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A note on the collapsing geometry of hyperkähler four manifolds

Dedicated to Professor Lo Yang on the Occasion of His 80th Birthday

Shouhei Honda[1], Song Sun[2,*] & Ruobing Zhang[3]

[1]*Mathematical Institute, Tohoku University, Sendai 980-8578, Japan;*
[2]*Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA;*
[3]*Department of Mathematics, Stony Brook University, Stony Brook, NY 11790, USA*

Email: shonda@m.tohoku.ac.jp, sosun@berkeley.edu, ruobing.zhang@stonybrook.edu

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Abstract We make some observations concerning the one-dimensional collapsing geometry of four-dimensional hyperkähler metrics.

Keywords hyperkähler metrics, collapsing, K3 manifold

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1 Introduction

In this paper, a K3 manifold $\mathcal{K}$ is, by definition, a four-dimensional oriented differentiable manifold diffeomorphic to the underlying differentiable manifold of a complex K3 surface. It is a well-known fact that all complex K3 surfaces are oriented diffeomorphic to each other, so this definition makes sense. Also it is known that $\pi_1(\mathcal{K}) = \{1\}$, and $H^2(\mathcal{K};\mathbb{Z})$ is isomorphic to the lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$, where $U = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ denotes the rank 2 lattice with the symmetric bilinear form given by $(e_i, e_j) = 1 - \delta_{ij}$, and E_8 denotes the negative definite E_8 lattice. In particular, the cup-product on $H^2(\mathcal{K};\mathbb{Z})$ has signature $(3, 19)$.

A hyperkähler metric g on $\mathcal{K}$ is by definition a Riemannian metric with holonomy group equal to Sp(1), which is isomorphic to SU(2). Denote by $\mathcal{M}$ the space of all hyperkähler metrics on $\mathcal{K}$ modulo diffeomorphisms. Given a hyperkähler metric g, we get a 3-dimensional positive definite subspace in $H^2(\mathcal{K},\mathbb{R})$ given by $\mathbb{H}^+_g$, the space of self-dual harmonic 2-forms on $\mathcal{K}$ with respect to g. This defines a period map

$$\mathcal{P} : \mathcal{M} \to \mathbb{R}^+ \times (\Gamma\backslash \mathrm{O}(3,19)/(\mathrm{O}(3)\times \mathrm{O}(19))); \quad g \mapsto (\mathrm{Vol}(g), \mathbb{H}^+_g), \tag{1.1}$$

where Γ is the automorphism group of the lattice $H^2(\mathcal{K};\mathbb{Z})$. The global Torelli theorem states that $\mathcal{P}$ is injective, and is surjective if we include all the possible orbifold degenerations of the hyperkähler metrics.

Recall a hyperkähler metric has vanishing Ricci curvature so convergence theory in Riemannian geometry can be applied to study their degenerations. Given a sequence of hyperkähler metrics g_j on $\mathcal{K}$,

* Corresponding author

we can rescale the metric so that g_j has diameter 1, and then we can rescale the volume measure of g_j to a probability measure $\nu_j = \mathrm{Vol}(\mathcal{K}, g_j)^{-1}\,\mathrm{dvol}_{g_j}$. By passing to a subsequence we may take a *measured Gromov-Hausdorff limit* $(X_\infty, d_\infty, \nu_\infty)$, where d_∞ denotes the limit metric and ν_∞ denotes the (re-normalized) limit probability measure. When the total volume of g_j has a uniform positive lower bound, then we are in the *non-collapsing* situation, and standard theory implies the limit X_∞ is a four-dimensional Riemannian orbifold and these are exactly the objects whose periods lie in the complement of the image of $\mathcal{P}$. Now if the total volume of g_j tends to zero, then we are in the *collapsing* situation and the limit space has Hausdorff dimension smaller than 4. In this case, the second factor of their periods has to diverge to infinity. In reality, the limit can be of dimensions 1, 2, or 3, and all these cases do occur. This can be easily seen by using the Kummer construction: taking a sequence of four-dimensional flat tori collapsing to a flat torus $\mathbb{T}^k$ of dimension $k \in \{1, 2, 3\}$, then applying the Kummer construction to each torus in this sequence, then by taking a diagonal subsequence we obtain a family of hyperkähler metrics on $\mathcal{K}$ that collapse to $\mathbb{T}^k/\mathbb{Z}_2$. Notice that as topological spaces, $\mathbb{T}^1/\mathbb{Z}_2$ is a one-dimensional interval and $\mathbb{T}^2/\mathbb{Z}_2$ is a 2-dimensional sphere. Also topologically all the known collapsing limits are given by these.

Our goal in this note is to make some initial observations towards classifying all the possible degenerations in the collapsing situation. More precisely, we will study the case when the limit space is of Hausdorff dimension 1. It may seem that nothing interesting could be said in this case, but we shall show that indeed one can still obtain new geometric structures and we also ask sensible questions. This has much to do with the fact that we are collapsing hyperkähler metrics rather than just an arbitrary sequence of metrics with Ricci curvature bounded below.

Theorem 1.1. *Let g_j be a sequence of hyperkähler metrics on the $K3$ manifold $\mathcal{K}$ with diameter 1, and let $\nu_j = \mathrm{Vol}(\mathcal{K}, g_j)^{-1}\,\mathrm{dvol}_{g_j}$ be the renormalized volume measure. Suppose they converge under the measured Gromov-Hausdorff topology*

$$(\mathcal{K}, g_j, \nu_j) \xrightarrow{mGH} (X_\infty, d_\infty, \nu_\infty),$$

where (X_∞, d_∞) is the Gromov-Hausdorff limit, and ν_∞ is the renormalized limit measure. Assume that the Hausdorff dimension of (X_∞, d_∞) is strictly less than 2. Then the following properties hold:

(1) *(X_∞, d_∞) is isometric to a closed unit interval $(\mathbb{I}, g_\infty)$ with the standard metric.*

(2) *There is a canonical affine structure on X_∞, i.e., there is a natural choice of coordinate function z on $\mathbb{I}$, unique up to affine transformations of the form $z \mapsto az + b$ for $a, b \in \mathbb{R}$.*

(3) *There are an integer $m \geqslant 0$, points $p_0 < p_1 < \cdots < p_{m+1}$ in $\mathbb{I}$ with $\partial\mathbb{I} = \{p_0, p_{m+1}\}$, and a concave piecewise affine function (with respect to the above affine structure) V on $\mathbb{I}$ which is smooth on $\mathbb{I} \setminus \{p_0, \ldots, p_{m+1}\}$, and positive on $\mathbb{I} \setminus \partial\mathbb{I}$, such that*

$$g_\infty = c_1 V dz^2,$$

and

$$\nu_\infty = c_2 V dz$$

for some constants $c_1, c_2 > 0$.

(4) *The collapsing is with uniformly bounded curvature on compact subsets in $\mathbb{I} \setminus \{p_0, \ldots, p_{m+1}\}$.*

(5) *$(X_\infty, d_\infty, \nu_\infty)$ is an $\mathrm{RCD}(0, \frac{4}{3})$ space. Moreover, $\frac{4}{3}$ is the optimal dimension if V vanishes at least one endpoint of $\mathbb{I}$.*

We make a few remarks regarding the theorem.

Remark 1.2. The main interest for us is the existence of the canonical affine structure. This is analogous to the study of the case of large complex structure limits where one expects to obtain on the limit space an integral affine structure with singularities. We expect the slopes of V to be integral up to scaling (see the discussion in Subsection 2.6), and this is related to more refined understanding of the collapsing geometry, in particular near a singular point p_k.

