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DEGENERATIONS AND MODULI SPACES IN KÄHLER GEOMETRY

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Abstract

We report some recent progress on studying degenerations and moduli spaces of canonical metrics in Kähler geometry, and the connection with algebraic geometry, with a particular emphasis on the case of Kähler–Einstein metrics.

1 Introduction

One of the most intriguing features in Kähler geometry is the interaction between differential geometric and algebro-geometric aspects of the theory. In complex dimension one, the classical uniformization theorem provides a unique conformal metric of constant curvature -1 on any compact Riemann surface of genus bigger than 1. This analytic result turns out to be deeply connected to the algebraic fact that any such Riemann surface can be embedded in projective space as a *stable* algebraic curve in the sense of geometric invariant theory. It is also well-known that the moduli space of smooth algebraic curves of genus bigger than 1 can be compactified by adding certain singular curves with nodes, locally defined by the equation $xy = 0$, which gives rise to the *Deligne–Mumford compactification*. This is compatible with the differential geometric compactification using hyperbolic metrics, in the sense that whenever a node forms, locally the corresponding metric splits into the union of two hyperbolic cusps with infinite diameter. Intuitively, one can view the latter as a canonical differential geometric object associated to a nodal singularity.

On higher dimensional Kähler manifolds, a natural generalization of constant curvature metrics is the notion of a *canonical Kähler metric*, which is governed by some elliptic partial differential equation. In this article we will mainly focus on Calabi's *extremal Kähler metrics*. Let X be a compact Kähler manifold of dimension n, and let $\mathcal{H}$ be a

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Kähler class on X, i.e. the space of Kähler metrics on X in a given de Rham cohomology class (which we shall denote by $[\mathcal{H}] \in H^2(X;\mathbb{R})$). For example, any ample line bundle L gives rise to a Kähler class with $[\mathcal{H}] = 2\pi c_1(L)$. If we fix one Kähler metric $\omega \in \mathcal{H}$, then any other Kähler metric in $\mathcal{H}$ is of the form $\omega + i\partial\bar{\partial}\phi$ for some smooth function ϕ.

A Kähler metric ω is called *extremal* if it is the critical point of the Calabi functional $Ca(\omega) = \int_X S(\omega)^2\omega^n$ on $\mathcal{H}$, where $S(\omega)$ denotes the scalar curvature function of ω. Equivalently, by calculating the Euler–Lagrange equation, this means that the gradient vector field $\nabla_\omega S(\omega)$ is holomorphic. A special case is when the vector field vanishes, then ω is a *constant scalar curvature Kähler* metric. If moreover the first Chern class $c_1(X)$ is proportional to $[\mathcal{H}]$, say $2\pi c_1(X) = \lambda[\mathcal{H}]$, then this further reduces to the *Kähler–Einstein equation* $Ric(\omega) = \lambda\omega$.

There are several well-known fundamental questions that aim to build connections between the analytic theory of these metrics and algebraic geometry.

(1). **Existence**: When does $(X, \mathcal{H})$ contain an extremal Kähler metric?

To find an extremal Kähler metric amounts to solving a difficult non-linear elliptic PDE. The famous *Yau–Tian–Donaldson conjecture* states that the solvability of this PDE is equivalent to *K-stability* of $(X, \mathcal{H})$. K-stability is a complex/algebro-geometric notion; roughly speaking, it is tested by the positivity of certain numerical invariant, the *Donaldson–Futaki invariant*, associated to $\mathbb{C}^*$ equivariant *flat* degenerations of $(X, \mathcal{H})$. It is analogous to the Hilbert–Mumford criterion for stability in geometric invariant theory.

The Yau–Tian–Donaldson conjecture extends the *Calabi conjecture* on the existence of Kähler–Einstein metrics. In this special case we have a positive answer.

Theorem 1.1. *Let X be a compact Kähler manifold and $\mathcal{H}$ be a Kähler class with $2\pi c_1(X) = \lambda[\mathcal{H}]$.*

- *(Yau [1978]) If $\lambda = 0$, then there is a unique Kähler metric $\omega \in \mathcal{H}$ with $Ric(\omega) = 0$. This ω is usually referred to as a Calabi–Yau metric.*
- *(Yau [ibid.], Aubin [1976]) If $\lambda < 0$ (in which case X is of general type), then there is a unique Kähler metric $\omega \in \mathcal{H}$ with $Ric(\omega) = -\lambda\omega$.*
- *If $\lambda > 0$ (in which case X is a Fano manifold), then there is a Kähler metric $\omega \in \mathcal{H}$ with $Ric(\omega) = \lambda\omega$ if and only if X is K-stable. The "only if" direction is due to various authors in different generality including Tian [1997], Stoppa [2009], and R. J. Berman [2016], and the "if" direction is due to Chen–Donaldson–Sun (c.f. Chen, Donaldson, and Sun [2015a,b,c]), and later other proofs can be found in Datar and Székelyhidi [2016], Chen, Sun, and B. Wang [2015], and R. Berman, Boucksom, and Jonsson [2015].*

(2). **Compactification of moduli space and singularities**

It is known (c.f. Chen and Tian [2008] and R. J. Berman and Berndtsson [2017]) that extremal metric in a Kähler class $\mathcal{H}$, if exists, is unique up to holomorphic transformations of X, so it canonically represents the complex geometry of $(X, \mathcal{H})$. Locally one can deform either the Kähler class $\mathcal{H}$ or the complex structure on X, and the corresponding deformation theory of extremal Kähler metrics has been well-studied. Globally to compactify the moduli space one needs to know how to take limits and what is the structure of the limits, especially near the singularities. This has interesting connection with the compactification from the algebro-geometric viewpoint, and is also intimately related to the existence question in (1).

