then $\Omega' = e^{-\lambda\phi}\Omega$, the Kähler-Einstein equation is equivalent to the volume form equation

$$(\omega + i\partial\bar{\partial}\phi)^n = Ce^{-\lambda\phi}\Omega, \tag{6}$$

where C is a constant. Then we have the following fundamental existence results

- Aubin [1], Yau [67] 1976: If $c_1(X) < 0$, then there is a unique Kähler metric ω on X with $Ric(\omega) = -\omega$.
- Yau 1976 [67]: If $c_1(X) = 0$, then there is a unique Kähler metric ω with $Ric(\omega) = 0$ in each Kähler class of X. Such a metric is called a *Calabi-Yau metric.*

In the Fano case, the problem is much more difficult. From the PDE point of view the sign of λ is crucial in obtaining *a priori estimates* via maximum principle argument. For example, at the maximum of ϕ we get from the equation that $Ce^{-\lambda\phi}\Omega \leq \omega^n$. If $\lambda < 0$ then we get from this an upper bound on ϕ; but if $\lambda > 0$ then we get the *wrong* sign in terms of obtaining a useful bound. On the other hand, the strict uniqueness fails when $c_1(X) > 0$, since in this case there could be nontrivial holomorphic vector fields on X with zeroes, and pulling back by the holomorphic automorphisms generated by these will yield genuinely different Kähler forms ω satisfying the same equation. Notice by Theorem 4 there are no non-zero holomorphic vector fields when $c_1(X) < 0$, so the group $\mathrm{Aut}(X)$ of holomorphic transformations of X is discrete and by the uniqueness statement in the Aubin-Yau theorem it preserves the Kähler-Einstein metric, so must indeed be finite. Similarly when $c_1(X) = 0$ it can be shown that $\mathrm{Aut}(X)$ is compact and acts by isometries with respect to the Calabi-Yau metric; it can be non-discrete though (for example, when X is a complex torus). But when $c_1(X) > 0$, the group $\mathrm{Aut}(X)$ can be non-compact (for example, when $X = \mathbf{CP}^1$), so can not preserve a Kähler-Einstein metric (assuming there is one).

Bando-Mabuchi [2] proved the above is the only way to cause non-uniqueness. Let $\mathrm{Aut}_0(X)$ be the identity component of $\mathrm{Aut}(X)$. For a Kähler metric ω we denote by $\mathrm{Iso}(X, \omega)$ the group of holomorphic transformations that preserve ω. It is a compact subgroup of $\mathrm{Aut}(X)$. Taking the complexified Lie algebra of the identity component $\mathrm{Iso}_0(X, \omega)$ inside the Lie algebra of $\mathrm{Aut}(X)$, we obtain a *complex* Lie group $\mathrm{Iso}_0(X, \omega)^{\mathbf{C}} \subset \mathrm{Aut}_0(X)$. If X is Fano, then $\mathrm{Aut}(X)$ acts naturally on K_X^{-1} so on $H^0(X, K_X^{-k})$ for all k. Hence for k large we can view $\mathrm{Aut}(X)$ as a subgroup of the group of projective linear transformations of $\mathbf{P}(H^0(X, K_X^{-k}))$, and $\mathrm{Iso}(X, \omega)$ as a subgroup of the corresponding projective unitary group (with respect to the natural L^2 hermitian inner product on $H^0(X, K_X^{-k})$ defined by ω). In this case it follows that $\mathrm{Iso}_0(X, \omega)^{\mathbf{C}}$ is indeed the *complexification* of $\mathrm{Iso}_0(X, \omega)$, in particular, the complex dimension of the former equals the real dimension of the latter.

Theorem 6 (Bando-Mabuchi)**.** *Suppose X is Fano, and ω_1, ω_2 are Kähler-Einstein metrics in $2\pi c_1(X)$, then there is an element $f \in Iso_0(X, \omega_1)^{\mathbf{C}} \subset Aut_0(X)$ such that $\omega_2 = f^*\omega_1$.*

So in any case if a Kähler-Einstein metric exists, then it is *canonical* in the sense the geometry is uniquely determined.

It is known that not every Fano manifold admits a Kähler-Einstein metric. The following is proved by Matsushima in 1957 [53].

Theorem 7 (Matsushima)**.** *Suppose X is Fano, ω is a Kähler metric with $Ric(\omega) = \omega$, then the group $Aut(X)$ is reductive.*

More precisely speaking, the Lie algebra of $\mathrm{Aut}(X)$ is naturally the complexification of the Lie algebra of the compact subgroup of holomorphic isometries of (X, ω). The original proof uses Bochner formula. We now explain briefly that this also follows from the uniqueness Theorem 6. Note indeed the original proof of Theorem 6 actually uses Theorem 7. However we will explain later that there is a more geometric proof of Theorem 6 and more importantly, its extension to singular varieties, without using Theorem 7.

Let $F \in \mathrm{Aut}(X)$, then $F^*\omega$ is also Kähler-Einstein, so by Theorem 6, we can find $G \in \mathrm{Iso}_0(X, \omega)^{\mathbf{C}}$, such that $F^*\omega = G^*\omega$, so $F \circ G^{-1} \in \mathrm{Iso}(X, \omega)$, and $F \in \mathrm{Iso}(X, \omega)\mathrm{Iso}_0^{\mathbf{C}}(X, \omega)$. If $F \in \mathrm{Aut}_0(X)$, then it follows that $F \in \mathrm{Iso}_0(X, \omega)^{\mathbf{C}}$. Notice we actually proved that $\mathrm{Aut}(X) = \mathrm{Iso}(X, \omega)\mathrm{Iso}_0^{\mathbf{C}}(X, \omega)$.

Theorem 7 provides an algebro-geometric obstruction to the existence of Kähler-Einstein metric on a Fano manifold X. One can use this to obtain examples of Fano manifolds not admitting any Kähler-Einstein metric. For example, let X be the blown up of $\mathbf{CP}^2$ at one point. Then

$$\mathrm{Aut}(X) = \{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \}/\mathbf{C}^*$$

Its maximal compact subgroup is

$$K = \{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \}/S^1$$

It is easy to see this is not reductive, since K has real dimension 4, while $\mathrm{Aut}(X)$ has complex dimension 6. So by Theorem 7 we know X does not not admit any Kähler-Einstein metric. Similar arguments also apply to the blown up of $\mathbf{CP}^2$ at two points.

We end this introduction by giving a heuristic reason why not every Fano manifold can admit a Kähler-Einstein metric. This has to do with the possibility of *jumping of complex structures*. Namely, there could exist a family of Fano manifolds $\pi : \mathcal{X} \to \Delta$ over the unit disc in $\mathbf{C}$, such that $X_t := \pi^{-1}(t)$ are all isomorphic for $t \neq 0$, but X_0 is genuinely different. If this happens then X_0 necessarily admits non-trivial holomorphic vector fields, which is possible when $c_1(X) > 0$ but not possible when $c_1(X) < 0$ (by Theorem 4).

Fix the underlying smooth manifold X, let $\mathcal{J}$ be the space of all integrable complex structures on X with positive first Chern class. There is a natural action of the group $\mathrm{Diff}(X)$ on $\mathcal{J}$. Then the existence of the above family implies the quotient space $\mathcal{J}/\mathrm{Diff}(X)$ is not Haudorff.

Let $\mathcal{K}$ be the space of all Kähler structures (J, ω) on X such that $Ric(\omega) = \omega$. Then there is a natural continuous map

$$\mathcal{K}/\mathrm{Diff}(X) \to \mathcal{J}/\mathrm{Diff}(X).$$

Suppose every Fano manifold admits a Kähler-Einstein metric, then this map is surjective. By Theorem 6, the map is injective as well, and the fact that the Kähler-Einstein metric is canonical would suggest that the inverse map should be continuous (this is why the argument is only “heuristic”). Now the contradiction follows since the space $\mathcal{K}/\mathrm{Diff}(X)$ is always Hausdorff. This is easy to see, using the fact that the space of isometries between two compact Riemannian manifolds is always compact.

As we will see in the next section, the above jumping phenomenon is indeed related to the obstructions to the existence of Kähler-Einstein metrics on Fano manifolds.

2. Formal picture and K-stability

We first give two variational interpretations of Kähler-Einstein metrics. These lead to a formal picture concerning the existence problem, which motivates the definition of K-stability. Even though this picture also holds in the more general setting of constant scalar curvature Kähler metrics, we will here restrict ourselves to the case that X is a Fano manifold and $\omega \in 2\pi c_1(X)$. Consider the space of all Kähler potentials in the Kähler class $\mathcal{K}_\omega$

$$\mathcal{H} = \{\phi \in C^\infty(X; \mathbf{R}) | \omega_\phi := \omega + i\partial\bar{\partial}\phi > 0\}.$$

This is an infinite dimensional Fréchet manifold modelled on $C^\infty(X; \mathbf{R})$. There is a Riemannian metric, usually referred to as the *Mabuchi-Semmes-Donaldson* metric, or *MSD* metric in short, on $\mathcal{H}$, given by

$$\langle \phi_1, \phi_2 \rangle_\phi := \int_X \phi_1 \phi_2 \omega_\phi^n,$$

where $\phi_1, \phi_2 \in T_\phi \mathcal{H} = C^\infty(X; \mathbf{R})$.