Remark 1.3. From the RCD geometric point of view, in general the density function of the renormalized limit measure cannot be C^1, as shown by the examples in [24, 35]. In higher-dimensional case of complex structure collapsing of Calabi-Yau manifolds, one can also get the interval limit where the optimal RCD dimension is $\frac{2n}{n+1}$ (see Proposition 3.8). These provide concrete examples of Ricci limit spaces for studying RCD geometry.

2 Proof of the theorem

2.1 Gromov-Hausdorff limits

Item (1) of Theorem 1.1 is an easy consequence of the following lemma.

Lemma 2.1. *Let (M^n_j, g_j) be a sequence of closed Riemannian manifolds with*

$$\mathrm{Ric}_{g_j} \geqslant -(n-1), \quad \mathrm{diam}_{g_j}(M^n_j) \leqslant D,$$

and $|\pi_1(M^n_j)| < \infty$ such that

$$(M^n_j, g_j) \xrightarrow{GH} (X_\infty, d_\infty).$$

Assume that $0 < \dim_{\mathcal{H}}(X_\infty) < 2$. Then X_∞ is isometric to a closed interval in $\mathbb{R}$.

Thanks to Theorem 3.2 in Section 3 it suffices to prove that X_∞ is not isometric to a one-dimensional circle. This then follows directly from a result of Sormani and Wei [32]. Indeed, by [32, Theorem 1.1] there exists a surjective homomorphism

$$\rho_* : \pi_1(M^n_j) \to \pi_1(X_\infty)$$

for all sufficiently large j. In particular, X_∞ must have finite fundamental group, so cannot be a circle.

By Cheeger-Colding's harmonic splitting map, we will be able to obtain a direct proof. Since the construction used in this proof will be applied in our later discussion, we would like to provide this alternative proof of Lemma 2.1 below.

Proof of Lemma 2.1. Again by Theorem 3.2 it suffices to rule out the limit S^1. We will argue by contradiction and suppose that we have the Gromov-Hausdorff convergence

$$(M^n_j, g_j) \xrightarrow{GH} (S^1, d_\infty).$$

In the first step, we will construct a continuous Gromov-Hausdorff approximation

$$\Phi_j : M^n_j \to S^1.$$

Fix a small positive constant $0 < \delta \ll \mathrm{diam}(S^1)$. Let $\{z_\alpha\}_{\alpha=1}^N \subset S^1$ be a $\frac{\delta}{2}$-dense subset satisfying

$$d_\infty(z_\alpha, z_\beta) \geqslant \frac{\delta}{4}, \quad 1 \leqslant \alpha, \beta \leqslant N.$$

By assumption, let j be sufficiently large such that

$$d_{GH}(M^n_j, S^1) < \epsilon_j \leqslant \frac{\delta^2}{100}$$

and there exists a $3\epsilon_j$-Gromov-Hausdorff approximation $F_j : M^n_j \to S^1$. So correspondingly we choose $\{y_\alpha\}_{\alpha=1}^N$ with $F(y_\alpha) = z_\alpha$ such that $\{B_\delta(y_\alpha)\}_{\alpha=1}^N$ gives a covering of M^n_j. By Cheeger and Colding [9], for each $1 \leqslant \alpha \leqslant N$, there exists a harmonic splitting map

$$\Phi_{j,\alpha} : B_\delta(y_\alpha) \to B_\delta(z_\alpha)$$

which is a $\delta \cdot \tau(\delta|D)$-Gromov-Hausdorff approximation. Here $\lim_{\delta\to 0} \tau(\delta|D) = 0$. Next, it is rather standard that the above harmonic maps can be patched together and gives a global Gromov-Hausdorff map. In fact, consider the partition of unity yielding to the cover $\{B_\delta(y_\alpha)\}_{\alpha=1}^N$:

$$1 \equiv \sum_{\alpha=1}^N \Psi_\alpha(x), \quad x \in M^n_j, \quad B_{\delta/2}(y_\alpha) \subset \mathrm{Supp}(\Psi_\alpha) \subset B_\delta(y_\alpha)$$

and the following estimate holds:

$$\delta^2 \cdot (|\nabla \Psi_\alpha| + |\Delta \Psi_\alpha|) \leqslant C_0,$$

where $C_0 > 0$ is independent of δ. Now we define

$$\Phi_j \equiv \sum_{\alpha=1}^{N} \Psi_\alpha \cdot \Phi_{j,\alpha}.$$

So it is straightforward to see that $\Phi_j : M_j^n \to S^1$ is a $\delta \cdot \tau(\delta|D)$-Gromov-Hausdorff approximation. Here the summation notation is understood in terms of a choice of identification S^1 with $\mathbb{R}/\mathbb{Z}$.

In the next step, we will see that contradiction arises from the topological assumption on M_j^n. Notice that $|\pi_1(M_j^n)| < \infty$ and $\pi_1(S^1) \cong \mathbb{Z}$ is torsion-free. So the universal covering map $\pi : \mathbb{R} \to S^1$ gives a lifting $\widetilde{\Phi}_j : M_j^n \to \mathbb{R}$ of Φ_j such that $\Phi_j = \pi \circ \widetilde{\Phi}_j$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ M_j^n \arrow[ur, "\widetilde{\Phi}_j"] \arrow[r, "\Phi_j"'] & S^1. \end{tikzcd}$$

Therefore, $\widetilde{\Phi}_j$ is a continuous function on M_j^n. So we suppose that $\widetilde{\Phi}_j$ achieves its maximum $\tilde{y}_0 \in \mathbb{R}$ with $y_0 \equiv \pi(\tilde{y}_0) \in S^1$. Let

$$B_\delta(y_0) \equiv \pi([\tilde{y}_0 - \delta, \tilde{y}_0 + \delta]) \subset S^1$$

and denote by

$$B_\delta^+(y_0) \equiv \pi((\tilde{y}_0, \tilde{y}_0 + \delta])$$

the projection image of the half interval $(\tilde{y}_0, \tilde{y}_0 + \delta]$. Since $\tilde{\Phi}_j \leqslant \tilde{y}_0$ on M_j^n, it follows that

$$\Phi_j(M_j^n) \subset S^1 \setminus B_\delta^+(y_0).$$

But the contradiction arises from the fact that Φ_j is an ϵ-Gromov-Hausdorff map, where $\epsilon \equiv \delta \cdot \tau(\delta|D)$ and $\lim_{\delta \to 0} \tau(\delta|D) = 0$. □

Applying the Chern-Gauss-Bonnet theorem (see [6] for example), for any Einstein metrics on the K3 manifold $\mathcal{K}$, we have

$$\frac{1}{8\pi^2} \int_{\mathcal{K}} |\operatorname{Rm}_{g_j}|^2 \operatorname{dvol}_{g_j} = \chi(\mathcal{K}) = 24.$$

Then by Cheeger-Tian ϵ-regularity theorem in [13] we know that there are finitely many points

$$p_0 < \cdots < p_{m+1} \in \mathbb{I},$$

with p_0, p_{m+1} the endpoints, such that away from this set the collapsing is with uniformly bounded curvatures. Hence by Fukaya's theorem [19] the regular region is collapsing along an infranil fibration.

We denote

$$\mathcal{S} \equiv \{p_0, p_1, \ldots, p_{m+1}\}$$

and

$$\partial \mathbb{I} \equiv \{p_0, p_{m+1}\}.$$

The rest of this section is devoted to the proof of Items (2) and (3) of Theorem 1.1.