(3). **Optimal degenerations**

There are two important geometric evolution equations, namely, the *Kähler–Ricci flow*, and the *Calabi flow*, that try to evolve an arbitrary Kähler metric towards a canonical metric. When $(X, \mathcal{H})$ is K-stable, then one expects the flow to exist for all time and converge to a canonical metric; when $(X, \mathcal{H})$ is not K-stable, then one expects the flow to generate a degeneration to some canonical geometric object associated to $(X, \mathcal{H})$, which is *optimal* in suitable sense.

We refer the readers to the article Donaldson [2018] in this proceeding for an overview on the existence question, and related discussion on K-stability. In this article we will report progress towards (2) and (3), mostly focusing on the case of Kähler–Einstein metrics.

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2 Gromov–Hausdorff limits

We consider a sequence of compact Kähler manifolds (X_i, ω_i) of dimension n. Throughout this section we will impose the following hypothesis

- $[\omega_i] = 2\pi c_1(L_i)$ for some ample line bundle L_i over X_i.

- $|Ric(\omega_i)| \le \Lambda$ for some fixed $\Lambda > 0$.
- The diameter of (X_i, ω_i) is uniformly bounded above by $D > 0$.

These ensure the sequence of metrics to satisfy a *volume non-collapsing condition*, i.e. there exists $\kappa > 0$ such that for all i and all $p \in X_i$, $Vol(B(p,r)) \ge \kappa r^{2n}$ for all $r \in (0,1]$.

By the convergence theory of Riemannian manifolds one can pass to a subsequence and extract a *Gromov–Hausdorff limit* X_∞, which is a compact length space. This seems to be a very rough process at first sight but the important fact is that it is an intrinsic limit, which does not require a choice of coordinate systems on X_i. By the work of Cheeger, Colding, and Tian [2002], we know X_∞ can be written as the union of the regular part $\mathcal{R}$, which is a smooth open manifold endowed with a $C^{1,\alpha}$ Kähler structure $(J_\infty, \omega_\infty)$, and the singular set $\mathcal{S}$, which has Hausdorff codimension at least 4. In the case when $n = 2$ it has been known by Anderson [1989], Bando, Kasue, and Nakajima [1989], and Tian [1990b] that X_∞ has only isolated orbifold singularities. By possibly passing to a further subsequence, we may assume the Chern connection on L_i converges modulo gauge transformations to the Chern connection of a hermitian holomorphic line bundle $L_{\mathcal{R}}$ on $\mathcal{R}$. The following result establishes a basic connection with algebraic geometry

Theorem 2.1 (Donaldson and Sun [2014]). *(1) X_∞ is naturally homeomorphic to a normal projective variety in such a way that the algebraic singularities are a subset of the metric singularities, and, by passing to subsequence the convergence of X_i to X_∞ can be realized in a fixed Hilbert scheme.*

(2) If furthermore we assume ω_i is Kähler–Einstein, then the two singular sets are equal, and the algebraic singularities of X_∞ are log terminal in the sense of minimal model program. Moreover, ω_∞ extends to a global Kähler current that satisfies a singular Kähler–Einstein equation in the sense of pluripotential theory, c.f. Eyssidieux, Guedj, and Zeriahi [2009].

Remark 2.2.

- The projective algebraic structure on X_∞ is intrinsically determined by $(\mathcal{R}, L_{\mathcal{R}})$. Namely, denote by $\iota : \mathcal{R} \to X_\infty$ the natural inclusion map, and we can define $\mathcal{O}_{X_\infty} := \iota_* \mathcal{O}_{\mathcal{R}}$ and $L_\infty := \iota_* L_{\mathcal{R}}$. Then the precise meaning of (1) is that $(X_\infty, \mathcal{O}_{X_\infty})$ is a normal complex analytic space and L_∞ is an ample $\mathbb{Q}$-line bundle on X_∞. Moreover, by passing to a subsequence, (X_i, L_i) and (X_∞, L_∞) can be fit into a *flat* family.
- In the Kähler–Einstein case with $\lambda = 1$, the line bundle L_i is given by $K_{X_i}^{-1}$, and the diameter bound is automatically satisfied by Bonnet–Myers's theorem. In this case X_∞ is a smoothable $\mathbb{Q}$-*Fano variety*. Theorem 2.1 thus provides a *topological*

compactification of the moduli space of Kähler–Einstein Fano manifolds in each dimension, by adding all the possible Gromov–Hausdorff limits.

A key ingredient in the proof of Theorem 2.1 is to obtain the *partial C^0 estimate*, conjectured by G. Tian in ICM 1990 (c.f. Tian [1990a]). Recall, in general that if (X, L) is a polarized Kähler manifold and ω is a Kähler metric in $2\pi c_1(L)$, then ω determines a hermitian metric $|\cdot|$ on L, unique up to constant multiple. For all k we have an induced L^2 norm $\|\cdot\|$ on $H^0(X, L^k)$ (defined with respect to the volume form of $k\omega$). To compare these we define the *density of state function* (or the *Bergman function*)

$$\rho_{X,\omega,k}(x) = \sup_{s\in H^0(X,L^k), s\neq 0} \frac{|s(x)|^2}{\|s\|^2}.$$

By the Kodaira embedding theorem this is a positive function for k sufficiently large. It is easy to see for each $x \in X$, the supremum is achieved on a unique one dimension subspace $\mathbb{C}_x \subset H^0(X, L^k)$, and $\rho_{X,\omega,k}$ is a smooth function on X. Moreover, we have a C^∞ asymptotic expansion (c.f. Zelditch [1998])

$$\rho_{X,\omega,k} = 1 + \frac{S(\omega)}{2}k^{-1} + \cdots. \tag{2-1}$$

Its importance can be seen as follows

- Denote $N_k = \dim H^0(X, L^k)$, and choose an L^2 orthonormal basis $\{s_1, \cdots, s_{N_k}\}$ of $H^0(X, L^k)$, then there is an alternative expression

  $$\rho_{X,\omega,k}(x) = \sum_{i=1}^{N_k} |s_i(x)|^2.$$

  So we get $N_k = \int_X \rho_{X,\omega,k}(k\omega)^n/n!$, and (2-1) can be viewed as a local version of the Riemann–Roch formula.