By a formal calculation (without putting rigorous topology) one gets

Lemma 8. *The Levi-Civita connection on $\mathcal{H}$ is given by*

$$\nabla_{\phi_1} \phi_2 = -\frac{1}{2} \langle \nabla \phi_1, \nabla \phi_2 \rangle_\phi,$$

where $\phi_1, \phi_2 \in C^\infty(X, \mathbf{R})$ are viewed naturally as local vector fields on $\mathcal{H}$ around ϕ.

Proof. First by definition the Levi-Civita connection is determined by the formula

$$\langle \nabla_{\phi_1} \phi_2, \phi_3 \rangle = \frac{1}{2} (\delta_{\phi_1} \langle \phi_2, \phi_3 \rangle_\phi + \delta_{\phi_2} \langle \phi_1, \phi_3 \rangle_\phi - \delta_{\phi_3} \langle \phi_1, \phi_2 \rangle_\phi),$$

where $\phi_1, \phi_2, \phi_3 \in C^\infty(X; \mathbf{R})$ are viewed as local vector field in a neighborhood of ϕ, and $\delta.$ is the variation along the $\cdot$ direction. It is easy to see

$$\delta_{\phi_1} \langle \phi_2, \phi_3 \rangle_\phi = \int_X \phi_2 \phi_3 \Delta \phi_1 \omega_\phi^n$$

So we obtain

$$\nabla_{\phi_1} \phi_2 = -\frac{1}{2} \langle \nabla \phi_1, \nabla \phi_2 \rangle$$

□

It then follows from a direct calculation that the curvature operator is given by

$$K_\phi(\phi_1, \phi_2)\phi_3 = -\frac{1}{4} \{\{\phi_1, \phi_2\}_\phi, \phi_3\}_\phi,$$

where $\{\cdot, \cdot\}_\phi$ is the Poisson bracket defined with respect to the symplectic form ω_ϕ. A convenient way to check this is to use local holomorphic coordinates.

In particular, the sectional curvature is

$$K_\phi(\phi_1, \phi_2) = -\frac{1}{4} \frac{\|\{\phi_1, \phi_2\}\|_\phi^2}{\|\phi_1\|_\phi^2 \|\phi_2\|_\phi^2} \leq 0.$$

Moreover, the curvature tensor is co-variantly constant.

So formally $\mathcal{H}$ is an infinite dimensional negatively curved Riemannian symmetric space. This fact is also a consequence of the moment map picture of Fujiki-Donaldson [33, 26], which we will not discuss here in detail.

A smooth path $\phi(t)(t \in [0,1])$ in $\mathcal{H}$ is a *geodesic* if it satisfies

$$\frac{d^2}{dt^2}\phi(t) - \frac{1}{2}|\nabla_{\omega_{\phi(t)}}(\frac{d}{dt}\phi(t))|^2_{\omega_{\phi(t)}} = 0. \tag{7}$$

If we complexify the variable t by setting $u := t+is$, and define $\Phi(u) = \phi(t)$, then the geodesic equation becomes a degenerate complex Monge-Ampère equation on $X \times [0,1] \times \mathbf{R}$ (see for example [27])

$$(\pi^*\omega + i\partial\bar{\partial}\Phi)^{n+1} = 0,$$

where $\pi : X \times [0,1] \times \mathbf{R} \to X$ is the natural projection.

Define two 1-forms α on $\mathcal{H}$ by

$$\alpha(\psi) = -\int_X \psi(Ric(\omega_\phi) - \omega_\phi) \wedge \omega_\phi^{n-1},$$

$$\beta(\psi) = \int_X \psi(-V^{-1}\omega_\phi^n + e^{-\phi}\Omega/(\int_X e^{-\phi}\Omega)),$$

where $V = \int_X \omega^n$ and Ω is the volume form determined by ω as in the previous section. By (5), (6), the zeroes of both forms are exactly the Kähler-Einstein potentials.

Lemma 9. *Both α and β are closed.*

Proof. By (4) we can calculate that given $\psi_1, \psi_2 \in C^\infty(X;\mathbf{R})$

$$\begin{aligned}\delta_{\psi_2}\alpha(\psi_1) &= -\int_X \psi_1(-\sqrt{-1}\partial\bar{\partial}\Delta\psi_2 - \sqrt{-1}\partial\bar{\partial}\psi_2) \wedge \omega_\phi^{n-1} \\ &- \quad (n-1)\psi_1(Ric(\omega_\phi) - \omega_\phi) \wedge \sqrt{-1}\partial\bar{\partial}\psi_2 \wedge \omega_\phi^{n-2}\end{aligned}$$

A simple integration by parts shows that this is symmetric in ψ_1 and ψ_2. It follows that $d\alpha = 0$. Similarly one can check $d\beta = 0$. □

Since $\mathcal{H}$ is contractible, there are two functions $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{H}$, well-defined up to addition of a constant, such that

$$\alpha = d\mathcal{E}, \qquad \beta = d\mathcal{F}$$

$\mathcal{E}$ is called the *Mabuchi functional*, and $\mathcal{F}$ is called the *Ding functional.*

The connection with the Mabuchi-Semmes-Donaldson geometry of $\mathcal{H}$ lies in the fact that

Proposition 10. *Both $\mathcal{E}$ and $\mathcal{F}$ are convex along smooth geodesics.*

To be more precise, let $\phi(t)$ be a geodesic in $\mathcal{H}$, then we have

$$\frac{d^2}{dt^2}\mathcal{E}(\phi(t)) = \int |\mathcal{D}(\dot{\phi}(t))|^2\omega^n_{\phi(t)},$$

where $\mathcal{D}f$ is the $(0,2)$ component of Hess(f) (with respect to $\omega_{\phi(t)}$), called the *Lichnerowicz Laplacian.* An important fact is that for a real valued function f, $\mathcal{D}f = 0$ if and only if $J\nabla f$ is a holomorphic Killing field.

For the Ding functional we have

$$\frac{d^2}{dt^2}\mathcal{F}(\phi(t)) = \int_X [\frac{1}{2}|\nabla(\dot{\phi}(t))|^2 - (\dot{\phi}(t) - \frac{\int_X \dot{\phi}(t)e^{-\phi(t)}\Omega}{\int_X e^{-\phi(t)}\Omega})^2]e^{-\phi(t)}\Omega$$

The non-negativity follows from the weighted Poincaré inequality proved by Futaki [36, 38]. This was also proved by a different approach and in a more general context by Berndtsson [6]. Denote $\underline{\dot{\phi}}(t) = \dot{\phi}(t) - \frac{\int_X \dot{\phi}(t)e^{-\phi(t)}\Omega}{\int_X e^{-\phi(t)}\Omega}$, then one can write the right hand side as

$$\int_X \underline{\dot{\phi}}(t)L(\underline{\dot{\phi}}(t))e^{-\phi(t)}\Omega,$$

where L is a second order positive elliptic operator acting on functions f with $\int_X fe^{-\phi}\Omega = 0$ (self-adjoint with respect to the L^2 inner product defined using the measure $e^{-\phi}\Omega$). It is also a fact that $L(f) = 0$ if and only if $J\nabla f$ is a holomorphic Killing field.

Now the uniqueness Theorem 6 follows at once if we know that any two points $\phi_1, \phi_2 \in \mathcal{H}$ can be connected by a smooth geodesic. It is not hard to show the existence of a unique *weak* solution Φ to (7) in the sense of pluripotential theory. Chen [14] proved Φ is always $C^{1,1}$ in the sense that Φ is in $C^{1,\alpha}$ for all $\alpha < 1$ and $i\partial\bar{\partial}\Phi$ is uniformly bounded; but in general one should not expect Φ to be smooth, as shown by Lempert-Vivas [42] and Darvas-Lempert [22].

The Ding functional is more amenable for our purpose, since it involves fewer derivatives on ϕ than the Mabuchi functional. This corresponds to the fact that a Kähler-Einstein metric is one with constant scalar curvature; the general constant scalar curvature equation is of fourth order, while the Kähler-Einstein Monge-Ampére equation is of second order. Indeed, Berndtsson [6] proved the convexity of the Ding functional along weak geodesics using the positivity of direct image bundles, and he used this to give a more geometric proof of the Bando-Mabuchi uniqueness result. More significantly, this has an important extension to singular Fano varieties, by Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [4].

Notice there is a natural action of $\mathrm{Aut}(X)$ on $\mathcal{H}$ given by pulling back Kähler metrics. Given any holomorphic vector field $V \in Lie(\mathrm{Aut}(X))$, it thus defines a Killing vector field v on $\mathcal{H}$. Clearly v preserves α, i.e., $\mathcal{L}_v\alpha = 0$, so $d(\iota_v\alpha) = 0$. Hence $\iota_v\alpha$ is independent of the choice of $\phi \in \mathcal{H}$. We define this to be the *Futaki invariant*

$$Fut : Lie(\mathrm{Aut}(X)) \to \mathbf{C}; V \mapsto \iota_v\alpha$$

The precise formula is

$$Fut(V) = -\int_X H(Ric(\omega_\phi) - \omega_\phi) \wedge \omega_\phi^{n-1},$$

where H is the Hamiltonian function generating the action of V, with respect to the symplectic form ω_ϕ.