2.2 Collapsing and Nil geometry in dimension 3

For the convenience of our discussion, we recall some basic facts regarding the nilpotent Lie groups in dimension 3.

By definition, a Heisenberg algebra $\mathfrak{h}$ is the Lie algebra generated by $\{\zeta_1, \zeta_2, \zeta_3\}$ with

$$[\zeta_1, \zeta_2] = \zeta_3, \quad [\zeta_1, \zeta_3] = [\zeta_2, \zeta_3] = 0.$$

More explicitly, a Heisenberg Lie algebra $\mathfrak{h}$ is the Lie algebra of matrices of the form

$$\mathfrak{h} \equiv \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

such that

$$\zeta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \zeta_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is well known that a 3-dimensional real nilpotent Lie algebra is either the Heisenberg algebra $\mathfrak{h}$ as the above or the abelian Lie algebra. Similarly, any connected simply connected 3-dimensional nilpotent Lie group N^3 is isomorphic to either the Heisenberg group $\mathbb{H}$ or the abelian Lie group $\mathbb{R}^3$, where

$$\mathbb{H} \equiv \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\}.$$

So the Heisenberg group $\mathbb{H}$ is a 2-step nilpotent group and it fits into a nontrivial extension

$$1 \to Z(\mathbb{H}) \to \mathbb{H} \to \mathbb{H}/Z(\mathbb{H}) \to 1, \tag{2.1}$$

where the center $Z(\mathbb{H}) = [\mathbb{H}, \mathbb{H}] \cong \mathbb{R}$ and $\mathbb{H}/Z(\mathbb{H}) \cong \mathbb{R}^2$. Moreover, the Lie algebra of $\mathbb{H}$ is $\mathfrak{h}$.

Any co-compact lattice $\Gamma \leqslant \mathbb{H}$ in a 3-dimensional nilpotent Lie group is generated by three elements e_1, e_2, e_3 with $e_1 \in Z(\mathbb{H})$, $[e_2, e_3] = e_1^k$ for some $k \in \mathbb{Z} \setminus \{0\}$. Explicitly in terms of unipotent upper triangular matrices, Γ is (abstractly) isomorphic to the subgroup consisting of all matrices of the form

$$\begin{pmatrix} 1 & m & \frac{p}{k} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

with $m, n, p \in \mathbb{Z}$. Denote such a lattice by Γ_k and notice that Γ_k and Γ_{-k} are isomorphic. This leads to the definition of a nilmanifold and infra-nilmanifold. Notice that these definitions work in general dimensions, but for our purpose and for simplicity, we are only working with the 3-dimensional case.

Definition 2.2 (Nilpotent manifold). A 3-dimensional nilmanifold $\mathrm{Nil}_k \equiv \Gamma_k \backslash N^3$ of degree $k \neq 0$, is given by the quotient of left action of some co-compact lattice Γ_k on N^3. In particular, it is an S^1-bundle over $\mathbb{T}^2$ of degree $\pm k$ (depending on the orientation). We also make the convention that Nil_0 denotes a 3-dimensional torus, given by the quotient of $\mathbb{R}^3$ by a lattice $\mathbb{Z}^3$.

Definition 2.3 (Infranilmanifold). An infranilmanifold $M^3 \equiv \Gamma \backslash N^3$ is a finite quotient of some nilmanifold $\Gamma_k \backslash N^3$ such that its fundamental group Γ is a finite extension of $\Gamma_k \leqslant N^3$ satisfying the exact sequence

$$1 \to \Gamma_k \to \Gamma \to Q \to 1,$$

where Q is finite and $\Gamma \leqslant \mathrm{Aut}(N^3) \ltimes N^3$.

Now let $N^3 \equiv \mathbb{H}$ be the 3-dimensional Heisenberg group. Then its automorphism group $\mathrm{Aut}(N^3)$ is isomorphic to the semi-direct product $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. The action of $\mathbb{R}^2$ corresponds to inner automorphisms given by conjugations, and the action of $\mathrm{GL}(2,\mathbb{R})$ descends to the natural action on the quotient $\mathbb{R}^2$. The semi-direct product $\mathrm{Aut}(N^3) \ltimes N^3$ can be understood in an alternative way. In fact, there is a canonical connection ∇^{can} on N^3, called the affine connection, such that all left invariant vector fields on N^3 are parallel. Denote by $\mathrm{Aff}(N^3, \nabla^{\mathrm{can}})$ the group of all affine transformations of $(N^3, \nabla^{\mathrm{can}})$. Then it follows that

$$\mathrm{Aut}(N^3) \ltimes N^3 \cong \mathrm{Aff}(N^3, \nabla^{\mathrm{can}}).$$

At the next stage, we will review some facts about the Riemannian geometry of $\mathbb{H}$. Let g be a left invariant metric on $\mathbb{H}$. Then g is determined by an inner product on $\mathfrak{h} \cong T_e N^3$. Notice that one can explicitly write down a basis of left invariant 1-forms

$$dx, \quad dy, \quad \theta \equiv dt - x dy.$$

Therefore, a left-invariant metric g is of the form

$$g = \epsilon_1 dx^2 + \epsilon_2 dy^2 + \epsilon_3 (dt - x dy)^2.$$

Suppose $\epsilon_1 = \epsilon_2 = \epsilon_3$. Then we get the so-called *3D Nil geometry*. The isometry group is given by $\mathrm{O}(2) \ltimes N^3$. This is a subgroup of the group of affine transformations, $\mathrm{Aut}(N^3) \ltimes N^3$.

Notice that given any infranilmanifold M^3, one can construct a family of metrics g_ϵ satisfying

$$|\sec_{g_\epsilon}| \leqslant 1, \quad \mathrm{diam}_{g_\epsilon}(M^3) \leqslant \epsilon \to 0,$$

which arises from *inhomogeneous rescaling* of left-invariant metrics on the universal cover N^3.

2.3 Hyperkähler metric with Heisenberg symmetries

Suppose (Ω, g) is a simply connected four-dimensional hyperkähler manifold. Then we have a triple of self-dual symplectic 2-forms $\omega_1, \omega_2, \omega_3$ with

$$\begin{cases} d\omega_\alpha = 0, \\ \dfrac{1}{2}\omega_\alpha \wedge \omega_\beta = \delta_{\alpha\beta} \, \mathrm{dvol}_g \, . \end{cases} \tag{2.2}$$

Each ω_α defines a parallel complex structure I_α such that $g(I_\alpha \cdot, \cdot) = \omega_\alpha(\cdot, \cdot)$. There is a natural ordering of ω_α's such that $I_1 I_2 = I_3$. Under this ordering, the above triple is unique up to the obvious $\mathrm{SO}(3)$ hyperkähler rotation.

Now suppose Ω is endowed with a free action of the Heisenberg group $\mathbb{H}$ which preserves the hyperkähler structure. This means the action preserves each ω_α. Notice this is a stronger assumption than the action being isometric.

