Collapsing of Calabi-Yau metrics and degeneration of complex structures

Song Sun*

UC Berkeley

November 7, 2019

Abstract

This is a survey article on recent study of collapsing of Calabi-Yau metrics under complex structure degenerations and related questions. An important role is played by the construction of possibly incomplete Calabi-Yau metrics with symmetry.

1 Introduction

A Calabi-Yau metric is by definition a Ricci-flat Kähler metric. Yau's solution of the Calabi conjecture [22] produces a unique Calabi-Yau metric in each Kähler class on a compact Kähler manifold X with vanishing first Chern class. These metrics have local holonomy group contained in $SU(n)$. They are fundamental objects in geometry.

Yau's original construction is based on solving a fully nonlinear complex Monge-Ampère equation via a priori estimates. A typical example is when X is an n dimensional smooth projective variety with trivial canonical bundle. In this case one can often write down explicitly a holomorphic volume form Ω. A Calabi-Yau metric ω solves the equation $\omega^n = C\Omega\wedge\bar{\Omega}$ for some constant C, but the solution ω in general can not be explicitly written down. Yau's result then generates an interesting question on understanding the geometry

*sosun@berkeley.edu. Partially supported by a Simons Collaboration Grant on Special Holonomy in Geometry, Analysis and Physics (No,488633, S.S.), and NSF grant DMS-1916520

of these metrics. Compact Ricci-flat metrics are difficult objects to visualize, partly because they can not admit non-trivial continuous symmetries.

Notice once the differentiable structure on a compact manifold X is fixed, Calabi-Yau metrics on X are naturally parametrized by two sets of algebraic data: the complex structure J and the Kähler class $\alpha \in H^2(X;\mathbb{R})$. By moving towards the boundary of moduli space one can often gain geometric description of these metrics. In connecting with algebraic geometry there are two important special classes of degenerations.

- **Kähler degenerations**. We work on a fixed underlying compact complex manifold X. The Kähler cone $\mathcal{C}$ is an open convex cone in $H^{1,1}(X;\mathbb{R})$. Given a sequence $\alpha_i \in \mathcal{C}$ which limits to a point α_∞ on $\partial\mathcal{C}$, we then obtain a sequence of degenerating Calabi-Yau metrics $\omega_i \in \alpha_i$. A typical situation arises as follows. Suppose $\pi : X \to Y$ is a holomorphic map onto a possibly singular variety Y which contracts at least one non-trivial holomorphic cycle. Given a Kähler class α on X and a Kähler class α' on Y, then we can take positive numbers $t_i \to 0$ and take $\alpha_i = t_i\alpha + \pi^*\alpha'$.

- **Complex degenerations**. This involves flat polarized degenerating families $\pi : (\mathcal{X}, \mathcal{L}) \to \Delta$, where $\Delta \subset \mathbb{C}$ is a disk. A typical situation arises as follows. Consider Calabi-Yau submanifold $X_{f_1,\cdots,f_k}$ in $\mathbb{CP}^N$ defined as the complete intersection of hypersurfaces $\{f_1 = \cdots f_k = 0\}$ with $\sum_j \deg(f_j) = N+2$. Then varying the coefficients of the defining equations can yield complex degenerations. A special case involving *maximal degenerations* is related to the SYZ formulation of mirror symmetry.

In terms of Riemannian geometry, by Gromov's precompactness theorem, for a sequence of n dimensional Calabi-Yau manifolds (X_i, g_i) with base points p_i, we can always pass to a subsequence and obtain a (pointed) Gromov-Hausdorff limit (X_∞, p_∞), which is a possibly non-compact length space. Then we may distinguish the geometric behavior into two cases. We say this sequence is *volume non-collapsing* if

$$\mathrm{Vol}(B_{g_i}(p_i, 1)) \geq \kappa \tag{1.1}$$

for some $\kappa > 0$ for all i; otherwise we say this sequence is *volume collapsing* In the non-collapsing situation one can appeal to the well-developed Cheeger-Colding regularity theory, which in particular implies that X_∞ has the structure of a smooth Calabi-Yau manifold off a closed subset of small Hausdorff dimension. In the collapsing case when we assume the sectional curvature of g_i is uniformly bounded, then the theory of Cheeger-Fukaya-Gromov asserts that the collapsing is essentially along certain nilpotent directions. Only in real four dimensions there is a Cheeger-Tian ϵ-regularity result for collapsing Einstein metrics, which in particular implies that the collapsing is with bounded curvature away from finitely many points, provided that the L^2 norm of curvature is uniformly bounded. A general foundational theory on collapsing of Einstein metrics is yet to be developed.