- Using an orthonormal basis of $H^0(X, L^k)$, for k large we get an embedding $\iota_k : X \to \mathbb{P}^{N_k-1}$, unique up to unitary transformations. Then we have

  $$\iota_k^*\omega_k = k\omega + i\partial\bar{\partial}\log\rho_k. \tag{2-2}$$

  So by (2-1) we know $k^{-1}\iota_k^*\omega_k$ converges smoothly to ω as k tends to infinity. This is the *Kähler quantization* picture, which is essential in studying the relationship between constant scalar curvature Kähler metrics and algebraic stability (c.f. Donaldson [2001]).

Theorem 2.3 (Donaldson and Sun [2014]). *Under the hypothesis in the beginning of this section, there are k_0 and $\epsilon_0 > 0$ that depend only on n, Λ and D, such that $\rho_{X_i,\omega_i,k_0} \geq \epsilon_0$ for all i.*

Remark 2.4. (1). When $n = 2$ this is proved by Tian [1990b], using the fact that in this case the Gromov–Hausdorff limit is an orbifold; for general n this is conjectured by Tian [1990a]. Indeed Tian's original conjecture is stated under more general assumptions, and there has been substantial progress on this, see for example Theorem 1.9 in Chen and B. Wang [2014].

(2). One may ask if there is a uniform asymptotic behavior of the density of state function as k tends to infinity. In general the expansion (2-1) can not hold uniformly independent of i, but a weaker statement is plausible, see Conjecture 5.15 in Donaldson and Sun [2014].

(3). It follows from Theorem 2.3 and (2-2) that the embedding map ι_{k_0} has a uniform Lipschitz bound for all i, hence one can pass to limit and obtain a Lipschitz map from the Gromov–Hausdorff limit X_∞ to a fixed projective space. This is the starting point to prove Theorem 2.1.

The proof of Theorem 2.3 is based on Hörmander's construction of holomorphic sections on definite powers of L_i with control. The idea is to first find holomorphic sections of Gaussian type in a local model, then graft them to the manifolds X_i to get approximately holomorphic sections, and finally correct these to genuine holomorphic sections by solving a $\bar\partial$ equation. The solvability of $\bar\partial$ equation with uniform estimates only uses one global geometric assumption, namely the lower bound of Ricci curvature of ω_i.

Here we briefly describe the notion of model Gaussian sections. Suppose first we are at a *smooth* point $x \in X_i$ for some *fixed* i. If we dilate the metric ω_i to $k\omega_i$ based at x, then as k tends to infinity we get in the limit the standard flat metric on $\mathbb{C}^n$. The dilation has the effect of replacing the line bundle L_i by L_i^k and it is also important to notice that the corresponding limit line bundle is the trivial holomorphic bundle on $\mathbb{C}^n$ endowed with the non-trivial hermitian metric $e^{-|z|^2/2}$ whose curvature is exactly the flat metric. The obvious trivial section is then naturally a Gaussian section. Using this one can construct for k large a holomorphic section of L_i^k over X_i, whose L^2 norm has a definite upper bound and its pointwise norm at x has a definite positive lower bound. This implies $\rho_{X_i,\omega_i,k}(x) \geq \epsilon > 0$. The difficulty in the proof of Theorem 2.3 is then to obtain uniform estimate on both k and ϵ, which is not a priori clear since it is conceivable that when the metric ω_i starts to form singularities the region where the Gaussian model behaves well for a fixed k shrinks to one point. For this purpose we use the Gromov–Hausdorff limit X_∞ and consider a sequence of points $x_i \in X_i$ that tend to a point $x_\infty \in X_\infty$. If x_∞ is a smooth point then previous argument goes through with little change to yield $\rho_{X_i,\omega_i,k}(x_i) \geq \epsilon > 0$ for k and ϵ independent of i. If x_∞ is singular, then we can dilate the metric on X_∞ based at x_∞. By passing to a subsequence we can obtain a *pointed* Gromov–Hausdorff limit, called a *tangent cone*. It is a *metric cone* $C(Y)$ over a compact length space Y, which in particular admits a dilation action.

By Cheeger, Colding, and Tian [2002] we also know that $C(Y)$ is a $C^{1,\alpha}$ Kähler manifold outside a closed subset of Hausdorff codimension at least 4 which is invariant under dilation. On this smooth part, being a metric cone implies that the metric can be written as $g = \frac{1}{2}\mathrm{Hess}(r^2)$, where r is the distance to the cone vertex. Now the crucial point in our Kähler setting is that the Kähler form can correspondingly be written as $\omega = \frac{1}{2}i\partial\bar{\partial}r^2$, so in particular it is the curvature of the trivial holomorphic bundle with hermitian metric $e^{-r^2/2}$. This is exactly the generalization of the above local model on $\mathbb{C}^n$. Thus we also have the candidate Gaussian section in this case, but there are various technical difficulties that had to be overcome in Donaldson and Sun [2014], mainly due to the appearance of singularities on the tangent cones $C(Y)$. One point to notice is that the proof does *not* require the tangent cone at x_∞ to be unique, even though this turns out to be true and we will discuss more in the next section.