We identify the Lie algebra $\mathfrak{h} = \mathrm{Lie}(\mathbb{H})$ with the space of Killing fields on Ω it generates. Choosing a generator of the center $\mathrm{Lie}(Z(\mathbb{H})) \simeq \mathbb{R}$, which can be viewed as a Killing field ∂_t on Ω. Since Ω is simply connected, we can find the hyperkähler moment map

$$(x_1, x_2, x_3) : \Omega \to \mathbb{R}$$

with respect to the action of ∂_t. They satisfy

$$dx_\alpha = -\partial_t \lrcorner \omega_\alpha.$$

Denote the function $V = |\partial_t|^{-2}$. Then we can write

$$\begin{cases} \omega_1 = V dx_2 \wedge dx_3 + dx_1 \wedge \theta, \\ \omega_2 = V dx_3 \wedge dx_1 + dx_2 \wedge \theta, \\ \omega_3 = V dx_1 \wedge dx_2 + dx_3 \wedge \theta, \end{cases} \tag{2.3}$$

and

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\theta^2,$$

where θ is a 1-form with $\theta(\partial_t) = 1$, and $d\theta = *dV$ is viewed as 2-forms on the local quotient $\mathbb{R}^3_{x_1,x_2,x_3}$ with the Euclidean metric. In particular $V > 0$ is a harmonic function on $\mathbb{R}^3$. This is the classical *Gibbons-Hawking construction* for hyperkähler 4 manifolds with a continuous symmetry. Now in our setting by definition any $\zeta \in \mathrm{Lie}(\mathbb{H})$ generates a Killing field which commutes with ∂_t, hence we see V and θ are both invariant under the local $\mathbb{H}$ action, so V descends to a one variable function on the local quotient $\Omega/\mathbb{H}$.

Notice the choice of x'_α a priori is only unique up to an SO(3) rotation, depending on the choice of basis $\{\omega_1, \omega_2, \omega_3\}$ satisfying (2.2). Given any $\zeta \in \mathrm{Lie}(\mathbb{H})$, since $[\zeta, \partial_t] = 0$, we have

$$d(\mathcal{L}_\zeta x_\alpha) = \mathcal{L}_\zeta(dx_\alpha) = -L_\zeta(\partial_t \lrcorner \omega_\alpha) = 0,$$

which implies $\mathcal{L}_\zeta x_\alpha$ is constant for all ζ and α. We may rotate $\{\omega_1, \omega_2, \omega_3\}$ by an element in $SO(3)$, and assume $\mathcal{L}_\zeta x_3$ is zero at a given point in Ω for all ζ. Then it follows that the same is true at all points. Since the local action of $\mathbb{H}$ is free, this choice of coordinate x_3 is unique up to scaling, and this freedom is due to the choice of ∂_t and a $\pm$ sign. The coordinate function x_3, up to affine transformations of the form $ax_3 + b$, can also be characterized as the function on Ω which is $\mathbb{H}$-invariant and harmonic. In particular, we get a local affine structure on the quotient.

We then have

$$V = \lambda x_3 + \mu$$

for some λ and μ. The quotient metric is

$$g = V dx_3^2.$$

Up to gauge transform we may assume θ is of the form

$$\theta = dt - \lambda x_2 dx_1.$$

Then it is not difficult to see that Lie($\mathbb{H}$) corresponds to Killing fields spanned by $\partial_t,\ \partial_{x_1} + \lambda x_2 \partial_t,\ \partial_{x_2}$.

2.4 Collapsing geometry in the regular region

Let $\mathcal{K}$ be the K3 manifold. In our setting, by Theorem 2.1, there is a Gromov-Hausdorff convergence

$$(\mathcal{K}, g_j) \xrightarrow{GH} (\mathbb{I}, dt^2).$$

For simplicity we shall denote by X_j the hyperkähler manifold $(\mathcal{K}, g_j)$.

Suppose $p \in X_\infty \setminus \mathcal{S}$. Then there is a neighborhood $\mathbb{I}'$ of p, such that $\mathbb{I}'$ is the Gromov-Hausdorff limit of the open neighborhoods $U_j \subset X_j$ with uniformly bounded curvature. Let $\tilde{U}_j \to U_j$ be the universal cover, and Γ_j be the deck transformation group. Then by the result of Fukaya [20] (see also [21]) we know by passing to a subsequence $(\tilde{U}_j, \tilde{g}_j, \Gamma_j)$ converges to a limit $(\tilde{U}_\infty, \tilde{g}_\infty, G_\infty)$, in the equivariant Gromov-Hausdorff topology. Here G_∞ is a 3-dimensional nilpotent Lie group acting freely on $\tilde{U}_\infty$. Moreover, it is a finite extension of its identity component G^0_∞, which is a connected and simply connected 3-dimensional nilpotent Lie group. So G^0_∞ must be isomorphic to either the Heisenberg group $\mathbb{H}$ or the abelian group $\mathbb{R}^3$. Any G_∞ orbit is diffeomorphic to G^0_∞.

Since each X_j is hyperkähler, by passing to a further subsequence we may assume the self-dual 2-forms on $\tilde{U}_j$ have limits on $\tilde{U}_\infty$, hence the limit $\tilde{U}_\infty$ is hyperkähler and the action of G_∞ preserves the hyperkähler structure. In this case we claim that G_∞ must be connected. To see this, suppose first $G^0_\infty = \mathbb{H}$. Then we are in the setting of Subsection 2.3, and we have a global hyperkähler moment map whose image is given by $\mathbb{R}^2_{x_1,x_2} \times \mathbb{I}'_{x_3}$, and the fiber is $\mathbb{R}_t$. Taking any element $\phi \in G_\infty$, since $\phi^* dx_3 = dx_3$, we see

$$\phi^* dx_1 = dx_1, \quad \phi^* dx_2 = dx_2, \quad \phi^* \theta = \theta.$$

In particular,

$$\phi^* x_1 = x_1 + c_1, \quad \phi^* x_2 = x_2 + c_2, \quad \phi^* t = t + \lambda c_2 x_1 + c_3$$

for constants c_1, c_2, c_3, hence ϕ is given by translation of an element in $\mathbb{H} = G_\infty^0$. Similarly, if $G_\infty^0 = \mathbb{R}^3$, then the limit metric is flat and we also have a similar expression of the self-dual 2-forms as in (2.3) with V being a constant. This proves the claim.

By [19] we know for i large, there is a smooth fibration $\pi_i : U_i \to \mathbb{I}'$ whose fiber is almost flat and hence a 3-dimensional infranilmanifold. For simplicity of notation we denote by z the coordinate function x_3 on $\mathbb{I}'$, and view it as the projection function $z = \pi_\infty : \tilde{U}_\infty \to \mathbb{I}'$.

Lemma 2.4. *The re-normalized limit measure on $\mathbb{I}'$ is given by $\nu_\infty = cV dz$ for some constant $c > 0$.*

Proof. Since we are collapsing with uniformly bounded curvatures, we may construct the above fibration maps $\pi_j : U_j \to \mathbb{I}'$ such that the lifting $\tilde{\pi}_j : \tilde{U}_j \to \mathbb{I}'$ converges smoothly to the function x_3 under the equivariant convergence of $\tilde{U}_j$ to $\tilde{U}_\infty$. This can be done quickly, for example, again by using Cheeger-Colding theory [9]. One can construct harmonic functions $\pi_j : U_j \to \mathbb{I}'$ by solving a Dirichlet problem and obtain smooth convergence on the lifting to $\tilde{U}_j$ since we have uniformly geometry bound.