Since the Ricci-flat equation is invariant under rescaling, we can rescale the sequence g_i by a sequence of positive numbers λ_i, then we may obtain different Gromov-Hausdorff limits. If $\lim_i \lambda_i = \infty$ then we are moving into a smaller scale; if $\lim_i \lambda_i = 0$ then we are moving into a larger scale; if $\log \lambda_i$ stays bounded, then we are essentially not changing the scale. This multi-scale phenomenon is an important aspect in our later discussion. Naively as common in geometric analytic contexts, one expects the structure of a *bubble tree* if we collect the limits in all possible scales.

Notice if the original manifolds X_i are compact, then there is a maximal scale defined by making the diameters of g_i to be a fixed constant. This scale is maximal in the sense that if we scale down further then we just get a point as the Gromov-Hausdorff limit. When we talk about volume non-collapsing without referring to scales, we always mean that (1.1) holds in the maximal scale. By Bishop-Gromov volume comparison theorem, if a sequence is volume non-collapsing in the maximal scale, then it is also volume non-collapsing in any smaller scales.

In this article we shall explain some recent progress towards understanding the metric collapsing of Calabi-Yau metrics and the connection with complex degenerations in algebraic geometry. In particular we shall explain one situation [19] where definitive statements can be said in all dimensions, and the construction of local models for large complex structure limits in complex dimension 3 [16]. In complex dimension 2 on K3 surfaces Calabi-Yau metrics are *hyperkähler*, and this is special in the sense that one can mix together complex degenerations and Kähler degenerations in terms of *hyperkähler rotation*. In this case one can even forget the choice of complex structures and simply talk about the collapsing of Ricci-flat metrics. We will explain various geometric construction of collapsing on K3 manifolds, including the recent work [10] which is motivated by the study of complex degenerations.

We point out that due to its length this article is *not* supposed to be inclusive of all results on collapsing of Calabi-Yau metrics. For example we will not discuss the case of Kähler degenerations (except in complex dimension 2). There are many interesting recent work in this direction, and we refer to [21, 15, 12] and the references therein.

In the construction of [10, 19] a key role is played by Calabi-Yau metrics with torus symmetry. One philosophy for our study is that even though compact Ricci-flat metrics do not admit non-trivial continuous symmetries, when they become degenerate one often observes *approximate* symmetries. In the non-collapsing situation this is reflected by the fact that at every singularity there are metric tangent cones, which admit dilation symmetry; in the Kähler setting using the complex structure J the dilation symmetry is turned to Killing symmetry and this property underpins the study of singularities of non-collapsed limits of Kähler-Einstein metrics. In the collapsing case as mentioned above if we assume curvature is uniformly bounded then we see approximate nilpotent symmetry.

On the complex geometry side one can observe complex torus symmetry in the following local model degenerations. Let D be a complex manifold with a holomorphic volume form Ω_D. Given $k+1(k \geq 1)$ ample holomorphic line bundles $L_0, \cdots, L_k$ over D, we let E be the total space of the vector bundle $L_0 \oplus \cdots \oplus L_k$, and we fix a holomorphic section f of $L_0 \otimes \cdots \otimes L_k$, generic in the sense that $\{f = 0\}$ defines a smooth hypersurface in D. Notice E is endowed with a natural $(\mathbb{C}^*)^{k+1}$ action given by the $\mathbb{C}^*$ multiplication on each factor L_j. Let $\mathcal{N}$ be the subvariety in $E \times \mathbb{C}$ defined by the equation

$$s_0 \otimes \cdots \otimes s_k + tf(x) = 0, \tag{1.2}$$

Let $\mathcal{N}_t = \mathcal{N} \cap (E \times \{t\})$, then $\mathcal{N}_t$ is smooth and all isomorphic for $t \neq 0$ and $\mathcal{N}_0$ is the union of the hypersurfaces $\{s_j = 0\}$ in E. So we can view $\mathcal{N}$ as the total space of a degenerating family of complex manifolds. Using residues we can also explicitly write down a family holomorphic volume forms Ω_t on $\mathcal{N}_t(t \neq 0)$ and on the smooth locus of $\mathcal{N}_0$, in terms of Ω_D. In reality when we consider polarized degeneration of compact manifolds, we are not exactly in the above situation, but as explained in [19], to leading order in generic cases the above local model provides a good approximation.