We mention that the extension of Theorem 2.1 and 2.3 to the case of Kähler–Einstein metrics with cone singularities plays a key role in the proof of the Yau–Tian–Donaldson conjecture for Fano manifolds (c.f. Chen, Donaldson, and Sun [2015a,b,c]). There are also other applications to the study of Gromov–Hausdorff limits of Kähler–Einstein metrics in the Calabi–Yau and general type case, c.f. Rong and Zhang [2011], Tosatti [2015], and Song [2017]. In both cases the diameter bound (hence the volume non-collapsing condition) is not automatically satisfied, and is essentially equivalent to the algebraic limit having at worst log terminal singularities.

There are a few interesting directions that require further development:

(1). Prove a local version of Theorem 2.1. Namely, let X_∞ be a Gromov–Hausdorff limit of a sequence of (incomplete) Kähler manifolds (X_i, ω_i) with diameter 1, uniformly bounded Ricci curvature, and satisfying a uniform volume non-collapsing condition, can we prove X_∞ is naturally a complex analytic space?

Notice in general one can not expect to fit X_i and X_∞ into a flat family in the absence of the line bundles, since one can imagine certain holomorphic cycles being contracted under the limit process. The answer to the above question will have applications in understanding convergence of Calabi–Yau metrics when Kähler class becomes degenerate, see Collins and Tosatti [2015]. In a related but different context, G. Liu [2016] studied the case when the Ricci curvature bound is replaced by a lower bound on the bisectional curvature, and used it to make substantial progress towards Yau's uniformization conjecture for complete

Kähler manifolds with non-negative bisectional curvature. Most generally, one would expect that in the above question to draw the complex-analytic consequence the most crucial assumption is a uniform lower bound on Ricci curvature, and the upper bound on Ricci curvature is more related to the regularity of the limit metrics.

(2). In general collapsing is un-avoidable. Can we describe the complex/ algebro-geometric meaning of the Gromov–Hausdorff limits?

Collapsing with uniformly bounded Riemannian sectional curvature has been studied extensively in the context of Riemannian geometry through the work of Cheeger–Fukaya–Gromov and others. In general one expects a structure of fibrations (in the generalized sense). Even in this case the above question is not well-understood. In general with only Ricci curvature bound there has been very few results on the regularity of the limit space itself (see Cheeger and Tian [2006] for results when $n = 2$).

Ultimately one would like to understand the case of constant scalar curvature or extremal Kähler metrics, where we are currently lacking the analogous foundations of the Cheeger–Colding theory which depends on comparison geometry of Ricci curvature, and so far we only have results in the non-collapsed case, see for example Tian and Viaclovsky [2008], Chen and Weber [2011].

3 Singularities

In this section we focus on finer structure of the Gromov–Hausdorff convergence studied in Theorem 2.1, and restrict to the case when (X_i, ω_i) is Kähler–Einstein. It is a folklore picture that when singularities occur certain non-compact Ricci-flat spaces must bubble off. To be more precise, suppose $p \in X_\infty$ is a singular point, and $p_i \in X_i$ is a sequence of points that converge to p. Take any sequence of integers $k_i \to \infty$ and consider the rescaled spaces $(X_i, L_i^{k_i}, k_i\omega_i, p_i)$, then by passing to subsequence, we always get a *pointed Gromov–Hausdorff limit* (Z, p_∞), which is a non-compact metric space.

Theorem 3.1 (Donaldson and Sun [2017])**.** *Any such limit Z is naturally a normal affine algebraic variety which admits a singular Ricci-flat Kähler metric.*

To explain the meaning of this, similar to Remark 2.2, we know the complex-analytic structure on Z is determined by the regular part of Z (in the sense of Cheeger, Colding, and Tian [2002]). To understand intrinsically the affine structure, we denote by $R(Z)$ the ring of holomorphic functions on Z that grow at most polynomially fast at infinity, then it is proved in Donaldson and Sun [2017] that $R(Z)$ is finitely generated and $\mathrm{Spec}(R(Z))$ is complex-analytically isomorphic to Z.

In dimension two, Z is an ALE Ricci-flat space possibly with orbifold singularities. In higher dimensions, by the volume non-collapsing condition we know Z is *asymptotically* *conical*. There has been extensive study on these spaces in the case when the tangent cone at infinity is smooth, see for example van Coevering [2011], Conlon and Hein [2015], C. Li [2015].

Let $\mathfrak{B}$ be the set of all *bubbles* at p, i.e. the set of pointed Gromov–Hausdorff limits of $(X_i, k_i\omega_i, p_i)$ with $p_i \to p$ and $k_i \to \infty$. It is an interesting question to find complex/algebro-geometric characterization of this set, which potentially forms a *bubble tree* structure at p. For each $Z \in \mathfrak{B}$, the Bishop–Gromov volume comparison defines an invariant $v(Z)$, namely, the asymptotic volume ratio of Z

$$v(Z) = \lim_{r\to\infty} Vol(B(p_\infty, r))r^{-2n}$$

A special element in $\mathfrak{B}$ is a metric tangent cone $C(Y)$ at p, which has the *smallest* asymptotic volume ratio among all the bubbles at p. The metric cone structure imposes an extra dilation symmetry on $C(Y)$, and this has the corresponding algebro-geometric meaning

Theorem 3.2 (Donaldson and Sun [2017]). *A tangent cone $C(Y)$ is naturally a normal affine algebraic cone.*