For each j, we denote $F_j(z) = \pi_j^{-1}(z)$, and let H_j be the mean curvature function on $F_j(z)$ in X_j. We then know that H_j uniformly converges to the function H, which is the mean curvature of $\pi_\infty^{-1}(z)$. It is straightforward to compute

$$H = \frac{1}{2} V^{-3/2} \partial_z V,$$

which depends only on z. By definition we have

$$\frac{d}{dz} \log \mathrm{Vol}(F_j(z)) = \frac{1}{\mathrm{Vol}(F_j(z))} \int_{F_j(z)} \frac{H_j}{|\nabla \pi_j|}. \tag{2.4}$$

By our convergence we know H_j uniformly converges to H, and $|\nabla \pi_j|$ uniformly converges to $|\nabla z| = V^{-1/2}$. So we see the right-hand side of (2.4) converges to $\frac{1}{2} V^{-1} \partial_z V$. Choosing a base point we then see that there exist constants $C_j \to \infty$ such that

$$\lim_{j \to \infty} C_j \cdot \mathrm{Vol}(F_j(z)) = V^{1/2}.$$

Now by co-area formula

$$\mathrm{Vol}(z_1 \leqslant z \leqslant z_2) = \int_{z_1}^{z_2} dz \int_{F_j(z)} \frac{1}{|\nabla \pi_j|}.$$

So

$$\lim_{j \to \infty} C_j \mathrm{Vol}(z_1 \leqslant z \leqslant z_2) = \int_{z_1}^{z_2} V dz.$$

This then gives the desired formula. □

Recall the Bakry-Émery Laplacian on the corresponding weighted Riemannian manifold $(\mathbb{I}', dt^2, \nu_\infty)$ is defined to be

$$\Delta_{BE} \equiv \Delta + \frac{1}{2} \nabla (\log V) \cdot \nabla.$$

It is easy to see that for a function f on the quotient, we have

$$\Delta_{BE} f = V^{-1} \frac{\partial^2 f}{\partial z^2}.$$

So f is harmonic in the Bakry-Emery sense if and only if f is an affine function in z.

2.5 Global affine structure

Fixing a point $q \in \mathcal{S} \setminus \partial \mathbb{I}$, we choose $\delta > 0$ such that the ball $B = B_\delta(q)$ satisfies $B \cap \mathcal{S} = \{q\}$. We may assume B is the limit of balls $B_j \equiv B_\delta(q_j)$ for $q_j \in X_j$.

Lemma 2.5. *By choosing $\delta > 0$ even smaller if necessary, for j sufficiently large, there is a harmonic function z_j on B_j, and a non-constant Lipschitz function z on B, such that $|\nabla z_j| \leqslant C$ for some constant $C > 0$ and z_j converges uniformly to z.*

Proof. This is a simple application of the construction of Cheeger and Colding [9]. Choose a fixed number ϵ such that $B_\epsilon(q) \cap \mathcal{S} = \{q\}$. Suppose $\partial B_\epsilon(q) = \{q', q''\}$, and choose a sequence of points $q_j', q_j'' \in X_j$ that converge to q', q'' correspondingly. Since the tangent cone of $\mathbb{I}$ at q is $\mathbb{R}$, by [9], for $\delta > 0$ sufficiently small, for j sufficiently large we can find a harmonic function z_j on $B_{2\delta}(q_j)$ such that

$$|z_j - (d_j(q_j', \cdot) - d_j(q_j', q_j))| \leqslant \frac{\delta}{10}.$$

For j large we then get that

$$\sup_{B_\delta(q_j)} z_j - \inf_{B_\delta(q_j)} z_j \in [\delta/10, 10\delta],$$

and by the Cheng-Yau gradient estimate we also have

$$\sup_{B_{3\delta/2}(q_j)} |\nabla z_j| \leqslant C.$$

The conclusion then follows. □

Lemma 2.6. *z is an affine function on $B \setminus \{q\}$ with respect to the above affine structures.*

Proof. Take any $q' \in B \setminus \{q\}$, and $q_j' \in X_j$ converging to q'. Then by the discussion in Subsection 2.4 we can find a neighborhood U_j of q_j', such that its universal cover $\tilde{U}_j$ converges to a smooth limit $\tilde{U}_\infty$ with the action of a 3-dimensional nilpotent group $\mathbb{H}$ or $\mathbb{R}^3$. Then the lift of the above harmonic functions z_j converge smoothly to an invariant harmonic function w on $\tilde{U}_\infty$. In particular we know w is an affine function in the quotient $\tilde{U}/\mathbb{H}$. On the other hand, naturally w descends to the function z. □

Remark 2.7. Since a function f on $B \setminus \{q\}$ is affine if and only if it is harmonic in the Bakry-Emery sense, the lemma also follows from more general statement [17, Lemma 3.17] (or [26, Theorem 1.3], [2, Corollary 4.5]).

By the discussion in Subsection 2.4 we know that on $B \setminus \{q\}$, z is a well-defined smooth coordinate function and we have

$$|\nabla z|^{-2} = V$$

is a positive affine function in z. It then follows that V is bounded above on $B \setminus \{q\}$; also V^{-1} is bounded above by Lemma 2.5. We may write the limit Riemannian metric as

$$g_\infty = V dz^2$$

on $B \setminus \{q\}$. Also the re-normalized limit measure can be written as

$$\nu_\infty = c_\infty V dz$$

for some $c_\infty > 0$ which is constant on the two connected components of $B \setminus \{q\}$. Notice priori it is not obvious that c_∞ is a constant. By [11, Theorem 4.15], we know the limit measure ν_∞ is absolutely continuous with respect to the Lebesgue measure on B. So we see that on B, ν_∞ has positive and bounded density the with respect to the volume measure of g_∞.

Lemma 2.8. *V extends to a continuous function on B.*

Proof. Let z_j be the sequence of harmonic functions constructed in Lemma 2.5. Since $\Delta z_j = 0$, by Weitzenböck formula we get

$$\Delta |\nabla z_j|^2 = 2|\nabla^2 z_j|^2 \geqslant 0.$$

By [9, Theorem 6.33] we have a good cut-off function $\chi_j : B_j \to [0, 1]$, which is compactly supported, with $\chi_j = 1$ on $B_{\delta/2}(q_j)$ and

$$|\nabla \chi_j| + |\Delta \chi_j| \leqslant C$$

for a constant $C > 0$ independent of j. Then we get

$$\int_{B_{\delta/2}(q_j)} |\nabla|\nabla z_j||^2 \leqslant C \int_{B_\delta(q_j)} \chi_j |\nabla^2 z_j|^2 = C \int_{B_\delta(q_j)} \Delta\chi_j \cdot |\nabla z_j|^2 \leqslant C\mathrm{Vol}(B_\delta(q_j)).$$

Letting $j \to \infty$ with [2, Theorems 4.2 and 4.6] it follows that

$$\int_{B_{\delta/2}(q)\setminus\{q\}} |\nabla|\nabla z||^2 \nu_\infty < \infty,$$

hence we get

$$\int_{B_{\delta/2}(q)} |\partial_z V^{-1/2}|^2 dz < \infty.$$

In particular we see $V^{-1/2}$ (hence V) extends continuously across q. □

Since $|\nabla z|^2 = V^{-1}$ is positive on B, it follows that z gives a C^1 coordinate with respect to the arc-length coordinate on B. Using the RCD theory explained in Section 3 (see Theorem 3.2) we know ν_∞ has continuous density function with respect to the Lebesgue measure because the limit metric measure space is an $\mathrm{RCD}(0,4)$ space (which comes from a fact that $\mathrm{RCD}(K,N)$-spaces are closed with respect to mGH-convergence) , so the continuity of V also implies that c_∞ has to be a constant on B.