Notice for all $t, \mathcal{N}_t$ is invariant under the action of $(\mathbb{C}^*)^k = \{(\lambda_0, \cdots, \lambda_k) \in (\mathbb{C}^*)^{k+1} | \lambda_0 \cdot \cdots \cdot \lambda_k = 1\}$, and this action preserves Ω_t. It then makes sense to ask for a $T^k \subset (\mathbb{C}^*)^k$ invariant Calabi-Yau metric ω_t on $\mathcal{N}_t$ for $t \neq 0$. In general one only expects ω_t to be incomplete and only defined on compact subsets of $\mathcal{N}_t$. Such family of metrics ω_t are important local models for understanding metric collapsing of complex structure degenerations for compact Calabi-Yau manifolds, and this serves as our starting point for the constructions in [10, 19]. Notice for $t \neq 0$, the $(\mathbb{C}^*)^k$ action on $\mathcal{N}_t$ is only free off the singular set of the total space $\mathcal{N}$, which is given by the union of $\Pi_{ij} = \{(x, [s_0, \cdots, s_k]) \in E | f(x) = 0, s_i = s_j = 0\}$ for all $i \neq j$.

This article is organized as follows. In Section 2 we review the classical Gibbons-Hawking construction in four dimensions, with an emphasis on some important examples of incomplete metrics including the Ooguri-Vafa metric and its recently found cousin in [10]. In Section 3 we discuss degeneration of hyperkähler metrics on K3 manifolds. In Section 4 we discuss the relationship between complex degenerations and collapsing of Calabi-Yau metrics in higher dimensions.

2 Four dimensional hyperkähler metrics with S^1 symmetry

2.1 Gibbons-Hawking construction

Fix the Euclidean space $\mathbb{R}^3$ with an orientation. Take a domain Q in $\mathbb{R}^3$ and a positive harmonic function V on Q. Then $*dV$ is a closed 2-form. Suppose

$$\frac{1}{2\pi}[*dV] \in H^2(Q; \mathbb{Z}), \tag{2.1}$$

then there is a principal $U(1)$ bundle P over Q and a connection 1-form $-\sqrt{-1}\theta$ on P with curvature $-\sqrt{-1} * dV$. Denote by $\pi : P \to Q$ the projection map. On P we can define an explicit Riemannian metric

$$g = V\pi^* g_{\mathbb{R}^3} + V^{-1}\theta \otimes \theta.$$

We fix an orthogonal splitting

$$\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R} \tag{2.2}$$

with complex coordinate $x + \sqrt{-1}y$ and real coordinate z, so that $g_{\mathbb{R}^3} = dx^2 + dy^2 + dz^2$ and $dx \wedge dy \wedge dz$ is compatible with the fixed orientation on $\mathbb{R}^3$. Then the complex valued 2-form

$$\Omega = -\sqrt{-1}(dx + \sqrt{-1}dy) \wedge (Vdz + \sqrt{-1}\theta) \tag{2.3}$$

is closed, hence it defines an integrable complex structure with respect to which Ω becomes holomorphic. Furthermore, g is Kähler with respect to this complex structure and the corresponding Kähler form is given by

$$\omega = Vdx \wedge dy + dz \wedge \theta. \tag{2.4}$$

The pair (ω, Ω) satisfies the complex Monge-Ampère equation

$$\omega^2 = \frac{1}{2}\Omega \wedge \bar{\Omega}. \tag{2.5}$$

In particular ω is a Calabi-Yau metric. Now varying the splitting $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ yields an S^2 family of parallel compatible complex structures, making g a *hyperkähler metric*. In other words, the holonomy group of g is contained in $\mathrm{SU}(2) = \mathrm{Sp}(1)$.

The natural S^1 action rotating the fibers of π preserves both forms ω and Ω. Conversely, any hyperkäher metric with an S^1 symmetry is locally (away from the fixed points) given by this construction. One should view the coordinates z and $x + \sqrt{-1}y$ as the real and complex moment map for the S^1 action with respect to the real and complex 2-forms ω and Ω, and V^{-1} as the norm squared of the corresponding Killing field.