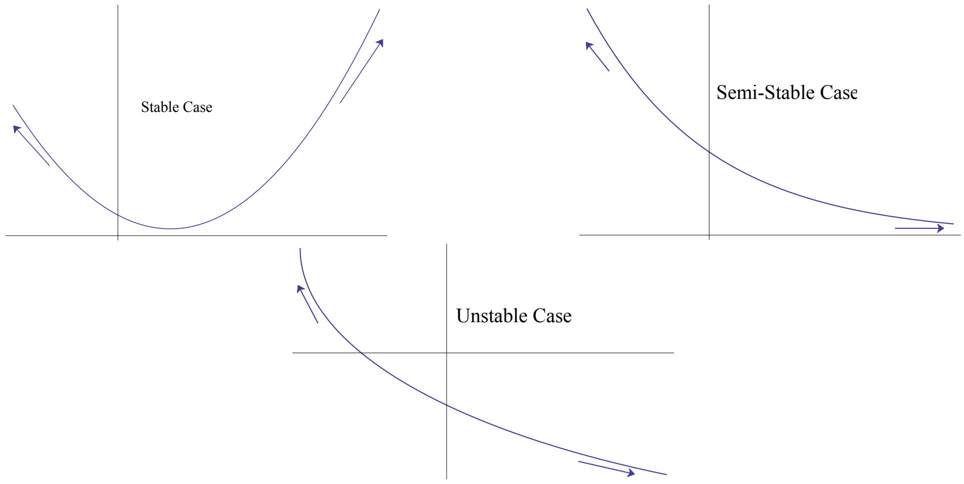

FIGURE 1

The Futaki invariant is indeed a Lie algebra homomorphism: given $V, W \in Lie(\mathrm{Aut}(X))$, we have

$$Fut([V, W]) = \alpha([v, w]) = \mathcal{L}_v(\alpha(w)) - (\mathcal{L}_v\alpha)(w) = 0.$$

Similarly, one can do the same for β, and can check that $\alpha(v) = C_n\beta(v)$, where C_n is a dimensional constant, so the Ding functional also yields essentially the Futaki invariant.

If X admits a Kähler-Einstein metric, then the Futaki invariant must vanish. So the Futaki invariant is an algebro-geometric obstruction to the existence of Kähler-Einstein metric on X.

From the above formal picture we know searching for Kähler-Einstein metrics on X amounts to finding critical points of a geodesically convex functional on $\mathcal{H}$. To illustrate this problem we consider the model case of a strictly convex function f on $\mathbf{R}$. There are three typical behaviors, see Figure 1. In the first case there is a unique critical point, and the derivatives of f at infinity along both directions are positive, we call this the *stable* case; in the second case f has no critical point, but f is globally bounded from below, and one can imagine there is a critical point at infinity, we call this the *semi-stable case*; in the third case f has no critical point, even at ∞, and the derivative of f at $+\infty$ is negative, we call this the *unstable case.*

Formally we hope the existence of Kähler-Einstein metrics on X is equivalent to the derivative at infinity of E along a geodesic ray in $\mathcal{H}$ is positive. This is the notion of *geodesic stability* formulated by Donaldson [27]. However such a condition seems impossible to verify since the geodesic rays in $\mathcal{H}$ are transcendent objects which are difficult to understand.

The notion of K-stability is an algebraization of this idea. We define a *test configuration* to be a flat family of polarized families $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbf{C}$ which is $\mathbf{C}^*$ equivariant and relatively ample, such that $(X_1, L_1) = (X, K_X^{-r})$ for some positive integer r. The central fiber $\mathcal{X}_0$ could be singular in general. This plays the role of a geodesic ray–the idea is that we are "degenerating" the complex manifold X as $t \to 0$.

When X_0 is smooth, the derivative at infinity of $\mathcal{E}$ should be given by the Futaki invariant on X_0 of the holomorphic vector field generating the $\mathbf{C}^*$ action. To define the Futaki invariant when X_0 is not smooth, one uses the fact the Futaki invariant has a purely algebro-geometric definition in terms of Riemann-Roch formula [28]

Let $L_0 = K_{X_0}^{-r}$, and d_k be the dimension of $H^0(X_0, L_0^k)$, w_k be the total weight of the $\mathbf{C}^*$ action on $H^0(X_0, L_0^k)$, then we have asymptotic expansions

$$d_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

We define the Donaldson-Futaki invariant of the test configuration $\mathcal{X}$ by

$$DF(\mathcal{X}) = 2(a_1 b_0 - a_0 b_1)/a_0.$$

By Riemann-Roch formula when X_0 is smooth, this agrees with the previous definition of $Fut(V)$, where V is the holomorphic vector field on X_0 generating the $\mathbf{C}^*$ action. There are also intersection theoretic formulae for the Donaldon-Futaki invariant, by Wang [66] and Odaka [54].

The following definition due to Tian [64] and Donaldson [28] is now natural, given the above model picture

Definition 11. *A Fano manifold X is* K-semistable *if $DF(\mathcal{X}) \geq 0$ for all test configuration $\mathcal{X}$; it is* K-stable *if $DF(\mathcal{X}) > 0$ for all non-trivial test configurations $\mathcal{X}$. X is* K-unstable *if it is not K-semistable.*

The meaning of being "non-trivial" is slightly technical and we will not go into the details here. In the previous section we discussed the phenomenon of jumping of complex structure for Fano manifolds, and explained that this implies that some Fano manifolds can not admit Kähler-Einstein metrics. The notion of K-stability help deal with this issue, in the sense that suppose there is a test configuration for X, with a smooth central fiber X_0 which is not isomorphic to X, then from the definition X and X_0 can not be K-stable simultaneously. In other words, the moduli space of K-stable Fano manifolds is expected to be Hausdorff.

In fact, K-stability is exactly the algebro-geometric criterion for the existence of Kähler-Einstein metrics on a Fano manifold.

Theorem 12 (Chen-Donaldson-Sun 2012)**.** *A Fano manifold X admits Kähler-Einstein metric if and only if X is K-stable.*

This proves a conjecture that goes back to Yau [68], and is a special case of the more general *Yau-Tian-Donaldson conjecture*. The "only if" direction is proved by Tian [64], Stoppa [60], Mabuchi [51] Berman, [3]. The "if" direction is proved by Chen-Donaldson-Sun [15, 16, 17, 18]. Also from the proof it follows that to check that X is not K-stable, one only needs to consider *special test configurations* in the sense of Ding-Tian [25], where the central fiber X_0 is assumed to be a $\mathbf{Q}$-*Fano* variety; this is also proved purely algebraically in [49], using the minimal model program in birational geometry. The notion of K-stability extends to more general varieties with singularities, and it is intimately related to singularities [54]. There are important recent advances towards understanding K-stability from the algebro-geometric point of view, see for example [9, 24, 34, 35, 44]

The rough sketch of the proof is as follows. Given X, we can always find a smooth divisor D in the class $|-\lambda K_X|$ for some $\lambda > 1$, and we can solve a unique Kähler-Einstein metric ω_β on X with cone angle $2\pi\beta$ along D for some $\beta = 1/p$ (where p is large positive integer). ω_β satisfies the equation

$$Ric(\omega_\beta) = (1 - (1-\beta)\lambda)\omega_\beta + (1-\beta)2\pi[D],$$

where $[D]$ is the current of integration along D. This is not difficult because the metric ω_β has indeed negative Ricci curvature on $X \setminus D$ (one can think that the cone angle introduces positive curvature transverse to D), and the singularities along D are of orbifold type, so we can essentially adapt the Aubin-Yau theorem.

Then we want to increase the cone angle towards $\beta = 1$ and deform the metrics ω_β correspondingly. By an implicit function theorem based on linear estimate of the Laplace operator on conical metrics [29], Donaldson proved that one can always increase β by a small amount $\epsilon > 0$. So if we can not solve the original Kähler-Einstein equation the deformation must break down at some angle $\beta_\infty \in (0, 1]$, namely, as β goes up to β_∞, the metrics ω_β do not converge in the obvious way to a limit metric on X with cone angle $2\pi\beta_\infty$ along D.

Now the essential part is to contradict K-stability of X if this divergence would occur; i.e., we need to construct a *de-stabilizing* test configuration $\mathcal{X}$ with $DF(\mathcal{X}) \leq 0$. So we want to achieve the following

(A) Construct the central fiber X_0;
(B) Construct the $\mathbf{C}^*$ equivariant family $\mathcal{X}$ (and prove it is non-trivial);
(C) Show that $Fut(\mathcal{X}) \leq 0$ (such an $\mathcal{X}$ is usually called a *de-stabilizing test configuration*).