Corollary 2.9. *V is a concave piecewise affine function in z.*

Proof. As in the proof of Lemma 2.8, applying Weitzenböck formula again with good cut-off functions constructed in [9] we see that local $H^{2,2}$-norms of z_j are uniformly bounded. Note that for any non-negatively valued Lipschitz function ϕ on B with compact support, there exists a sequence of non-negatively valued Lipschitz functions ϕ_j on B_j with compact supports such that ϕ_j converge to ϕ in the $H^{1,2}$-sense (see for example the proof of [2, Theorem 4.4]). Then since

$$\int_{B_j} \langle \nabla|\nabla z_j|^2, \nabla\phi_j\rangle \leqslant 0,$$

taking the limit $j \to \infty$ with [2, Theorem 4.6 and Corollary 4.3] (see also [26, Theorem 4.9]) yields

$$\int_B \langle \nabla|\nabla z|^2, \nabla\phi\rangle \nu_\infty \leqslant 0.$$

Notice by definition

$$\langle \nabla|\nabla z|^2, \nabla\phi\rangle \nu_\infty = \partial_z(V^{-1})\partial_z\phi|\nabla z|^2 \cdot c_\infty V dz = c_\infty \partial_z(V^{-1})\partial_z\phi dz.$$

It follows that in z coordinate V^{-1} is a convex function. Since V is piecewise affine, this is equivalent to saying that V is concave. □

Similarly we also have the following property.

Corollary 2.10. *A continuous function f on B is harmonic* (*in the weak sense*) *if and only if it is affine in z.*

Proof. Suppose f is harmonic. Then on $B \setminus \{q\}$ we know f is smooth and affine in z. Now f being harmonic in the weak sense means that

$$\int_B \langle \nabla f, \nabla\phi\rangle \nu_\infty = 0$$

for all Lipschitz function ϕ on B with compact support. So it follows that f must extend smoothly across q, hence is a global affine function on B. The converse direction is easy to see similarly. □

Now we can define a canonical affine structure on the limit space $\mathbb{I}$. Over each interval (p_k, p_{k+1}) we use the natural affine coordinate x_3 coming from Subsection 2.4, which is unique up to affine transformations. The above discussion implies that using the local harmonic function z in a neighborhood

p_k ($k \in \{2, \dots, m\}$) we can naturally glue the two affine structures on the two intervals (p_{k-1}, p_k), and obtain a global affine structure. Intrinsically speaking, the affine structure can be defined in terms of the sheaf of harmonic functions in the interior of $\mathbb{I}$. This proves Items (2) and (3) of Theorem 1.1.

Notice however in a neighborhood $B_\delta(q)$ of a boundary point q in $\mathbb{I}$, there is no non-constant harmonic functions. Indeed, suppose the affine coordinate is given by z. Then a harmonic function $f(z)$ in $B_\delta(q)\backslash\{q\}$ is of the form $az+b$ for constants a and b. Now being harmonic on $B_\delta(q)$,

$$\int f'(z)\psi'(z)dz = 0$$

for all $\psi \in C_0^\infty(B_\delta(q))$. This forces $a = 0$.

2.6 Some remarks

We finish this section with some remarks. By the gluing construction in [14, 24, 35] we have examples of collapsing limit of hyperkähler metrics on K3 manifolds such that V is either a constant, or V is (up to constant multiple) of the form on $[0, 1]$,

$$V(z) = \begin{cases} k_1 z, & z \in \left[0, \dfrac{k_2}{k_2 - k_1}\right], \\ k_2 z - k_2, & z \in \left[\dfrac{k_2}{k_2 - k_1}, 1\right], \end{cases}$$

for integers $-9 \leqslant k_2 < k_1 \leqslant 9$. It is expected that the construction in [24] can be generalized to yield limit spaces with V given by an arbitrary piecewise affine function on $[0, 1]$ with $V(0) = V(1) = 0$, and each slope of V is an integer in $[-9, 9]$. Motivated by these it makes sense to ask the following questions which we leave for future exploration.

• In the case of collapsing hyperkähler metrics on K3 manifolds, can we show that if V is not locally constant near a boundary point $p \in \partial\mathbb{I}$, then $V(p) = 0$?

This is certainly compatible with the known gluing constructions. Notice this is not the case if we consider general Ricci limit space. As an example, we consider the boundary of the ϵ-neighborhood Σ_ϵ of $\{(x_1, 0, 0); 0 \leqslant x_1 \leqslant 1\}$ in $\mathbb{R}^3$, which is homeomorphic to $\mathbb{S}^2$. Since the induced Riemannian metric g_ϵ on Σ_ϵ is not smooth, taking a suitable smoothing of g_ϵ (the smoothed part is included in a small neighborhood of $\{(0, x_2, x_3); x_2^2 + x_3^2 = 1\} \cup \{(1, x_2, x_3); x_2^2 + x_3^2 = 1\}$) yields that there exists a smooth Riemannian metric $\hat{g}_\epsilon$ on $\mathbb{S}^2$ with non-negative sectional curvature such that $(\mathbb{S}^2, \hat{g}_\epsilon, \mathrm{vol}_{\hat{g}_\epsilon}/\mathrm{vol}_{\hat{g}_\epsilon}\mathbb{S}^2)$ mGH-converge to $([0, 1], d_{[0,1]}, \mathcal{L}^1)$ as $\epsilon \to 0^+$, where this convergence can be checked by considering the canonical projection π_ϵ from Σ_ϵ to $[0, 1]$. In particular, we see that $(\mathbb{S}^2, \hat{g}_\epsilon, \mathrm{e}^{-x_1}\mathrm{vol}_{\hat{g}_\epsilon}/\mathrm{vol}_{\hat{g}_\epsilon}\mathbb{S}^2)$ mGH-converge to $([0, 1], d_{[0,1]}, \mathrm{e}^{-x}\mathcal{L}^1)$. On the other hand by [29, Theorem 3] $(\mathbb{S}^2, \hat{g}_\epsilon, \mathrm{e}^{-x_1}\mathrm{vol}_{\hat{g}_\epsilon}/\mathrm{vol}_{\hat{g}_\epsilon}\mathbb{S}^2)$ is a Ricci limit space because the (finite-dimensional) Bakry-Émery Ricci tensor has a definite lower bound. Therefore $([0, 1], d_{[0,1]}, \mathrm{e}^{-x}\mathcal{L}^1)$ is also a Ricci limit space.

• Is it true that V must be singular at a point in $\mathcal{S}$? Again this is the case for examples from the gluing constructions and if the answer to this question is yes, then it shows that the singular point of the collapsing (not the singular point of the limit space) can be detected from the singularity of the renormalized limit measure.

• Over each interval (p_k, p_{k+1}) the function V is affine, but its slope s_k is not well-defined since the coordinate function z is only well-defined up to affine transformations. However since we have a global affine structure on $\mathbb{I}$, given two intervals (p_k, p_{k+1}) and (p_l, p_{l+1}), the ratio $s_k/s_l \in [0, \infty]$ is well-defined, and one can ask for the intrinsic meaning of this number in terms of the collapsing geometry. From the known gluing construction, it is natural to expect that when this number is in $(0, \infty)$, it should be rational and coincides with the degree of the corresponding collapsing nilmanifold fibers over each interval. Also the fact that the limiting group G_∞ in Subsection 2.4 is connected also indicates that the collapsing fiber in the regular region is likely to be a nilmanifold instead of a general infranilmanifold.

3 Optimal dimension as RCD space

A triple (X, d, m) is said to be a *metric measure space* if (X, d) is a complete separable metric space and m is a Borel measure on X with full support. Throughout this section we assume that all metric measure space (X, d, m) is not trivial, i.e., X is not a single point.