To make interesting topology we should allow V to have singularities. Suppose $p_0 \in Q \subset \mathbb{R}^3$ and V is a positive harmonic function defined on $Q \setminus \{p_0\}$, then by Bôcher's theorem we can write

$$V = \frac{A}{2|p - p_0|} + F$$

for a constant A and a smooth harmonic function F on Q. To apply the Gibbons-Hawking construction on $Q \setminus \{p_0\}$, the integrality condition (2.1) forces A to be an integer. If $A = 1$ then we can smoothly compactify the corresponding hyperkähler metric on P by adding a point $\hat{p}_0$ which maps to p_0 under the projection π. The S^1 action can be extended so that this added point is a fixed point. The local model is given as follows. Consider the flat metric on $\mathbb{C}^2$ with the S^1 action

$$\lambda \cdot (z_1, z_2) = (\lambda^{-1} z_1, \lambda z_2). \tag{2.6}$$

This corresponds to the case $Q = \mathbb{R}^3$, $p_0 = 0$, and $V = \frac{1}{2|p|}$. The projection map is given by the Hopf projection

$$\pi : \mathbb{C}^2 \to \mathbb{C} \oplus \mathbb{R}; (z_1, z_2) \mapsto (z_1 z_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)). \tag{2.7}$$

In general for the above V with $A = 1$, one can locally use (z_1, z_2) as coordinates in a neighborhood of $\hat{p}_0$ and it is not difficult to see that the hyperkähler metric extends smoothly across $\hat{p}_0$. We can view the function V as a function across p_0 and satisfies the distributional equation

$$\Delta V = -2\pi \delta_{p_0}. \tag{2.8}$$

If $A > 1$ one can also compactify the space as an orbifold, locally of the form $\mathbb{C}^2/\mathbb{Z}_A$.

Now we choose two points $p_1 \neq p_2$ in $\mathbb{R}^3$ and apply the Gibbons-Hawking construction to the function on $\mathbb{R}^3$ of the form

$$V = \frac{1}{2|p - p_1|} + \frac{1}{2|p - p_2|}. \tag{2.9}$$

Then we obtain a hyperkähler metric which contains a 2-sphere S, given as the pre-image under π of the straight line segment p_1p_2 in $\mathbb{R}^3$ connecting p_1 and p_2. It has self-intersection -2. The resulting metric is known as the *Eguchi-Hanson space*, and the underlying differentiable manifold is diffeomorphic to T^*S^2. The underlying complex manifold depends on the choice of the complex structure, hence on the choice of the orthogonal splitting (2.2). If the segment p_1p_2 is not contained in the $\mathbb{R}$ factor of the splitting $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$, then the resulting complex manifold is a smooth quadric in $\mathbb{C}^3$. Otherwise the 2-sphere S is holomorphic and the underlying complex manifold is isomorphic to the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^1$. If we let p_1 and p_2 come together, then the 2-sphere S shrinks to a point and the limiting metric is the flat orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The latter can be seen as obtained by applying the Gibbons-Hawking construction to the function $\frac{1}{|p|}$ on $\mathbb{R}^3$.

It is straightforward to generalize this construction using $k \geq 2$ points, and obtain *multi-Eguchi-Hanson spaces.* For generic choice of complex structures the underlying complex manifold is a smooth hypersurface in $\mathbb{C}^3$ defined by the equation $x_1x_2 = f(x_3)$ where f is degree k polynomial. Letting points collide corresponds to contraction of 2-sphere's and in the extreme case when all the k points collide we obtain the flat orbifold $\mathbb{C}^2/\mathbb{Z}_k$.

Now notice that the above family of metrics admits an extra deformation. For example, for the flat metric on $\mathbb{C}^2$ we can consider instead the function

$$V_T = \frac{1}{2|p|} + T, \quad T > 0. \tag{2.10}$$

Then we obtain a family of hyperkähler metrics Z_T, and they correspond to homothetic rescaling of a single *Taub-NUT metric* g_{TN} which corresponds to $T = 1$. The underlying manifold is easily seen to be diffeomorphic to $\mathbb{R}^4$, but the geometry is quite different from the standard flat metric. At infinity, the length of the fibers of the Hopf projection (2.7) tends to a fixed constant, and the volume of a ball of radius R grows like $O(R^3)$. If we let $T \to \infty$, then the geometry corresponds to scaling down the fixed metric g_{TN}, and in the limit the space collapses to $\mathbb{R}^3$ with the flat metric in the Gromov-Hausdorff sense. However, this convergence is not with bounded curvature, due to the existence of the singular fibers of the Hopf projection. This also suggests a different way of interpreting (2.10) for $T \gg 1$, as a perturbation of the constant solution T by a fixed *singular* harmonic function, which has the effect of changing the topology from a trivial S^1 bundle to a non-trivial one.