Among the three (A) is the most essential part, and is done by producing algebraic structure out of a Gromov-Haudorff limit of the metrics ω_β as $\beta \to \beta_\infty$ (these limits are a priori only metric spaces). This was first constructed in [31] for the Gromov-Hausdorff limit of a sequence of smooth Kähler-Einstein Fano manifolds, which we will say more in the next section, and was later extended to the case with cone singularities in [17, 18] which, not surprisingly, involves much more delicate analysis. The outcome is that the Gromov-Hausdorff limit is naturally a normal $\mathbf{Q}$-Fano variety, and this will be our X_0.

The proof of (B) and (C) involves more refined understanding of the $\mathbf{Q}$-Fano variety X_0, geometric invariant theory, and the recent development in the pluripotential theory of Monge-Ampère equations [3, 6, 4].

After Theorem 12 was proved there are also new proofs by Datar-Székelyhidi [23] using the classical continuity path

$$Ric(\omega_t) = t\omega_t + (1-t)\alpha$$

for a fixed Kähler form $\alpha \in 2\pi c_1(X)$, and deform from $t = 0$ (whose solution is guaranteed by Yau's Theorem 3) to $t = 1$; and by Chen-Sun-Wang [19]

using the Ricci flow
$$\frac{\partial}{\partial t}\omega(t) = \omega(t) - Ric(\omega(t))$$
starting from any smooth initial Kähler form $\omega(0) \in 2\pi c_1(X)$, and studying the limit as $t \to \infty$. Both of these new proofs also depend on constructing the de-stabilizing test configuration from certain differential geometric limits. There is a fourth alternative proof by Berman-Boucksom-Jonsson [5], which proves a slightly weaker result with K-stability replaced by *uniform K-stability*, and which follows more closely the variational picture described in the beginning of this subsection.

3. Algebraic structure on Gromov-Hausdorff limits

In this section we discuss the ingredient (A) in the proof of Theorem 12 (appeared at the end of last section). We will first explain the construction of algebraic structure on Gromov-Hausdorff limits in a slightly different situation [31], which should give a clearer geometric picture, and which also have its own importance, for example, in the study of moduli compactification of Fano manifolds. At the end of this section we will explain briefly the extra complications involved in the actual proof of (A).

Let $(X_i, L_i = K_{X_i}^{-1}, \omega_i \in 2\pi c_1(L_i))$ be a sequence of n dimensional Kähler-Einstein Fano manifolds with $Ric(\omega_i) = \omega_i$. By Myers's theorem the diameter of (X_i, ω_i) is uniformly bounded above and by the Bishop-Gromov comparison theorem we also have a uniform non-collapsing property, i.e., there exists $\kappa > 0$ such that $Vol(B(p_i, r)) \geq \kappa r^{2n}$ for all i, $p_i \in X_i$, and $r \in (0, 1]$. One important consequence of the non-collapsing condition is a uniform *Sobolev inequality* (See for example [45])

$$\|f\|_{L^{\frac{2n}{n-1}}} \leq C(\|\nabla f\|_{L^2} + \|f\|_{L^2}).$$

By Riemannian convergence theory, passing to a subsequence we may obtain a *Gromov-Hausdorff limit* Z, which is a compact metric space. This is done by approximating each X_i uniformly by finite discrete metric spaces, and taking diagonal limits.

To see the connection with algebraic geometry we recall the classical Kodaira embedding theorem. Given a polarized Kähler manifold (X, L, ω) such that $-i\omega$ is the curvature of a hermitian metric h on L. This gives rise to a Hermitian inner product on $H^0(X, L^k)$ for any $k \geq 0$

$$\langle s_1, s_2 \rangle := \int_X \langle s_1, s_2 \rangle_h \frac{(k\omega)^n}{n!}.$$

Here notice when we study sections of L^k we use the corresponding Kähler form $k\omega \in 2\pi c_1(L^k)$.

The *density of state* function, or sometimes called *Bergman function*, is defined by

$$\rho_{k,X}(x) = \sup_{s \in H^0(X, L^k) \setminus \{0\}} \frac{|s(x)|_h}{\|s\|}.$$

The Kodaira embedding theorem claims that there is a $k > 0$ such that the associated map $F : X \to \mathbf{P}(H^0(X, L^k))^*$ is an embedding; in particular $\rho_k > 0$ everywhere on X. Using the L^2 inner product we can identify $\mathbf{P}(H^0(X, L^k)^*)$ with $\mathbf{P}^{N_k}$, where $N_k + 1 = \dim H^0(X, L^k)$, and the map $F : X \to \mathbf{P}^{N_k}$ is well-defined, up to the action by $U(N_k + 1)$.

Applied to our sequence we see that for each i there is a k_i such that X_i is embedded into some projective space using the sections of L^{k_i}. The key property we need is a uniformity on k_i.

Theorem 13 (Donaldson-Sun [31])**.** *There are $\epsilon > 0$, and $\underline{k} > 0$ depending only on the dimension n, such that $\rho_{\underline{k}, X_i}(x) \geq \epsilon$ for all i.*

Remark 14. *This was proved by Tian* [61] *in 1990 in dimension two using the fact that the limit Z in this case is an* orbifold*, and the above result is conjectured by Tian (called the* partial C^0 estimate*) in* [63].

Theorem 13 implies that the map $F_i : X_i \to \mathbf{P}(H^0(X_i, L^{\underline{k}})^*)$ is well-defined. Using the Bochner formula for the $\Delta_{\bar\partial}$ acting on $L^{\underline{k}}$-valued $(1, 0)$ form and the Moser iteration argument (which involves the Sobolev inequality) it is not hard to show that a holomorphic section $s \in H^0(X_i, L_i^{\underline{k}})$ with unit L^2 norm has $|\nabla s|_{L^\infty}$ uniformly bounded. It then follows that F_i has uniformly bounded derivative with respect to the natural Fubini-Study metric on the projective space. Then with extra work one can show that by replacing $\underline{k}$ with $m\underline{k}$ for some sufficiently large integer m also depending only n, the map $F_i : X_i \to \mathbf{P}(H^0(X_i, L^{\underline{k}}))^*$ is an embedding for all i. Passing to a further subsequence we can assume for all i the dimension of $H^0(X_i, L^{\underline{k}})$ is the same integer $N + 1$, then the maps $F_i : X_i \to \mathbf{P}^N$ converge to a continuous map $F_\infty : Z \to \mathbf{P}^N$. We can then prove that F_∞ is a homeomorphism onto a normal projective subvariety $W \subset \mathbf{P}^N$, and $F_i(X_i)$ converges to W in a fixed Hilbert scheme. Furthermore, W is indeed a $\mathbf{Q}$-Fano variety, with Kawamata-log-terminal singularities, and the singularities of W match with the metric singularities of Z. This class of singularities was first introduced and studied in the birational algebraic geometry and minimal model program. Furthermore the isomorphism class of W is independent of further increasing power $\underline{k}$ by $m\underline{k}$. It is in this sense we can say that

Theorem 15 (Donaldson-Sun [31])**.** *Z is naturally a $\mathbf{Q}$-Fano variety.*

We can also give a more intrinsic description of the algebraic structure on Z. Fixing a metric $\mathbf{d}$ on the disjoint union of X_i and Z which realizes the Gromov-Hausdorff convergence of X_i to Z. Then we can define a presheaf of rings of functions on Z by assigning to each open set $U \subset Z$ the space of all continuous functions on U that are naturally uniform limits of holomorphic functions on corresponding open subsets of X_i. Let $\mathcal{O}$ be the associated sheaf. Then the statement is that $\mathcal{O}$ exactly defines the sheaf of holomorphic functions on Z. One consequence is that $\mathcal{O}$ does not depend on the choice of the metric $\mathbf{d}$.

In terms of moduli theory, Theorem 15 gives a *topological* compactification of moduli space of Kähler-Einstein Fano manifolds by adding certain $\mathbf{Q}$-Fano varieties in the boundary. It is also known that Z admits a *weak* Kähler-Einstein metric, in the sense of pluripotential theory. It is also K-stable by [3] (notice the definition of K-stability in the previous section makes sense for any normal projective variety with $\mathbf{Q}$-Gorenstein singularities. There are further results in this direction towards understanding the algebraic structure on the moduli space itself, see [57, 55, 30, 58, 47, 48, 56].

A last remark is that even though we only stated the above results for Kähler-Einstein Fano manifolds, the technique applies also to polarized Kähler-Einstein manifolds with zero or negative Ricci curvature, but we need to impose the extra assumption that the volume is non-collapsed, since in general collapsing can possibly occur (even in complex dimension one).

Now we explain the idea in the proof of Theorem 13. Unravelling definition it amounts to constructing holomorphic section of $L^{\underline{k}}$ with a definite upper bound on the L^2 norm, and with a definite positive lower bound at a given point. We first digress to give here an analytic account of the proof of Kodaira embedding theorem, and mention the technical difficulties one has to deal with in order to extend it to prove Theorem 13.