This requires some explanation. Let $R(C(Y))$ be the affine coordinate ring of $C(Y)$. On the regular part of $C(Y)$, we have a *Reeb* vector field $\xi = Jr\partial_r$ which is a holomorphic Killing vector field. It generates holomorphic isometric action of a compact torus $\mathbb{T}$ on $C(Y)$. This action induces a weight space decomposition of $R(C(Y))$, which can be written as

$$R(C(Y)) = \bigoplus_{\mu\in\mathbb{H}} R_\mu(C(Y)).$$

Here $R_\mu(C(Y))$ is the space of homogeneous holomorphic functions f on $C(Y)$ satisfying $\mathfrak{L}_\xi f = i\mu f$ (i.e. homogeneous of degree μ), and the *holomorphic spectrum* $\mathbb{H}$ is the set of all $\mu \in \mathbb{R}_{\geq 0}$ such that $R_\mu(C(Y)) \neq 0$. In general we know $\mathbb{H}$ is contained in the set of algebraic numbers, but not necessarily a subset of $\mathbb{Q}$. This *positive* $\mathbb{R}$-grading on $R(C(Y))$ is the precise meaning for $C(Y)$ to be an affine algebraic *cone* in Theorem 3.2.

Theorem 3.3 (Donaldson and Sun [ibid.]). *Given any $p \in X_\infty$, there is a unique metric tangent cone at p, as an affine algebraic cone endowed with a singular Ricci-flat Kähler metric.*

One step in the proof is to show that the holomorphic spectrum $\mathbb{H}$ is a priori unique. The crucial observation is that the set of all possible tangent cones at p form a connected and compact set under the pointed Gromov–Hausdorff topology, and $\mathbb{H}$ consists of only algebraic numbers so must be rigid under continuous deformations. The latter uses the volume minimization principle of Martelli, Sparks, and Yau [2008] in Sasaki geometry. Recall that $C(Y)$ being Ricci-flat is equivalent to Y being *Sasaki–Einstein* possibly with singularities. Consider the open convex cone $\mathcal{C}$ in $Lie(\mathbb{T})$ consisting of elements η which also induce positive gradings on $R(C(Y))$. Then by Martelli, Sparks, and Yau [ibid.] we know ξ is the unique critical point of the (suitably normalized) *volume function* on $\mathcal{C}$, which is a convex rational function with rational coefficients (noticing there is a natural rational structure in $Lie(\mathbb{T})$). This fact leads to the algebraicity of $\mathbb{H}$.

To describe the relationship between the local algebraic singularity of X_∞ at p and the tangent cone $C(Y)$, we define a *degree* function on $\mathcal{O}_p$, the local ring of germs of holomorphic functions at p, by setting

$$d(f) = \lim_{r\to 0} \frac{\sup_{B(p,r)} \log|f|}{\log r}$$

By Donaldson and Sun [2014], for all nonzero f, $d(f)$ is always well-defined and it belongs to $\mathbb{H}$. Indeed d defines a *valuation* on $\mathcal{O}_p$. So we can define a graded ring associated to this

$$R_p = \bigoplus_{\mu\in\mathbb{H}} \{f \in \mathcal{O}_p | d(f) \geq \mu\}/\{f \in \mathcal{O}_p | d(f) > \mu\}.$$

Theorem 3.4 (Donaldson and Sun [2017])**.**

- *R_p is finitely generated and $Spec(R_p)$ defines a normal affine algebraic cone W, which can be realized as a weighted tangent cone of (X_∞, p) under a local complex analytic embedding into some affine space.*
- *There is an equivariant degeneration from W to $C(Y)$ as affine algebraic cones.*

Remark 3.5. The finite generation of R_p and the fact that W is normal put on very strong constraint on the valuation d. The proof depends on Theorem 3.2 and a three-circle type argument that relates elements in $\mathcal{O}_p$ and $R(C(Y))$. In the context of Theorem 3.1 there is a similar result relating Z and its tangent cone at *infinity*.

This situation is analogous to the Harder–Narasimhan–Seshadri filtration for unstable holomorphic vector bundles. It also suggests a *local* notion of stability for algebraic singularities, since using the extension of the Yau–Tian–Donaldson conjecture to affine algebraic cones by Collins and Székelyhidi [2012], one can view $C(Y)$ as a K-stable algebraic cone and W as a K-semistable algebraic cone.

One interesting point is that W is only an algebraic variety but does not support a natural metric structure. In Donaldson and Sun [2017] we conjectured that W and $C(Y)$ are both invariants of $\mathcal{O}_p$ and there should a purely algebro-geometric way of characterizing

W. C. Li [2015] made this conjecture more precise by giving an algebro-geometric interpretation of the volume of an affine cone, and formulating a corresponding conjecture that the valuation d should be the unique one that minimizes volume. This brings interesting connections with earlier work on asymptotic invariants and K-stability in algebraic geometry. It is an extension of the Martelli–Sparks–Yau volume minimization principle. There is much progress in this direction, see Blum [2016], C. Li and Y. Liu [2016], and C. Li and Xu [2017].

One of the motivation for studying metric tangent cones is related to precise understanding of the metric behavior of a singular Kähler–Einstein metric. The following result, which uses the result of Fujita [2015] and C. Li and Y. Liu [2016], gives the first examples of compact Ricci-flat spaces with isolated conical singularities.

Theorem 3.6 (Hein and Sun [2017]). *Let (X, L) be a $\mathbb{Q}$-Gorenstein smoothable projective Calabi–Yau variety with isolated canonical singularities, each locally complex-analytically isomorphic to a strongly regular affine algebraic cone which admits a Ricci-flat Kähler cone metric, then there is a unique singular Calabi–Yau metric $\omega \in 2\pi c_1(L)$ which is smooth away from the singular locus of X, and at each singularity is asymptotic to the Ricci-flat Kähler cone metric at a polynomial rate.*

Remark 3.7.