Complex geometrically, if we fix a splitting $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$, and denote by π_1 the projection onto the $\mathbb{C}$ factor, then the composition $\pi_1 \circ \pi : Z_T \to \mathbb{C}$ is a holomorphic map, with generic fiber $\mathbb{C}^*$ and the fiber over 0 given by the singular curve $x_1 x_2 = 0$ in $\mathbb{C}^2$. Using this one can show that the underlying complex manifold of the Taub-NUT metric is the same as $\mathbb{C}^2$ (see [14]). Moreover, one can write down the Kähler potential of the metric in terms of the coordinates $\{z_1, z_2\}$ as in the above Hopf projection (2.7) (but notice these are no longer holomorphic coordinates if $T > 0$).

$$\phi = \frac{1}{2}(|z_1|^2 + |z_2|^2) + \frac{T}{4}(|z_1|^4 + |z_2|^4). \tag{2.11}$$

As before one can work with more than one singular points and obtain *multi-Taub-NUT spaces.*

It is worth mentioning that the Gibbons-Hawking construction easily extends to domains in a flat 3 manifold with a global parallel orthonormal frame. In the following we shall explain interesting examples of both incomplete and complete hyperkähler metrics with S^1 symmetry.

2.2 Ooguri-Vafa metric and its cousin

We first discuss the Ooguri-Vafa metric following [8, 10]. Here the domain will be an open subset of the product $S^1 \times \mathbb{R}^2$ with the flat metric. Let V be a Green's function with a singularity at a point $p_0 \in S^1 \times \{0\}$. One can think of V as the electric potential of a point charge at p_0. In other words, V satisfies the equation

$$\Delta V = -2\pi \delta_{p_0}. \tag{2.12}$$

There are many ways to see the existence of such function V, for example by a direct separation of variables method [19]. Notice V is certainly not unique since one can add any harmonic function on $\mathbb{R}^2$. For us the key extra property, which makes V essentially unique, is that

$$V(x, u) + \log |u| = O(e^{-\delta |u|}), \quad |u| \to \infty, \tag{2.13}$$

for some $\delta > 0$ and $(x, u) \in S^1 \times \mathbb{R}^2$. Intuitively in terms of electric potential this is clear since when we are far away from p_0 the S^1 fiber direction is negligible and the situation is approximated by a point charge on $\mathbb{R}^2$. The property (2.13) can be established more precisely from the construction of V.

Now we can consider the family of harmonic functions

$$V_T = V + T, \quad T > 0.$$

Then the domains Q_T on which V_T is positive exhaust $S^1 \times \mathbb{R}^2$ as T tends to infinity. Applying the Gibbons-Hawking construction to V_T we then obtain an incomplete S^1 invariant hyperkähler metric on the fibration $\pi : X_T \to Q_T$. Denote by $\pi_{\mathbb{C}} : S^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ the natural projection map. Choosing the obvious splitting $\mathbb{R} \oplus \mathbb{C}$ on the universal cover of $S^1 \times \mathbb{R}^2$, then the composed projection $\pi_{\mathbb{C}} \circ \pi$ is holomorphic and this makes X_T an *elliptic fibration* over a domain in $\mathbb{C}$, with a singular fiber over 0, and the monodromy around 0 is

$$I_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{Z}). \tag{2.14}$$

The singular fiber is called an I_1 fiber in terms of the Kodaira classification.

One can perform a *hyperkähler rotation*, i.e., choose a different complex structure, so that $\pi_{\mathbb{C}} \circ \pi : X_T \to \mathbb{R}^2$ becomes a *special Lagrangian* fibration over a domain. For example, we can choose the splitting on the universal cover $\mathbb{R} \times \mathbb{R}^2$ as $\mathbb{C} \oplus \mathbb{R}$ (by fixing a line $\mathbb{R} \subset \mathbb{R}^2$). Then at the same time X_T is holomorphic cylinder fibration over a cylinder.