Given (X, L, ω), a point $p \in X$ and $k \geq 0$, we consider the rescaling $(X, L^k, k\omega)$ with base point p, then as $k \to \infty$, we see the obvious limit is $\mathbf{C}^n$ endowed with the standard flat metric ω_0. A slightly more non-trivial fact is that the Hermitian holomorphic line bundle converges smoothly to the trivial holomorphic line bundle L_0 over $\mathbf{C}^n$ endowed with the (non-trivial) hermitian metric $e^{-|z|^2/2}$. This is an easy consequence of the definition of being Kähler: we can choose local holomorphic coordinate around p and local holomorphic trivialization of L such that the corresponding metrics ω and h agrees with the standard metrics up to first order.

Let σ be the standard section of L_0, then $|\sigma(z)|^2 = e^{-|z|^2/2}$, and one can compute that $\|\sigma\|^2_{L^2(\mathbf{C}^n)} = (2\pi)^n$. For R sufficiently large we can choose a cut-off function χ_R on $\mathbf{C}^n$ that has value 1 on the ball B_R and is supported in the ball B_{R+1}, then it is easy to see

$$\|\chi_R\sigma\|^2_{L^2} = (2\pi)^n - \epsilon(R); \quad \|\bar{\partial}(\chi_R\sigma)\|^2_{L^2} = \epsilon(R),$$

where $\epsilon(R)$ denotes a function of R that decays faster than any polynomial rate as $R \to \infty$.

Now for k large, $\chi_R\sigma$ can be grafted to a smooth section of L^k over X, supported in a neighborhood of p, which we denote by β. It is approximately holomorphic in the sense that if we fix R and make k sufficiently large then we may assume

$$\|\beta\|^2_{L^2} = (2\pi)^n + \epsilon(R); \quad \|\bar{\partial}\beta\|^2_{L^2} = \epsilon(R).$$

Moreover β is exactly holomorphic in the ball of radius $R - 1$ around p (again, with respect to the metric $k\omega$).

The next step is to correct β by a small amount to make it holomorphic. The key is the following estimate

Lemma 16. *The operator $\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $\Omega^{0,1}(X, L^k)$ satisfies*

$$\Delta_{\bar{\partial}} = (\nabla'')^*\nabla'' + k^{-1}Ric(\omega) + 1.$$

In particular, for k sufficiently large we may assume $\Delta_{\bar{\partial}} \geq 1/2$ in the natural L^2 norm.

This is a version of Bochner formula in complex geometry. It is important that we only consider $(0,1)$ forms so that only Ricci curvature shows up, not the full sectional curvature.

We denote $\tau = \bar{\partial}^*\Delta^{-1}_{\bar{\partial}}\bar{\partial}\beta$, then

$$\|\tau\|^2_{L^2} = \langle \Delta^{-1}_{\bar{\partial}}\bar{\partial}\beta, \bar{\partial}\beta\rangle \leq 2\|\bar{\partial}\beta\|^2 = \epsilon(R).$$

Let $s := \beta - \tau$, then $\bar{\partial}s = 0$, and

$$\|s\|^2_{L^2} \leq (2\pi)^n + \epsilon(R).$$

Moreover, in the ball of radius one around p (with respect to the metric $k\omega$), we have $\bar{\partial}\tau = 0$, and so at these points by standard interior estimate for holomorphic functions we get

$$|\tau| \leq C\|\tau\|_{L^2} \leq \epsilon(R).$$

On the other hand, we know $|\beta(q)| \geq e^{-1/2} - \epsilon(R)$ for q in the ball of radius 1 , so if we a priori fix R to be large, and then choose $k \gg R$ we obtain

$$|s(q)| \geq \frac{1}{2}e^{-1/2}.$$

Therefore it follows that

$$\rho_{k,X}(q) \geq \frac{1}{4}e^{-1/2}(2\pi)^{-n}.$$

Notice since X is compact and the metric ω is smooth, we can obtain a uniform k so that the above arguments work for all points $p \in X$, so we get a uniform positive lower bound of $\rho_{k,X}$.

This implies the map $F : X \to \mathbf{P}(H^0(X, L^k)^*)$ is well-defined for such k. To improve F to an embedding one can replace k by mk for some positive integer m (notice $\rho_{k,X} > 0$ implies $\rho_{mk,X} > 0$ for all integer $m \geq 1$ due to the natural (non-linear) map $Sym^m H^0(X, L^k) \to H^0(X, L^{mk})$). First one can use similar arguments as above to make F an immersion–simply replacing the model section σ be $l \cdot \sigma$, where l is a linear holomorphic function on $\mathbf{C}^n$–so that F becomes a finite covering map. Now on a fiber $F^{-1}(x) = \{p_1, \cdots, p_l\}$, we can replace k again by a large multiple and construct holomorphic sections $s_1, \cdots, s_l$ of L^k, so that $|s_j(p_j)| \geq \frac{1}{2}e^{-1/2}(2\pi)^{-n}$, and $|s_j(p_i)| \leq 100^{-1}(2\pi)^{-n}$ (this is possible because the above grafted section β is zero outside the ball of radius R and $\|\tau\|_{L^2} = \epsilon(R)$). This implies that $p_1, \cdots, p_l$ have different images under the map F, and hence F is injective.

With more work and more machinery we indeed know that ρ_k admits an asymptotic expansion of the form (see for example [62, 11, 69])

$$\rho_k(x) = \frac{1}{(2\pi)^n}(1 + a_1k^{-1} + a_2k^{-2} + \cdots),$$

where a_i is a function on X which depends only on the local invariant of g, namely the curvature and its covariant derivatives. In particular by [50] $a_1 = \frac{1}{2}S(\omega)$. Using this expansion, one can show that

$$\frac{1}{k}F_k^*\omega_{FS} = \omega + O(k^{-2}).$$

This is the starting of point of "quantization techniques" in Kähler geometry.

Now we return to the proof of Theorem 13. From the above argument it is easy to see that we can obtain a uniform embedding if the Kähler metric (J, ω) vary in a compact set of smooth Kähler metrics in the C^l topology for some large l. However, in general the set of Fano Kähler-Einstein manifolds is not compact in such a topology and singularities will occur in general. In complex dimension two, the singularities are of *orbifold type*, i.e., locally modelled on the quotient $\mathbf{C}^2/\Gamma$ for a finite subgroup of $U(2)$; the readers are referred to [57] for explicit examples of formation of singularities.

We need some input from the convergence theory of Riemannian manifolds. Fix a metric $\mathbf{d}$ on the disjoint union of X_i's and Z. Then it follows from the work of Cheeger-Colding-Tian [13] that there is a decomposition $Z = \mathcal{R} \cup \Sigma$, where the *regular set* $\mathcal{R}$ is an open set of Z, which is endowed with a smooth Kähler-Einstein metric $(\omega_\infty, J_\infty)$, and the convergence on $\mathcal{R}$ is smooth in the sense that for any fixed compact subset K of $\mathcal{R}$, we can find smooth maps $\chi_i : K \to X_i$ for large i so that $\mathrm{d}(x, \chi_i(x)) \leq \epsilon_i$ and $(\chi_i^*\omega_i, \chi_i^* J_i)$ converges smoothly to $(\omega_\infty, J_\infty)$. The *singular set* Σ is a closed subset of real Hausdorff codimension at least four. Again in our setting we also obtain a smooth Hermitian holomorphic line bundle (L_∞, h_∞) over $\mathcal{R}$.

Now we notice in the above construction of holomorphic sections the only place we used the global Ricci curvature is through the invertion of $\bar{\partial}$ operator (indeed only the lower bound on Ricci curvature is used). It is then not difficult to see that the uniform estimate of the density of state functions still holds on points of X_i that are fixed distance away from Σ (with respect to $\mathbf{d}$). It is near Σ that we need to work much harder. Notice we can not use the model Gaussian sections on $\mathbf{C}^n$ anymore since as we approach Σ the region where this model is effective shrinks down to a point. Instead we use another important aspect of the Cheeger-Colding theory. At each point $p \in \Sigma$, we can dilate the limit metric d_∞ based on p, then again by general theory we obtain (pointed) Gromov-Hausdorff limits, called *tangent cones*. Cheeger-Colding [12] proved that any tangent cone is a metric cone, of the form $(C(Y), d)$ for Y a compact metric space of diameter at most π (called the *cross section*). Here $C(Y)$ is topologically a cone given by adding one point (the cone vertex) to $Y \times (0, \infty)$, and the distance is given by the formula

$$d((y_1, r_1), (y_2, r_2))^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos d_Y(y_1, y_2).$$

In fact the singular set Σ is defined exactly as those points where the tangent cones are not isometric to the smooth flat cone. Again a tangent cone admits similar regular-singular decomposition as before.

We will say more about the tangent cones in our setting in the next section. For the purpose here, it suffices to mention one important feature of tangent cones in the Kähler setting. Namely, on the regular part of the tangent cone, the Kähler metric can be written as $i\partial\bar{\partial}\frac{r^2}{2}$, which is the curvature form of the hermitian metric $e^{-r^2/2}$ on the trivial holomorphic line bundle. It is then clear that the constant section plays the role of σ_0 on the smooth cone $\mathbf{C}^n$.