- The notion of *strong regularity* is a technical assumption which is equivalent to that the affine algebraic cone coincides with its Zariski tangent cone at the vertex.
- An important special case is when the singularities of X are ordinary double points, in which case the Ricci-flat Kähler cone metric can be explicitly written down, and Theorem 3.6 also implies the existence of *special lagrangian vanishing spheres* on a generic smoothing of X.

The above general strategy has other applications, one is to the study of the asymptotic behavior of geometric flows, as we shall describe in Section 5, and the other is to the study of singularities of Hermitian-Yang–Mills connections (c.f. Chen and Sun [2017]).

4 Moduli spaces

Theorem 2.1 gives a *Gromov–Hausdorff compactification* of the moduli space of Kähler–Einstein Fano manifolds in each fixed dimension, as a topological space. By Theorem 1.1 this is the same as a compactification of the moduli space of K-stable Fano manifolds, so it is natural to ask for algebro-geometric meaning of this moduli space itself; furthermore one would like to characterize them explicitly since there are many concrete examples of families of smooth Fano manifolds. Understanding this would also lead to new examples of K-stable Fano varieties, including singular ones.

In dimension two, there are only four families of Fano manifolds with non-trivial moduli. They are *del Pezzo surfaces* of anti-canonical degree $d \in \{1, 2, 3, 4\}$, i.e. the blown-ups of $\mathbb{CP}^2$ at $9 - d$ points in very general position. Let $\mathfrak{M}_d$ be the Gromov–Hausdorff compactification of moduli space of Kähler–Einstein metrics on del Pezzo surfaces of degree d.

Theorem 4.1 (Odaka, Spotti, and Sun [2016]). *Each $\mathfrak{M}_d$ is naturally homeomorphic to an explicitly constructed moduli space $\mathfrak{M}_d^{alg}$ of certain del Pezzo surfaces with orbifold singularities.*

The construction of $\mathfrak{M}_d^{alg}$ depends on the classical geometry of del Pezzo surfaces, whose moduli is closely related to geometric invariant theory.

- Objects in $\mathfrak{M}_3^{alg}$ are GIT stable[1] cubics in $\mathbb{P}^3$. These were classified by Hilbert.
- Objects in $\mathfrak{M}_4^{alg}$ are GIT stable complete intersections of two quadrics in $\mathbb{P}^4$. These were classified by Mabuchi and Mukai [1993], who also proved Theorem 4.1 in this case, with a more involved argument.
- Objects in $\mathfrak{M}_2^{alg}$ are either double covers of $\mathbb{P}^2$ branched along a GIT stable quartic curve, or double covers of the weighted projective plane $\mathbb{P}(1,1,4)$ branched along a curve of the form $z^2 = f_8(x, y)$ for a GIT stable octic in 2 variables. This moduli was constructed by Mukai [1995].
- Objects in $\mathfrak{M}_1^{alg}$ are more complicated to describe, but a generic element is a double cover of $\mathbb{P}(1,1,2)$ branched along a sextic curve that is GIT stable in a suitable sense (even though the automorphism group of $\mathbb{P}(1,1,2)$ is not reductive). $\mathfrak{M}_1^{alg}$ is a certain birational modification of this GIT moduli space.

We briefly describe the general strategy in the proof, which also applies in higher dimensions (see Theorem 4.2). For a more detailed survey see Spotti [n.d.].

- Show $\mathfrak{M}_d$ is non-empty. This can be achieved by studying a special element in each family, for example through computation of α-invariant (see Tian and Yau [1987]), or alternatively Arezzo, Ghigi, and Pirola [2006], or by a glueing construction.
- Rough classification of Gromov–Hasudorff limits. The Bishop–Gromov volume comparison yields that at any singularity of the form $\mathbb{C}^2/\Gamma$, we have $|\Gamma| < 12/d$. For larger d we get stronger control on Γ hence the corresponding $\mathfrak{M}_d$ is simpler. Using this one can estimate the *Gorenstein index* of the Gromov–Hausdorff limits,

[1] The stability in this article means *polystablity* in the usual literature.

and then understand their anti-canonical geometry. This is related to the explicit determination of the smallest number k_0 in Theorem 2.3.

By R. J. Berman [2016] all the objects X in $\mathfrak{M}_d$ are K-stable, and a crucial ingredient in Odaka, Spotti, and Sun [2016] is that one can often compare K-stability with GIT stability. This implies X is also GIT stable in appropriate sense, and suggests that $\mathfrak{M}_d$ is closely related to GIT moduli.

- Construction of $\mathfrak{M}_d^{alg}$ so that the natural map from $\mathfrak{M}_d$ to $\mathfrak{M}_d^{alg}$ is a homeomorphism. The key point is to make sure $\mathfrak{M}_d^{alg}$ is Hausdorff and the map is well-defined, i.e. any possible Gromov–Hausdorff limit in $\mathfrak{M}_d$ is included in $\mathfrak{M}_d^{alg}$. In general one can start with a natural GIT moduli space, and perform birational modifications that are suggested by the previous step. For example when $d = 2$, the natural GIT moduli of quartic curves contains a point that corresponds to a reducible surface, which we know can not be in $\mathfrak{M}_d$; this motivates one to blow-up the point and the exceptional divisor turns out to correspond to a different GIT as described above.