Now we let $T \to \infty$. The limit geometry depends on the choice of scales. If we rescale so that the diameter is fixed, then we can see collapsing onto a metric on the unit disc in $\mathbb{R}^2$, given in the form

$$g_\infty = (-\log |u|)(dx^2 + dy^2), \qquad u = x + \sqrt{-1}y. \tag{2.15}$$

This metric is singular at the origin but the singularity is very mild. For example, its tangent cone is the flat $\mathbb{R}^2$. If we rescale so that the volume is fixed, then the metrics collapse to the flat $\mathbb{R}^2$, but the convergence is with bounded curvature only away from the origin. Notice in a very small neighborhood of p_0, V_T is given by $\frac{1}{2|p-p_0|} + T + F$ for a fixed smooth function F, so if we rescale further in a neighborhood of the fixed point of the S^1 action, then we obtain the Taub-NUT space as a *bubble*. In this sense we see the Taub-NUT metric naturally embeds into the Ooguri-Vafa space.

Just as before, when T is large, we can view V_T as a singular perturbation of the constant solution T, and the singularity is responsible for creating interesting topologies.

In [10] similar construction was applied to $T^2 \times \mathbb{R}$ instead of $S^1 \times \mathbb{R}^2$, with $p_0 \in T^2 \times \{0\}$. Here T^2 is endowed with a fixed flat metric so it also has a moduli. We normalize so that the area of T^2 is 1. There is a Green's function V with pole at p_0 and with asymptotics

$$V = -2\pi|z| + O(e^{-\delta z}) \tag{2.16}$$

where $\delta > 0$. Again applying the Gibbons-Hawking construction to the family of harmonic functions $V_T = V + T$ we obtain a family of incomplete hyperkähler metrics Y_T.

Again the underlying complex manifold has different models depending on the choice of complex structure. For example if we choose the splitting of the universal cover $\mathbb{R}^2 \times \mathbb{R}$ as $\mathbb{C} \oplus \mathbb{R}$, then the projection map to T^2 is holomorphic and we obtain a holomorphic cylinder fibration. The singular fiber becomes union of two discs attached at one point. One can also perform a hyperkähler rotation to make Y_T bi-holomorphic to the complement of an I_1 singular fiber in an elliptic fibration. Topologically we can always view Y_T as a singular fibration over an interval in $\mathbb{R}$, and the generic fiber is a principal S^1 bundle over T^2, which have different degrees on the two ends, and the singular fiber is a pinched nilmanifold, i.e. a singular S^1 fibration over T^2.

Now let $T \to \infty$. If we fix the diameter then the metrics collapse to a one dimensional interval with standard metric. But the convergence is only with bounded curvature away from the limit point of the fixed point of S^1 action. If we rescale suitably around this point then we obtain the Taub-NUT metric as a bubble. Hence the Taub-NUT metric also naturally embeds into the space Y_T. Again for T large, V_T is a singular perturbation of the constant solution T, and the singularity creates topology.

Notice in the above discussion in both cases one may use instead Green's function with more than one singular point to construct slightly more general spaces. In Section 3 we shall see the use of these incomplete metrics in the construction of collapsing families of hyerkähler metrics on K3 manifolds.

2.3 Model ends for complete metrics

We can also make hyperkähler metrics with both an complete and incomplete end, and these can model the infinity of certain complete hyperkähler metrics.

On $T^2 \times [1, \infty)$ and we can simply choose the function $V = 2\pi kz$ for some positive integer k, then we obtain a metric $\mathcal{C}$ with both a complete end and boundary. One can choose a complex structure so that this becomes a holomorphic fibration over T^2 with fibers isomorphic to punctured discs. Furthermore, the underlying complex manifold can be identified with a submanifold in the total space of a degree k holomorphic hermitian line bundle L (whose curvature is the flat metric on T^2), as the complement of the zero section in the unit disc bundle. Notice in this case the choice of the connection 1-form is not unique and this corresponds to choosing different holomorphic structures on L. It follows from (2.16) that this space models the boundary of the space Y_T in Section 2.2 for T large.