From this point there are a few technical difficulties to overcome if one wants to use the previous arguments to prove Theorem 13. Let p be a point in Z, and let $C(Y)$ be *one* tangent one at p (notice a priori we do not know if the tangent cone is unique; but that is not needed here). By a diagonal sequence argument we may assume that $C(Y)$ is the limit of $(X_i, m_i\omega_i)$ for a sequence $m_i \to \infty$. We first assume Y smooth. Then there are two points to deal with

- The convergence from $(X_i, m_i\omega_i)$ to $C(Y)$ is only smooth away from the vertex of $C(Y)$. This means that using the previous techniques we can at best control the norm of the constructed holomorphic section outside a small neighborhood of the vertex (with respect to $m_i\omega_i$). In order to control the norm globally, we need an estimate of the derivative of s. This simply follows from the uniform gradient estimate of s we mentioned before.
- We wrote down the hermitian metric on L_∞ directly, but it is not a priori known that the connection on $L_i^{m_i}$ converges smoothly to that on L_∞ (which is what we need since we want to compare the $\bar\partial$ operators), even though the curvature forms converge smoothly. There is a potential ambiguity of holonomy caused by a flat connection. This can be overcome since Y is smooth and has positive Ricci curvature, so that $\pi_1(Y)$ is finite and we can get rid of the holonomy problem by raising to a large, but definite uniform power of L_i.

Now in general Y itself can be singular, then there are extra complications. We briefly outline these, and one can find the details in [31].

(1) When we cut-off the section σ, we also need to cut-off along rays over the singular set of Y. This requires the size of the singular set Σ_Y of Y to be relatively small, and follows from the fact [13] that the singular set of Y has Haudorff co-dimension bigger than two. It implies the existence of a *good* cut-off function. Namely, for any $\epsilon > 0$, we can find a smooth non-negative function χ on the regular part of Y, which is supported outside a neighborhood of Σ_Y and is equal to one outside the ϵ-neighborhood of Σ_Y, and with $\|\nabla\chi\|_{L^2} \leq \epsilon$. Using this, the extra cut-off will not introduce a big error term when estimating $\bar\partial\beta$.

(2) The holonomy problem becomes more complicated. In three dimension one can show that Y has only orbifold singularities transverse to circles and the regular part of Y has finite fundamental group, and then use similar arguments as above. In general one can bypass the problem of understanding the topology of the regular part of Y, by using the Dirichlet approximation theorem in elementary number theory. The point is that for our argument to go through it is not necessary to make the holonomy trivial, but rather making the holonomy small.

Now going back to the item (A) in the end of last section. There we need to adapt the above arguments to the case of Kähler-Einstein metrics with cone singularities. Let β_i be a sequence going up to β_∞, and we want to understand the corresponding Gromov-Hausdorff limit Z.

- If $\beta_\infty < 1$, then we prove that there is also a regular-singular decomposition of Z as above. But now the singular set will have Hausdorff co-dimension two (the divisor D is singular set of ω_{β_i}. A large technical part of [17] is devoted to understanding better the codimension two part of the singular set, and constructing a *good* cut-off function.
- If $\beta_\infty = 1$, then this is the case when the cone singularity should *disappear*. Here extra difficulty emerges even in the first step, when one wants to prove that the limit Z contains a large open smooth subset. Details can be found in [18].

4. Singularities of Gromov-Hausdorff limits

This section is based on some part of [32], in which we make a deeper study of the tangent cones of non-collapsed Gromov-Haudorff limits of Kähler-Einstein manifolds. There are several motivations for this study. First of all, in many cases singular Kähler-Einstein metrics occur naturally, and it is an important question to understand quantitatively the metric behavior near the singularities, which we hope would help advance the theory in the more general Riemannian setting. Secondly, as shown in the two dimensional result of Odaka-Spotti-Sun [57], the study of tangent cones is expected to be important in classifying Gromov-Hausdorff limits in the moduli compactifcation of Kähler-Einstein Fano manifolds, and these can lead to explicit existence results of Kähler-Einstein metrics on certain *families* of Fano manifolds (this is a different approach from applying Theorem 12 and study K-stability explicitly). Also, as we shall describe below, this study has its own interest which yields potential interesting stability notion for a local algebraic singularity (more precisely, a *log terminal* singularity), and motivates some new questions in algebraic geometry.

We first digress to explain some background on Kähler cones. Let Y be a smooth compact manifold of dimension $2n-1$. Denote $C(Y) = Y \times \mathbf{R}^+$, and let r be the coordinate function on $\mathbf{R}^+ = (0, \infty)$.

Definition 17. *A Kähler cone structure on $C(Y)$ consists of a Kähler metric (g, J, ω) such that $g = dr^2 + r^2 g_Y$ for some Riemannian metric g_Y on Y. In particular g is a Riemannian cone.*

The induced structure on $Y = \{r = 1\}$ is usually referred to as a *Sasaki structure*; for our purpose it is more convenient to focus on the cone $C(Y)$ instead of Y. The *Reeb vector field* is given by $\xi = Jr\partial_r$. The following are a few basic, but important, properties of a Kähler cone. The first has already been used in the previous section

Lemma 18. *(1) $\omega = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}r^2 = \frac{1}{4}dd^c r^2$;*
(2) $\mathcal{L}_{r\partial_r} g = 2g$, $\mathcal{L}_{r\partial_r}\omega = 2\omega$;
(3) ξ is holomorphic, Killing and Hamiltonian, i.e., $\mathcal{L}_\xi J = \mathcal{L}_\xi \omega = \mathcal{L}_\xi g = 0$.

Proof. For (1) we notice that from definition $g = \frac{1}{2}\mathrm{Hess}(r^2)$, and it is easy to check that the $(1,1)$ part of Hessian on a Kähler manifold is given by $\sqrt{-1}\partial\bar{\partial}$. The first identity in (2) follows from definition, and for the second identity we compute using Cartan formula

$$\mathcal{L}_{r\partial_r}\omega = d(\iota_{r\partial_r}\omega).$$

Notice by definition $\iota_{\partial_r} g = dr$, so $\iota_{r\partial_r}\omega = -rJdr = \frac{1}{2}d^c r^2$, and hence $\mathcal{L}_{r\partial_r}\omega = 2\omega$. For (3), $\mathcal{L}_\xi J = 0$ follows from (2); Similar calculation as (2) which shows that $\iota_\xi \omega = -rdr = -d(r^2/2)$ hence $\mathcal{L}_\xi\omega = 0$ and the Hamiltonian function for ξ is $-r^2/2$. □

By item (3) ξ generates a holomorphic isometric action of some real torus $\mathbf{T}$ on $C(Y)$. Let k be the rank of $\mathbf{T}$. When $k = 1$, we obtain a holomorphic $\mathbf{C}^*$ action on $C(Y)$, and if the $\mathbf{C}^*$ action is free then by taking Kähler quotient we obtain a polarized Kähler manifold (V, L, ω_V); this is called the *regular case*. If the $\mathbf{C}^*$ action is not free then we obtain a *polarized orbifold*; this is called the *quasi-regular case*. If $k > 1$ then we are in the *irregular case*.

A regular cone $C(Y)$ is Ricci-flat if and only if (V, ω_V) is Kähler-Einstein. This reveals an intimate connection between the Calabi-Yau geometry and the Fano Kähler-Einstein geometry. The following is an analogue of the Kodaira embedding theorem for Kähler cones.

Theorem 19 (Van Coevering [65])**.** *A Kähler cone is naturally an affine algebraic cone.*

It is easy to see that one can complete the metric structure on $C(Y)$ by adding a *vertex* O, so that r becomes the distance function to O. Complex analytically, general theory of Grauert also shows that there is a unique way to put a structure of a complete analytic space on $C(Y) \cup \{O\}$. For simplicity of notation we will still denote by $C(Y)$ when we add the vertex O.

The theorem means that there is a holomorphic embedding of $C(Y)$ into some affine space $\mathbf{C}^N$ as an affine subvariety, and the $\mathbf{T}$ action extends to a diagonal linear action on $\mathbf{C}^N$. Moreover, there is a preferred real one parameter subgroup of $\mathbf{T}^{\mathbf{C}}$ (corresponding to the cone dilation) with respect to which every non-zero $\mathbf{T}$-homogeneous holomorphic function on $C(Y)$ has positive weight. The algebraicity depends essentially on the $\mathbf{T}$ symmetry, which allows us to decompose holomorphic functions into sum of eigenfunctions.

More intrinsically, we can describe the coordinate ring $R(C(Y))$ as the ring of holomorphic functions on $C(Y)$ with polynomial growth at infinity. It admits a decomposition

$$R(C(Y)) = \bigoplus_{d \in \mathcal{S}} R_d$$

where R_d consists all holomorphic functions f on $C(Y)$ which is homogeneous of degree d, i.e., $\mathcal{L}_{r\partial_r} f = df$, and $\mathcal{S}$ is the set of all $d \geq 0$ such that $R_d \neq \{0\}$. The cone structure of $C(Y)$ (the action of $r\partial_r = -J\xi$) is encoded in the grading by $\mathcal{S}$. Notice in general $\mathcal{S}$ may not be contained in $\mathbf{Q}$ (if $C(Y)$ is irregular). We can also write the decomposition as

$$R(C(Y)) = \bigoplus_{\alpha \in Lie(\mathbf{T})^*} R_\alpha$$

where for each d we can uniquely write $d = \langle \xi, \alpha \rangle$ for some $\alpha \in Lie(\mathbf{T})^*$. In this sense we can say that the affine variety $C(Y)$ is an affine algebraic cone, and ξ is the Reeb vector field of the affine algebraic cone.