Theorem 4.2. • *Spotti and Sun [2017]: For all n, the Gromov–Hausdorff compactification of the moduli space of Kähler–Einstein metrics on complete intersection of two quadrics in $\mathbb{P}^{n+2}$ is naturally homeomorphic to the GIT moduli space.*

- *Y. Liu and Xu [2017]: The Gromov–Hausdorff compactification of the moduli space of Kähler–Einstein metrics on cubics in $\mathbb{P}^4$ is naturally homeomorphic to the GIT moduli space.*

A major step is to prove the Gromov–Hausdorff limit has a well-defined canonical line bundle, i.e., is *Gorenstein*, and has large divisibility. This relies on the following deep theorem of K. Fujita proved via K-stability, and the later generalization by Y. Liu [2016] to give an improvement of the Bishop–Gromov volume comparison.

Theorem 4.3. • *Fujita [2015]: Let X be a Kähler–Einstein Fano manifold in dimension n, then $(-K_X)^n \leq (n+1)^n$, and equality holds if and only if X is isomorphic to $\mathbb{P}^n$.*

It is very likely that similar results to Theorem 4.2 can be established for most families of Fano threefolds, and some classes of higher dimensional Fano manifolds with large anti-canonical volume. There is a related conjecture on a local analogue of Theorem 4.3. For more on this see Spotti and Sun [2017].

Now we move on to discuss general abstract results in higher dimension concerning moduli space of Kähler–Einstein/K-stable manifolds. As applications of Chen, Donaldson, and Sun [2015a,b,c] we have

- Odaka [2012a] and Donaldson [2015]: The moduli space of K-stable Fano manifolds with discrete automorphism group is Zariski open.
- Spotti, Sun, and Yao [2016]: A smoothable $\mathbb{Q}$-Fano variety admits a singular Kähler–Einstein metric if and only if it is K-stable.
- C. Li, X. Wang, and Xu [2014] and Odaka [2015]: The Gromov–Hausdorff compactification of moduli space of Kähler–Einstein Fano manifolds in a fixed dimension is naturally a proper separated algebraic space.

We remark that one important technical aspect is still open, namely, the *projectivity* of the moduli space. There is a well-defined CM line bundle on the moduli space. Over the locus parametrizing smooth Fano manifolds it admits a natural Wei–Peterson metric of positive curvature, and this locus has been shown to be quasi-projective C. Li, X. Wang, and Xu [2015].

There is also recent progress in the general type case. Here we already have a compactification using minimal model program, namely, the *KSBA moduli space*, where the boundary consists of varieties with semi-log-canonical singularities (in one dimension it is the same as being nodal). Odaka [2013, 2012b] proved that these are exactly the ones which are K-stable, and R. J. Berman and Guenancia [2014] established the existence of a unique singular Kähler–Einstein metric in a suitable weak sense. It is further shown by Song [2017] that under a KSBA degeneration, the Kähler–Einstein metric on smooth fibers converges in the pointed Gromov–Hausdorff sense to the metric completion of the log terminal locus on the central fiber. So far the proof uses deep results in algebraic geometry but one certainly hopes for a more differential-geometric theory in the future.

5 Optimal degenerations

Let (X, L, ω) be a polarized Kähler manifold with $[\omega] = 2\pi c_1(L)$. In this section we focus on two natural geometric flows emanating from ω. We first consider the *Ricci flow*

$$\frac{\partial}{\partial t}\omega(t) = \omega(t) - Ric(\omega(t)) \tag{5-1}$$

and restrict to the case when X is Fano and $L = K_X^{-1}$. This case is immediately related to the Yau–Tian–Donaldson conjecture; the general case is also interesting and is related to the analytic minimal model program, but is beyond the scope of this article.

Clearly a fixed point of (5-1) is exactly a Kähler–Einstein metric. There are also self-similar solutions to (5-1), i.e., solutions $\omega(t)$ that evolve by holomorphic transformations of X. These correspond to *Ricci solitons*, which are governed by the equation $Ric(\omega) = \omega + \mathfrak{L}_V\omega$, where V is a holomorphic vector field on X.

It is well-known that in our case a smooth solution $\omega(t)$ exists for all time $t > 0$ with $[\omega(t)] = 2\pi c_1(K_X^{-1})$. A folklore conjecture, usually referred to as the *Hamilton–Tian conjecture*, states that as $t \to \infty$ by passing to subsequence one should obtain Gromov–Hausdorff limits which are Ricci solitons off a singular set of small size. This is now confirmed by Chen and B. Wang [2014].

Theorem 5.1 (Chen and B. Wang [ibid.]). *As $t \to \infty$, by passing to a subsequence $(X, \omega(t))$ converges in the Gromov–Hausdorff sense to a $\mathbb{Q}$-Fano variety endowed with a singular Kähler–Ricci soliton metric $(X_\infty, V_\infty, \omega_\infty)$.*

This can be viewed as a generalization of the Cheeger–Colding theory and the results in Section 2 to the parabolic case. The proof makes use of the deep results of Perelman. In connecting with algebraic geometry we have

Theorem 5.2 (Chen, Sun, and B. Wang [2015]).