Lott and Villani [30] and Sturm [33,34] introduced the notion of *Ricci bounds from below by* $K \in \mathbb{R}$ *and dimension bounds from above by* $N \in [1, \infty]$ for a metric measure space (X, d, m), so-called $\mathrm{CD}(K, N)$ *space*, independently. It is worth pointing out that N is not necessary to be an integer. Moreover they proved fundamental functional inequalities/comparison theorems including Bishop-Gromov type inequalities. In particular if (X, d, m) is a $\mathrm{CD}(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$, then for all $x \in X$,

$$\liminf_{r \to 0^+} \frac{m(B_r(x))}{r^N} > 0. \tag{3.1}$$

In a series of works by Ambrosio et al. [1], Ambrosio et al. [4], Erbar et al. [18] and Gigli [22,23], the notion $\mathrm{RCD}(K, N)$ *space* was introduced by adding "a Riemannian structure" to $\mathrm{CD}(K, N)$ spaces. In order to keep our short presentation, we skip the precise definition of $\mathrm{RCD}(K, N)$ spaces. Instead of that, combining previous known results, we introduce the complete list of "one-dimensional" RCD spaces below. Typical examples of $\mathrm{RCD}(K, N)$ spaces are weighted Riemannian manifolds; for n-dimensional complete Riemannian manifold (M^n, g) with $f \in C^2(M^n)$, the metric measure space $(M^n, d_g, \mathrm{e}^{-f}\mathrm{vol}_g)$ is $\mathrm{RCD}(K, N)$ if and only if it holds that $n \leqslant N$ and that

$$\mathrm{Ric}^g_{M^n} + \mathrm{Hess}^g_f - \frac{df \otimes df}{N - n} \geqslant Kg. \tag{3.2}$$

Note that if $n = N$, then the inequality (3.2) is understood as that f must be a constant with $\mathrm{Ric}^g_{M^n} \geqslant Kg$. Other typical examples are obtained by pointed measured Gromov-Hausdorff (pmGH) limit spaces of Riemannian manifolds with Ricci bounds from below by K, dimensions from above by N, and renormalized measures, so-called *Ricci limit spaces*.

Let us introduce a key notion in this section. For an $\mathrm{RCD}(K, N)$ space (X, d, m), where $N \in [1, \infty)$, it is proved in [7] that there exists a unique $k \in [1, N] \cap \mathbb{N}$ such that for m-a.e. $x \in X$, it holds that $(X, r^{-1}d, m/m(B_r(x)), x)$ pmGH converge to $(\mathbb{R}^k, d_{\mathbb{R}^k}, \omega_k^{-1}\mathcal{L}^k, 0_k)$ as $r \to 0^+$. This generalizes a result of [12] to $\mathrm{RCD}(K, N)$ spaces, where $\omega_k = \mathcal{L}^k(B_1(0_k))$. We denote k by $\dim_{d,m}(X)$ and call it the *essential dimension*.

Remark 3.1. More generally, for any $\mathrm{RCD}(K, N)$ space (X, d, m) $(N < \infty)$ and all $l \in \mathbb{N}$ the *l-dimensional regular set* $\mathcal{R}_l$ is defined by the set of all points $x \in X$ satisfying

$$(X, r^{-1}d, m(B_r(x))^{-1}m, x) \stackrel{pmGH}{\to} (\mathbb{R}^l, d_{\mathbb{R}^l}, \omega_l^{-1}\mathcal{L}^l, 0_l) \quad (r \to 0^+). \tag{3.3}$$

Then it follows from the Bishop-Gromov inequality that $\mathcal{R}_l = \emptyset$ for all $l \in (N, \infty) \cap \mathbb{N}$. It is still open problem (even for Ricci limit spaces) if $\mathcal{R}_l \neq \emptyset$ for some l, then $\mathcal{R}_n = \emptyset$ for all $n \neq l$, except for the case when $l = 1$ ([28], see also [15]) or $l = N$ ([16,27]). The well-definedness of the essential dimension means that (combining with [31]) there exists a unique k with $m(\mathcal{R}_k) > 0$.

It is worth pointing out that if $(X, r^{-1}d, x)$ pGH converge to $(R^l, d_{\mathbb{R}^l}, 0_l)$ as $r \to 0^+$, then (3.3) holds, that is, $x \in \mathcal{R}_l$. The proof is the same as that of [10, Proposition 1.35]. This tells us that the l-dimensional regular point is purely metric notion.

We are now in a position to introduce a classification of "one-dimensional" $\mathrm{RCD}(K, N)$ spaces. For that we prepare the distorsion coefficients; for all $K \in \mathbb{R}, K \in \mathbb{R}_{\geqslant 0}, s \in [0, 1]$ and $\theta \in \mathbb{R}_{>0}$,

$$\sigma^s_{K,N}(\theta) = \begin{cases} \infty, & \text{if } K\theta^2 \geqslant N\pi^2, \\ \dfrac{\sin(s\theta(K/N)^{1/2})}{\sin(\theta(K/N)^{1/2})}, & \text{if } 0 < K\theta^2 < N\pi^2, \\ s, & \text{if } K\theta^2 = 0, \\ \dfrac{\sinh(s\theta(-K/N)^{1/2})}{\sinh(\theta(-K/N)^{1/2})}, & \text{if } K\theta^2 < 0. \end{cases}$$

Although the following is just a combination of previous known results in [3, 5, 8, 28] (see also [15, 25] for Ricci limit spaces), to the best of our knowledge, we write down it explicitly with the sketch of the proof for reader's convenience.

Theorem 3.2. *Let (X, d, m) be an* $\mathrm{RCD}(K, N)$ *space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Then the following are equivalent*:

(a) $\mathcal{R}_1 \neq \emptyset$.

(b) $\dim_{d,m}(X) = 1$.

(c) $\dim_H(X, d) \in [1, 2)$, *where* $\dim_H$ *denotes the Hausdorff dimension.*

(d) *(X, d) is isometric to a one-dimensional complete Riemannian manifold (M^1, g_1) with possible boundary. Moreover there exists $h \in C^0(X)$ such that $m = h\mathrm{vol}_{g_1}$ and that for all minimal geodesic $\gamma : [0, 1] \to X$ with $x = \gamma(0)$ and $y = \gamma(1)$ (i.e., $d(\gamma(u), \gamma(v)) = d(x, y)|u - v|$ holds for all $u, v \in [0, 1]$) it holds that for all $s, t_i \in [0, 1]$,*

$$\begin{aligned} &h(\gamma(st_0 + (1 - s)t_1))^{1/(N-1)} \\ &\quad \geqslant \sigma^s_{K,N-1}(d(x, y))h(\gamma(t_0))^{1/(N-1)} + \sigma^{(1-s)}_{K,N-1}(d(x, y))h(\gamma(t_1))^{1/(N-1)}. \end{aligned} \tag{3.4}$$

Finally if a one-dimensional complete Riemannian manifold $(\hat{M}^1, \hat{g}_1)$ with possible boundary satisfies (3.4) for some $h \in C^0(\hat{M}^1)$ with $h > 0$ on $\hat{M}^1 \setminus \partial\hat{M}^1$, then $(\hat{M}^1, d_{\hat{g}_1}, h\mathrm{vol}_{\hat{g}_1})$ is an $\mathrm{RCD}(K, N)$ *space.*

Proof. It is trivial that (b) ⇒ (a) holds. Let us first prove (c) ⇒ (b). If (b) is not satisfied, then letting $k := \dim_{d,m}(X) \geqslant 2$ with [3, Theorem 4.1] yields that the k-dimensional reduced regular set $\mathcal{R}^*_k (\subset \mathcal{R}_k)$ has a positive m-measure and that m and $\mathcal{H}^k$ are mutually absolutely continuous on $\mathcal{R}^*_k$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. In particular, we have

$$\mathcal{H}^k(X) \geqslant \mathcal{H}^k(\mathcal{R}^*_k) > 0,$$

which implies $\dim_H(X, d) \geqslant k \geqslant 2$. Thus we have the desired implication.