Remark 20. *This notion of an affine algebraic cone is* different *from the usual meaning of affine cone in the algebraic geometry literature. The point is that we allow the grading is positive by not necessarily rational, so the rank of* $\boldsymbol{T}$ *can in general be bigger than one, whereas in the literature one is often restricted to consider a* $\mathbf{C}^*$ *action.*

As in [52, 20] we can define the index character

$$F(t,\xi)=\sum_{d\in\mathcal{S}} e^{-td}\dim R_d$$

and as a consequence of Riemann-Roch for orbifolds there is an asymptotic expansion

$$F(t,\xi)=\frac{a_0 n!}{t^{n+1}}+\frac{a_1(n-1)!}{t^n}+O(t^{1-n})$$

where

$$a_0=c_n\int_{C(Y)} e^{-r^2/2}$$

for some dimensional constant c_n. We define the *volume* of the Kähler cone to be

$$\mathrm{Vol}(\xi)=\int_{C(Y)} e^{-r^2/2}.$$

It is clear from the above formula, or by direct verification, that the volume is independent of the particular choice of the Kähler cone metric.

The item (1) in Lemma (18) means that one can fix the underlying complex structure of $C(Y)$ and study the deformation of Kähler cone metrics with possibly varying Reeb vector fields. The set of all possible Reeb vector fields in $Lie(\mathbf{T})$ is referred to as the *Reeb cone*, by [52, 8, 39, 20] and it admits an algebro-geometric description too as the set of all ξ such that if $R_\alpha\neq 0$, then $\langle\xi,\alpha\rangle>0$.

For a simple explicit example, we consider $\mathbf{C}^n$ as a complex manifold equipped with the standard holomorphic action of $\mathbf{T}=T^n$. So $Lie(T^n)=\mathbf{R}^n$. Then it is easy to see the Reeb cone $\mathcal{R}$ is equal to $(\mathbf{R}_{>0})^n$. This means that for any $\xi=(\xi_1,\cdots,\xi_n)$ where each ξ_i is positive, there is a Kähler cone metric on $\mathbf{C}^n$ with Reeb vector field equal to ξ. Such a cone metric is regular if and only if all ξ_i's are equal and the quotient Fano manifold is the $n-1$ dimensional projective space . If ξ is proportional to a rational vector then the cone metric is quasi-regular and the quotient orbifold is a *weighted projective space*. In the remaining case the metric is irregular. Notice as algebro-geometric object, the set of all weighted projective spaces in a fixed dimension is quite a *discrete* set; on the cone $\mathbf{C}^n$, by adding the set of irregular Kähler cone structures, this set is made *connected*.

Now we assume the Kähler cone $(C(Y),\xi)$ is Ricci-flat. The induced metric on Y is *Sasaki-Einstein*, with positive Ricci curvature. So $\pi(Y)$ is finite. The Ricci-flat metric induces a flat connection on the canonical bundle $K_{C(Y)}$, so we can find a positive integer $m>0$ such that $K_{C(Y)}^m$ has a nonzero parallel section Ω. In particular, the underlying algebraic cone is $\mathbf{Q}$-*Gorenstein*. Notice the Ricci-flat equation can be written as

$$\omega^n=C(\Omega\wedge\bar{\Omega})^{1/m}.$$

Since $\mathcal{L}_\xi\omega=2\omega$, it follows that Ω is homogeneous with

$$\mathcal{L}_\xi\Omega=\sqrt{-1}(n+1)m\Omega. \tag{8}$$

We define a distinguished hyperplane H in the Reeb cone $\mathcal{R}$ by

$$H = \{\xi' \in \mathcal{R} | \mathcal{L}_{\xi'}\Omega = \sqrt{-1}(n+1)m\Omega\}$$

and consider the volume functional

$$\mathrm{Vol} : H \to \mathbf{R}_{>0},$$

Here are a few properties proved by Martelli-Sparks-Yau (see [52, 39, 20])

- Vol is strictly convex and proper;
- ξ is a critical point of Vol;
- Vol has an algebraic formula as a rational function on H with rational coefficients.

These imply that ξ is an isolated critical point of a rational function with rational coefficients, so by elementary number theory we obtain

Lemma 21 ([52])**.** *The Reeb vector field $\xi \in Lie(\boldsymbol{T})$ is an algebraic vector, and hence the holomorphic spectrum $\mathcal{S}$ is algebraic.*

Now we go back to the sequence (X_i, L_i, ω_i) with Gromov-Hausdorff limit (Z, d_Z). Let $C(Y)$ be a tangent cone at a point $p \in Z$. It can be realized as the Gromov-Hausdorff limit of an appropriately rescaled sequence $(X_i, L_i^{a_i}, a_i\omega_i)$ for some $a_i \to \infty$. The regular part of $C(Y)$ is a smooth Ricci-flat Kähler cone, and if Y is smooth then we are essentially in the previous setting (notice here on a tangent cone the vertex is naturally there). In general Y can be singular, and we also have

Proposition 22 ([32])**.** *$C(Y)$ is naturally a normal affine algebraic cone with log terminal singularities.*

We will not prove this, but see [32]. Here we would like to explain the intrinsic meaning of the algebraic structure. We can define the sheaf $\mathcal{O}$ on $C(Y)$ by taking the push-forward of the sheaf of holomorphic functions on the regular part of $C(Y)$. Then it turns out that $(C(Y), \mathcal{O})$ defines a natural *normal* complex analytic structure, and then we can proceed as in the case Y is smooth to define intrinsically the coordinate ring $R(C(Y))$. The conclusion is that $R(C(Y))$ is finitely generated, and defines an embedding of $C(Y)$ as an affine algebraic variety in $\mathbf{C}^N$. Moreover, the Reeb vector field ξ on the regular part of $C(Y)$ generates an action of a compact torus $\mathbf{T}$, and this action extends to a holomorphic isometric action on the whole $C(Y)$. The dilation action on $C(Y)$ defines a grading on $R(C(Y))$ by the holomorphic spectrum $\mathcal{S}$.

It is interesting to ask for the algebro-geometric meaning of $C(Y)$ in terms of the singularity p. Notice a priori we do not know if there is a unique tangent cone at p, even though this is true in the end. However, a relatively simple observation gives the following, which is crucial in the discussion below.

Lemma 23 ([32])**.** *The holomorphic spectrum $\mathcal{S}$ is independent of the choice of tangent cones at p.*

The key is that the set of all tangent cones at p forms a compact and connected set under the natural topology, and $\mathcal{S}$ deforms continuously. Then the algebraicity of $\mathcal{S}$ (a generalization of Lemma 21) implies it is rigid hence allows no deformations.

Now we explain the relation between $\mathcal{O}_p$ and $R(C(Y))$. Intuitively, given a holomorphic function $f \in \mathcal{O}_p$, as we rescale the metric, we expect to be able to take a limit and obtain homogeneous functions on the tangent cones. But since we are essentially working on a non-compact space, it is possible that the limit we get is always zero, and there is no guarantee that the limit must be homogeneous.

We first need to establish a convexity property. Fix $\lambda \in (0,1)$, and denote by B_i the ball $B(p, \lambda^i)$ being rescaled to have radius one, and then B_i converges to unit balls in the tangent cones.