- *As $t \to \infty$, there is a unique Gromov–Hausdorff limit $(X_\infty, V_\infty, \omega_\infty)$.*
- *If X is K-stable, then X_∞ is isomorphic to X, $V_\infty = 0$, and ω_∞ is a smooth Kähler–Einstein metric on X_∞.*
- *If X is K-unstable, then the flow $\omega(t)$ defines a unique degeneration of X to a $\mathbb{Q}$-Fano variety $\bar{X}$ with a holomorphic vector field $\bar{V}$, and there is an equivariant degeneration from $(\bar{X}, \bar{V})$ to (X_∞, V_∞).*

The second item gives an alternative proof of the Yau–Tian–Donaldson conjecture for Fano manifolds. The third item requires slightly more explanation. The flow $\omega(t)$ induces a family of L^2 norms $\|\cdot\|_t$ on $H^0(X, K_X^{-k})$ for all k, and yields a notion of *degree* of a section $s \in H^0(X, L^k)$ by setting

$$d(s) = \lim_{t\to\infty} \frac{1}{t} \log \|s\|_t.$$

The main point in Chen, Sun, and B. Wang [ibid.] is that this is well-defined and gives a filtration of the homogeneous coordinate ring of (X, L); moreover, the associated graded ring defines a normal projective variety $\bar{X}$, and the grading determines a holomorphic vector field $\bar{V}$. So $(\bar{X}, \bar{V})$ is canonically determined by the flow $\omega(t)$, hence by the initial metric ω.

It is conjectured in Chen, Sun, and B. Wang [ibid.] that when X is K-unstable, both $(\bar{X}, \bar{V})$ and $(X_\infty, V_\infty, \omega_\infty)$ are invariants of X itself, i.e. are independent of the initial metric ω, and $(\bar{X}, \bar{V})$ defines a unique *optimal degeneration* of X. By W. He [2016], R. J. Berman and Nystrom [2014], and Dervan and Székelyhidi [2016] we know a partial answer to this, and it seems possible to eventually obtain a purely algebro-geometric characterization.

Now we turn to discuss the *Calabi flow*, starting from an initial metric ω with $[\omega] = 2\pi c_1(L)$. The equation takes the form

$$\frac{\partial}{\partial t}\omega(t) = -\Delta_{\bar\partial} Ric(\omega(t))(= -i\bar\partial\partial S(\omega(t))) \tag{5-2}$$

A fixed point is a Kähler metric with constant scalar curvature, and a self-similar solution is an extremal Kähler metric. From the viewpoint of the infinite dimensional moment map picture of Fujiki–Donaldson, the Calabi flow has an interesting geometric meaning, at least on a formal level. First it is the negative gradient flow of a geodesically convex functional (the *Mabuchi functional*) on the Kähler class $\mathcal{H}$, so it decreases a natural distance function on $\mathcal{H}$ (c.f. Calabi and Chen [2002]). If we transform it to a flow of integrable almost complex structures compatible with a fixed symplectic form then it is also the negative gradient flow of the *Calabi functional*.

A main difficulty in the study of Calabi flow (c.f. Chen and W. Y. He [2008]) arises from the fact it is a *fourth* order geometric evolution equation and the usual technique of maximum principle does not apply directly. There has been little progress on the problem of general long time existence. If we put aside all the analytic difficulties, then we do have a picture on the asymptotic behavior of the flow, analogous to Theorem 5.2.

Theorem 5.3 (Chen, Sun, and B. Wang [2015])**.** *Let $\omega(t)(t \in [0,\infty))$ be a smooth solution of the Calabi flow in the class $2\pi c_1(L)$, and assume the Riemannian curvature of $\omega(t)$ and the diameter are uniformly bounded for all t. Then*

- *There is a unique Gromov–Hausdorff limit $(X_\infty, L_\infty, V_\infty, \omega_\infty)$, where ω_∞ is a smooth extremal Kähler metric on a smooth projective variety X_∞ with $[\omega_\infty] = 2\pi c_1(L_\infty)$, and $V_\infty = \nabla S(\omega_\infty)$ is a holomorphic vector field.*

- *If (X, L) is K-stable, then (X_∞, L_∞) is isomorphic to (X, L), $V_\infty = 0$, and ω_∞ is a constant scalar curvature Kähler metric on X.*

- *If (X, L) is K-unstable, then it gives rise to an optimal degeneration of (X, L) to $(\bar X, \bar L, \bar V)$, which minimizes the normalized Donaldson–Futaki invariant, and there is an equivariant degeneration from $(\bar X, \bar L, \bar V)$ to $(X_\infty, L_\infty, V_\infty)$.*

There is also a generalized statement for extremal Kähler metrics Chen, Sun, and B. Wang [ibid.]. Notice in general we should not expect the Calabi flow to satisfy the strong geometric hypothesis in Theorem 5.3. First, the curvature may blow up and singularities can form, similar to the case of Ricci flow; second, the diameter can go to infinity (c.f. Székelyhidi [2009]) and collapsing may happen. The second issue is related to the folklore expectation that one may need to strengthen the notion of K-stability to certain *uniform*

K-stability in the statement of Yau–Tian–Donaldson conjecture. Nevertheless we expect Theorem 5.3 will lead to existence results of extremal Kähler metrics in concrete cases.

There are also recent progress on analytic study of the asymptotic behavior on the Calabi flow, related to the conjectural picture described by Donaldson [2004]. On one hand, in complex dimension two, H. Li, B. Wang, and Zheng [2015] proved that, assuming the existence of a constant scalar curvature Kähler metric ω_0 in the class $2\pi c_1(L)$, if a Calabi flow $\omega(t)$ in $2\pi c_1(L)$ exists for $t \in [0,\infty)$, then as $t \to \infty$ the flow must converge to ω_0, modulo holomorphic transformations of X. On the other hand, R. J. Berman, Darvas, and Lu [2017] proved a dichotomy for the behavior of *weak solution* to the Calabi flow in the sense of Streets [2014], to the effect that it either diverges to infinity with respect to a natural distance on $\mathcal{H}$, or it converges to a weak minimizer of the Mabuchi functional in a suitable sense. Finally, in Chen and Sun [2014] Calabi flow on a small complex structure deformation of a constant scalar curvature Kähler manifold is studied, which also leads to a generalized uniqueness result.

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