On the other hand, using the above holomorphic description, one can write down its Kähler potential (see for example [10])

$$\phi(\xi) = C(-\log |\xi|^2)^{\frac{3}{2}}, \quad \xi \in L, \quad C > 0, \tag{2.17}$$

where we have fixed a hermitian metric $|\cdot|$ on L whose curvature is the flat metric on T^2. This corresponds to the Calabi ansatz for constructing Kähler-Einstein metrics on line bundles over Kähler-Einstein manifolds. One can generalize this to higher dimensional case, replacing T^2 by an n dimensional Calabi-Yau manifold, L by an ample hermitian line bundle with curvature form given by the Calabi-Yau metrics, and $\frac{3}{2}$ by $\frac{n+1}{n}$. For simplicity we call these spaces *Calabi model spaces.*

These spaces can serve as models at infinity of a class of complete Ricci-flat Kähler metrics. The later were constructed by Tian-Yau [20], on the complement of a smooth anti-canonical divisor D in a smooth Fano manifold M. They exhibit interesting geometric behavior at infinity. As we move towards infinity, certain directions expand and other directions shrink. If we scale down the metric based at a fixed point, then the manifold collapses to $[0, \infty)$, but the collapsing is of *multi-scale.* This type of Tian-Yau spaces should not be confused with another type of complete Calabi-Yau metrics also constructed in [20], on the complement of a smooth anti-canonical divisor in a projective manifold, when the divisor has trivial normal bundle. These metrics are *asymptotically cylindrical* so do not exhibit the above multi-scale behavior at infinity.

Now consider instead $S^1 \times \mathbb{R}^2$, and use the harmonic function $V = k \log |u|$ for a positive integer k. Then it is positive for $|u| > 1$. We also obtain a hyperkähler metric with a complete end and a boundary. One can choose a complex structure so that the underlying complex manifold is an elliptic fibration over a punctured disc in $\mathbb{R}^2 = \mathbb{C}$, with monodromy type I_k.

It is an interesting question whether these metrics can serve as model at infinity of a complete hyperkähler metric. What is known is that some $\mathbb{Z}_2$ quotients of them indeed do so, and they provide the model at infinity of complete hyperkähler metrics constructed by Hein [9]. Choosing a suitable complex structure, the underlying complex manifold is the complement of an I_k^* singular fiber in a rational elliptic surface.

3 Hyperkähler metrics on K3 manifolds

A K3 manifold $\mathcal{K}$ is, by definition, a four dimensional oriented differentiable manifold underlying a complex K3 surface. It is known that the cup-product on $H^2(\mathcal{K};\mathbb{Z})$ has signature $(3,19)$. A hyperkähler metric g on $\mathcal{K}$ is by definition a Riemannian metric with holonomy group equal to $\mathrm{Sp}(1)=\mathrm{SU}(2)$. So choosing any compatible complex structure makes the metric a Calabi-Yau metric but we can also forget about the complex structures and study the moduli space of only the underlying Riemannian metrics.

Denote by $\mathcal{M}$ the space of all unit diameter hyperkähler metrics on $\mathcal{K}$ modulo diffeomorphisms. Given $g\in\mathcal{M}$, we obtain a three dimensional positive definite subspace in $H^2(\mathcal{K},\mathbb{R})$ given by $\mathbb{H}_g^+$, the space of cohomology classes of self-dual harmonic 2-forms on $\mathcal{K}$ with respect to g. This defines a period map

$$\mathcal{P}:\mathcal{M}\to\mathcal{D}=\Gamma\backslash\,\mathrm{O}(3,19)/(\mathrm{O}(3)\times\mathrm{O}(19)); g\mapsto\mathbb{H}_g^+, \tag{3.1}$$

where Γ is the automorphism group of the lattice $H^2(\mathcal{K};\mathbb{Z})$. The global Torelli theorem states that $\mathcal{P}$ is injective, and is surjective if we include all the possible orbifold degenerations of the hyperkähler metrics. The points in $\mathcal{D}\setminus Im(\mathcal{P})$ correspond precisely to the non-collapsing Gromov-Hausdorff limits of elements in $\mathcal{M}$.