Next, let us prove (a) ⇒ (d) only in the case when $K = 0$ because the other case is similar. By a result of [28, Theorem 1.1] we know that (X, d) is isometric to a one-dimensional Riemannian manifold with possible boundary and that there exists a nonnegatively valued $h \in C^0(\mathrm{Int}(X))$ such that $m = h\mathcal{H}^1$ and that $h^{1/N}$ is concave on each minimal geodesic in $\mathrm{Int}(X)$, where $\mathrm{Int}(X) = X \setminus \partial X$. In particular, h has a continuous extension to X. Then applying a similar argument in the proof of [8, Theorem 4.2] yields that (3.4) holds. Thus we have (d).

Finally, we prove the remaining statement for $(\hat{M}^1, d_{\hat{g}_1}, h\mathrm{vol}_{\hat{g}_1})$. If $\partial\hat{M}^1 \neq \emptyset$, or $(\hat{M}^1, \hat{g}_1)$ is isometric to $(\mathbb{R}, g_{\mathbb{R}})$, then the desired statement is justified from [8, Lemma 6.2]. If $(\hat{M}^1, \hat{g}_1)$ is isometric to $(\mathbb{S}^1(r), g_{\mathbb{S}^1(r)})$ for some $r > 0$, where

$$\mathbb{S}^1(r) := \{x \in \mathbb{R}^2 : |x| = r\}$$

with the standard Riemannian metric, then taking a finite open covering of arcs whose lengths are at most πr with [5, Theorem 6.14] and the observation above in the case when $\partial\hat{M}^1 \neq \emptyset$ yields the conclusion. □

Remark 3.3. As a corollary of Theorem 3.2 we see that if $(\mathbb{S}^1(r), d_{\mathbb{S}^1(r)}, m)$ is an $\mathrm{RCD}(0, N)$ space for some N, then $m = a\mathcal{H}^1$ for some $a > 0$. The proof is as follows. Thanks to Theorem 3.2 we have $m = f\mathcal{H}^1$ for some positively valued continuous function f on $\mathbb{S}^1(r)$. Let $c := \min f$ and let $A := f^{-1}(c) \neq \emptyset$. Thanks to (3.4) it is easy to check that any $x \in A$ is an interior point of A because of $\sigma^s_{0,N-1}(\theta) = s$. Thus A is an open subset. Since it follows from the continuity of f that A is a closed set, we have $A = \mathbb{S}^1(r)$ which shows that f is a constant.

Remark 3.4. In the condition (d) above if h is continuous on $\gamma([0, 1])$ and C^2 on $\gamma((0, 1))$, then (3.4) holds if and only if

$$(h^{1/(N-1)})'' + \frac{K}{N-1}h^{1/(N-1)} \leqslant 0 \tag{3.5}$$

holds. Moreover if $h = \mathrm{e}^{-f}$, then (3.5) holds if and only if

$$f'' - \frac{(f')^2}{N-1} \geqslant K \tag{3.6}$$

holds. Compare with (3.2).

Remark 3.5. It also follows from Theorem 3.2 and [8, Theorem 4.2] that if (X, d, m) is an RCD $(K, 1)$ space, then $m = a\mathcal{H}^1$ for some constant $a > 0$.

Definition 3.6. For an RCD(K_1, ∞) space (X, d, m) for some $K_1 \in \mathbb{R}$ let us define the *optimal dimension* $\dim_{\mathrm{RCD}}(X, d, m)$ as RCD space by the infimum $N \in (1, \infty]$ such that (X, d, m) is an RCD(K_2, N) space for some $K_2 \in \mathbb{R}$.

It follows from [3, Theorem 4.1] and [16, Corollary 1.5] that

$$\dim_{d,m}(X) \leqslant \dim_H(X, d) \leqslant [\dim_{\mathrm{RCD}}(X, d, m)],$$

where $[N]$ denotes the integer part of N. Note that for all $N \in [1, \infty)$, a metric measure space $([0, \pi], d_{\mathbb{R}}, \sin^{N-1} t\mathcal{L}^1)$ is an RCD$(N-1, N)$ space whose optimal dimension is N (see [3]) and that the Gaussian space $(\mathbb{R}^n, d_{\mathbb{R}^n}, \mathrm{e}^{-K|x|^2}\mathcal{L}^n)$ has the optimal dimension ∞ if $K \neq 0$.

Remark 3.7. It is natural to ask if $\dim_{\mathrm{RCD}}(X, d, m) = 1$, then $m = a\mathcal{H}^1$ for some $a > 0$. However this is not true (compare with Remark 3.5). In fact, for all closed interval I in $\mathbb{R}$ and any $K \neq 0$, thanks to (3.6), the metric measure space $(I, d_{\mathbb{R}}, \mathrm{e}^{-Kx}\mathcal{L}^1)$ has the optimal dimension 1. However, e^{-Kx} is not constant. More generally the optimal dimension of $(A, d_g, \mathrm{e}^{-f}\mathrm{vol}_g)$, where A is a closed convex subset of M^n with $f \in C^2(M^n)$ and $\mathrm{vol}_g(B_r(x) \cap A) > 0$ for all $x \in A$ and all $r > 0$, is equal to n because of (3.2).

Proposition 3.8. *For all $d_i > 0$ $(i = 1, 2)$ and $n \in \mathbb{N}$, a metric measure space*

$$(X, d, m) := \left(\left[-\frac{d_1}{d_1 + d_2}, \frac{d_2}{d_1 + d_2}\right], d_{\mathbb{R}}, f(x)^{\frac{n-1}{n+1}}\mathcal{L}^1\right)$$

is RCD$(0, \frac{2n}{n+1})$ *whose optimal dimension is* $\frac{2n}{n+1}$*, where*

$$f(x) := \begin{cases} \left(\dfrac{x}{d_1} + \dfrac{1}{d_1 + d_2}\right), & x \in \left[-\dfrac{d_1}{d_1 + d_2}, 0\right], \\ \left(-\dfrac{x}{d_2} + \dfrac{1}{d_1 + d_2}\right), & \textit{otherwise}. \end{cases}$$

Proof. It follows from (3.1) with a fact

$$\lim_{r \to 0^+} \frac{m(B_r(p))}{r^N} = 0$$

for all $N \in [1, \frac{2n}{n+1})$ that

$$\dim_{\mathrm{RCD}}(X, d, m) \geqslant \frac{2n}{n+1}$$

holds, where $p \in \partial X$. Thus it suffices to prove that (X, d, m) is an RCD$(0, \frac{2n}{n+1})$ space.

Since the function $x \mapsto f(x)$ is concave on X, we see that for all $t_i \in X$ $(i = 1, 2)$ and all $s \in [0, 1]$,

$$f(st_0 + (1-s)t_1) \geqslant sf(t_0) + (1-s)f(t_1),$$

which is equivalent to that (3.4) holds as

$$K = 0, \quad N = \frac{2n}{n+1}, \quad h(x) = f(x).$$

Therefore Theorem 3.2 yields that (X, d, m) is an RCD$(0, \frac{2n}{n+1})$ space. □

This proposition computes the optimal RCD dimension of the collapsing limits constructed in [24, 35]. Also Item (5) of Theorem 1.1 follows easily from this.

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