Proposition 24. *Given any $\underline{d} \notin \mathcal{S}$, there is an $I > 0$, such that if $i \geq I$, and a holomorphic function f defined on B_i satisfies $\|f\|_{L^2(B_{i+1})} \geq \lambda^{\underline{d}} \|f\|_{L^2(B_i)}$, then $\|f\|_{L^2(B_{i+2})} \geq \lambda^{\underline{d}} \|f\|_{L^2(B_{i+1})}$.*

Proof. For simplicity of presentation we only prove this under the assumption that all tangent cones at p have smooth cross sections; the general case requires more technical work [32]. We prove it by contradiction. Suppose the conclusion does not hold, then we can find a subsequence which we still denote by $\{i\}$, a holomorphic function f_i on B_i such that

$$\|f_i\|_{L^2(B_{i+1})} \geq \lambda^{\underline{d}} \|f_i\|_{L^2(B_i)},$$

but

$$\|f_i\|_{L^2(B_{i+2})} \leq \lambda^{\underline{d}} \|f_i\|_{L^2(B_{i+1})}.$$

Multiplying by a constant we may assume $\|f_i\|_{L^2(B_{i+1})} = 1$. By passing to a subsequence we may assume B_i converges to the unit ball B around the vertex in some tangent cone $C(Y)$. We may also assume f_i converges to a limit function f_∞ on B. The convergence is smooth away from the cone vertex and is uniform away from the boundary ∂B. It then follows that

$$\|f_\infty\|_{L^2(B)} \leq \lambda^{-\underline{d}},$$

and

$$\|f_\infty\|_{L^2(\lambda B)} = 1; \quad \|f_\infty\|_{L^2(\lambda^2 B)} \leq \lambda^{\underline{d}}.$$

Now on the cone $C(Y)$ one can use the weight decomposition to see that for any holomorphic function $f \in L^2(B)$,

$$\|f\|_{L^2(B)} \|f\|_{L^2(\lambda^2 B)} \geq \|f\|^2_{L^2(\lambda B)}$$

and equality holds if and only if f is homogeneous. Hence f_∞ is homogenous of degree $\underline{d}$. This contradicts the choice of $\underline{d}$. □

Remark 25. *On a cone $C(Y)$ it is easy to see the function* $\log \|f\|_{L^2(rB)}$ *is a convex function of* $\log r$*. The proposition essentially states that on B_i for i large an* almost convexity *holds.*

Using this and standard interior estimate for holomorphic functions, one can show that for any non-zero function $f \in \mathcal{O}_p$, the following is a well-defined number in $\mathcal{S} \cup \{\infty\}$

$$d_p(f) := \lim_{r \to \infty} \frac{\sup_{B_r} \log |f|}{\log r}$$

We call this the *degree* of f. If $d_p(f) < \infty$, then by similar arguments as in the proof of Proposition 24 one can show that f yields homogeneous functions of degree $d(f)$ on all the tangent cones.

Here are some properties of the degree function.

(1) $d_p(f) < \infty$ for all $f \neq 0$;
(2) $d_p(f) = 0$ if and only if $f(p) \neq 0$;
(3) $d_p(fg) = d_p(f) + d_p(g)$;
(4) $d_p(f+g) \geq \min(d_p(f), d_p(g))$.

Except (4) which follows directly from definition, the other properties are not trivial and indeed depend on the proof of Theorem 26 below. In terms of usual language in algebraic geometry, we can say d_p is a *valuation*.

Conversely, given a homogenous function $\tilde{f}$ on a tangent cone $C(Y)$ of degree d, one can use a local version of the Hörmander construction in the last section to find holomorphic functions f_i on B_i that converges naturally to $\tilde{f}$. Then one can use Proposition 24 to see that for i sufficiently large $d_p(f_i) \leq d$. However, in general strict inequality may be possible–this corresponds to the fact that in the above (4) only an inequality holds so $d(f)$ does not define a *grading* on the ring $\mathcal{O}_p$, but rather a filtration:

$$\mathcal{O}_p = I_0 \supset I_1 \supset I_2 \supset$$

where I_k is the ideal of $\mathcal{O}_p$ consisting of functions f with $d_p(f) \geq d_k$, and we have listed the elements of $\mathcal{S}$ in increasing order as $0 = d_0 < d_1 < d_2 \cdots$. Let $R_p = \bigoplus_k I_k/I_{k+1}$. It inherits a grading by $\mathcal{S}$.

Theorem 26 (Donaldson-Sun [32])**.**

- *There is a unique tangent cone $C(Y)$ at p, as a normal affine algebraic variety endowed with a (weak) Ricci-flat Kähler cone metric;*
- *R_p is finitely generated, and $W = Spec(R_p)$ is a normal affine algebraic cone endowed with the action of $\boldsymbol{T}$, and W can be realized as a weighted tangent cone in $\mathbf{C}^N$ under a complex-analytic embedding of the germ $\mathcal{O}_p$ in $\mathbf{C}^N$ (again, this seems to differ from the well-known notion of weighted tangent cone in the literature where one often considers only rational weight);*
- *There is a flat $\mathbf{C}^*$ equivariant family of affine algebraic cones $\pi : \mathcal{W} \to \mathbf{C}$, such that $\pi^{-1}(t)$ is isomorphic to W for $t \neq 0$ and $\pi^{-1}(0)$ is isomorphic to $C(Y)$, and there is a holomorphic action of $\boldsymbol{T}$ on $\mathcal{W}$, that restricts to the known action on W and $C(Y)$ on each fiber.*

We will not go into the details of proofs of this, but rather describe a geometric (but not completely rigorous) interpretation of the above statements. First we embed all tangent cones $C(Y)$ as affine algebraic cones in a fixed $\mathbf{C}^N$, with a common vertex 0 and the dilation actions on each $C(Y)$ is given by the restriction of a linear map Λ on $\mathbf{C}^N$. Then for i large we can find embeddings $F_i : B_i \to \mathbf{C}^N$ (this is not exactly what is done in [32] due to a technical point) so that p is always mapped to 0, and B_i converges to balls B_∞ in the tangent cones as local complex-analytic subsets of $\mathbf{C}^N$.

Notice there is a natural inclusion map $\Lambda_i : B_{i+1} \to B_i$, which converges to the dilation Λ on the tangent cones. At this point we can not say much on Λ_i since the embedding maps F_i and F_j are not a priori related for $i \neq j$.

The next idea is to simplify these Λ_i's by constructing a good set of functions on a fixed B_{i_0}, and then use certain linear combinations of these functions on each smaller B_i ($i \geq i_0$) to construct F_i. This step depends on what we described above and Proposition 24, but is indeed much more delicate. What we can achieve in the end is that Λ_i becomes linear and commute with Λ. We denote by G_Λ the group of linear transformations of $\mathbf{C}^N$ that commute with Λ, then $\Lambda_i \in G_\Lambda$.

Now we let W_i be the weighted tangent cone of B_i at 0, given by the limit $\lim_{t\to\infty} e^{t\Lambda}.B_i$. Since $\Lambda_i \in G_\Lambda$, it follow easily that $W_i = \Lambda_i.W_{i+1}$. In other words, all the W_i's are in the same G_Λ orbit (we can think of them in the *Hilbert scheme* of all affine algebraic cones in $\mathbf{C}^N$ with the same Reeb vector field). Moreover every tangent cone $C(Y)$ is in the closure of such an orbit.

Now the main result follows from geometric invariant theory, and the key point is that G_Λ is reductive (a generalization of Matsushima theorem that follows from the techniques of [6, 4]).

Theorem 26 gives an algebraic description of $C(Y)$ as a two-step degeneration from the germ $\mathcal{O}_p$, and W only depends on the Kähler-Einstein metric near p. Notice by [20] one can extends the notion of K-stability to affine cones. In terms of this, one expects that $C(Y)$ is K-stable and W is K-semistable. We have the following

Conjecture 27. *The degree function d_p, W, and $C(Y)$ all depend only on the complex-analytic germ $\mathcal{O}_p$, and is independent of the Kähler-Einstein metric. In terms of K-stability, d_p should be uniquely determined so that the corresponding W is K-semistable, and $C(Y)$ is then uniquely determined as the K-stable affine cone such that there is an equivariant degeneration from W to $C(Y)$ in the sense of item 3 in Theorem 26.*

There is an interesting class of examples. For $n \geq 3$ and $k \geq 1$ let X_k^n be the affine variety in $\mathbf{C}^{n+1}$ defined by the equation

$$x_0^{k+1} + x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$

There is a natural algebraic cone structure determined by the $\mathbf{C}^*$ action with weight $(2, k+1, \cdots, k+1)$. It is known [59, 37, 46] that X_k^n admits a compatible Ricci-flat Kähler cone metric if and only if $k+1 < 2\frac{n-1}{n-2}$. Let X_∞^n be the affine variety in $\mathbf{C}^4$ defined by

$$x_1^2 + x_2^2 + \cdots x_n^2 = 0.$$

Then X_∞^n is the product of $X_\infty^{n-1} \times \mathbf{C}$, so admits a Ricci-flat Kähler cone metric, and the cone structure is determined by the $\mathbf{C}^*$ action with weight $(1, 2\frac{n-1}{n-2}, \cdots, 2\frac{n-1}{n-2})$.

It is easy to see that this $\mathbf{C}^*$ action degenerates X_k^n to X_∞^n if and only if $k+1 > 2\frac{n-1}{n-2}$, when X_k^n does not admits a Ricci-flat Kähler cone metric compatible with its own natural algebraic cone structure. In this case we expect both W and $C(Y)$ agree with X_∞^n. This agrees with the uniqueness stated in the above conjecture. In the borderline case $k+1 = 2\frac{n-1}{n-2}$ the two cone structure agrees but there is a degeneration from X_k^n to X_{k+1}^n through a $\mathbf{C}^*$ action that commutes with this action. In this case we expect $W = X_k^n$ but $C(Y) = X_\infty^n$.

Recently Chi Li [43] has proposed a generalization of Martelli-Sparks-Yau volume minimization principle to characterize the above d_p in terms of a valuation with minimal volume, and has proposed an algebraic counterpart of Conjecture 27. One expects that Conjecture 27 is also related to a better understanding of the K-stability (for afffine cones). Notice Collins and Székelyhidi [21] generalized Theorem 12 to K-stable affine algebraic cones, and also described related picture to Conjecture 27.

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