A typical example of orbifold degenerations is given by the converse of the Kummer construction. Take a flat four torus T^4 and let X be its quotient by the involution $x\mapsto -x$. Then X has 16 orbifold singularities of type $\mathbb{R}^4/\mathbb{Z}_2$. Notice the Eguchi-Hanson spaces are asymptotic to the cone $\mathbb{R}^4/\mathbb{Z}_2$ at infinity so we can rescale the Eguchi-Hanson spaces suitably and glue 16 of them to the singularities of X, and then perturb to a genuine hyperkähler metric using the implicit function theorem. This is a typical example of a *gluing construction*. It turns out that under the period map $\mathcal{P}$, this construction can produce an open neighborhood $\mathcal{U}$ of the point in $\mathcal{D}\setminus Im(\mathcal{P})$ corresponding to X. There are different complex geometric models depending on the choice of complex structures. One model is given by taking a complex structure on T^4 and consider a minimal resolution of singularities $\mathcal{K} \to X$. Varying the Kähler classes on $\mathcal{K}$ so that the area of these exceptional curves tend to zero gives rise to Kähler degenerations, and these give certain slices in the above $\mathcal{U}$. On the other hand, we can take a Kummer quartic X_0 in $\mathbb{CP}^3$ with 16 nodes; then X_0 can be viewed as $T^4/\mathbb{Z}_2$ for suitable choices of complex structures on T^4. We can fix the Kähler class to be the cohomology class of the Fubini-Study metric, and deform the quartic to a smooth quartic, whose underlying differential manifold is a K3 manifold $\mathcal{K}$. Reversing this procedure we then obtain complex structure degenerations.

Now given a sequence of hyperkähler metrics $g_j \in \mathcal{M}$ which are volume collapsing, we can rescale the volume measure of g_j to a probability measure $\nu_j = \mathrm{Vol}(\mathcal{K}, g_j)^{-1}\mathrm{dVol}_{g_j}$. By passing to a subsequence we may take a *measured Gromov-Hausdorff limit* (X_∞, ν_∞) [1], where ν_∞ is the limit probability measure. In this case, the period $\mathcal{P}(g_j)$ has to diverge in $\mathcal{D}$. In reality, the limit X_∞ can be of dimension 1, 2, or 3, and all these cases do occur. This can be easily seen using the Kummer construction again: taking a sequence of 4 dimensional flat tori collapsing to a flat torus $\mathbb{T}^k$ of dimension $k \in \{1, 2, 3\}$, and applying the Kummer construction to each torus in this sequence, then by taking a diagonal subsequence we obtain a family of hyperkähler metrics on $\mathcal{K}$ that collapse to $\mathbb{T}^k/\mathbb{Z}_2$. Notice as topological spaces, $\mathbb{T}^1/\mathbb{Z}_2$ is a one dimensional interval and $\mathbb{T}^2/\mathbb{Z}_2$ is a 2 dimensional sphere.

There are more complicated collapsing behaviors. The geometric description below is mostly obtained via a gluing construction. Namely, taking as building blocks some complete and incomplete hyperkähler metrics, glue them together via a cut-off to obtain approximately hyperkähler metrics, and then perturb to genuine hyperkähler metrics by a quantitative implicit function theorem. Technically speaking, it is convenient to use the description of hyperkähler condition not in terms of the Calabi-Yau equation on (ω, Ω), but in terms of a triple of symplectic forms $\omega_1, \omega_2, \omega_3$ satisfying the equation

$$\omega_i \wedge \omega_j = \frac{1}{3}(\omega_1^2 + \omega_2^2 + \omega_3^2)\delta_{ij}.$$

This has the advantage of making the gluing construction more flexible (see [5, 6]), and the price to pay is the loss of the information on complex structures. In Section 4 when we indeed want to understand the connection with complex degenerations, we are in a much more rigid situation.

The known constructions of collapsing hyperkähler metrics on K3 manifolds falls into the following cases.

Case 1: $\dim X_\infty = 3$.

In this case among all known examples the Gromov-Hausdorff limit is always the flat orbifold $T^3/\mathbb{Z}_2$. Foscolo [6] constructed examples such that the collapsing has curvature blowing up at $8+n$ points (including the 8 orbifold points) for any $0 \leq n \leq 16$. The construction is via a gluing argument which we briefly sketch here. Consider the projection $\tau : T^3 \to T^3/\mathbb{Z}_2$, and denote by $S \subset T^3$ the 8 points which map to the orbifold points under τ. One takes a Green's function V on the flat T^3 with positive poles at $2n$ points away from S which are invariant under τ, and with possible negative poles at S. Notice since T^3 is compact there is at least one negative pole. Applying the Gibbons-Hawking construction to the function $1+\epsilon V$ on $T^3/\mathbb{Z}_2$, for $\epsilon > 0$ small, produces an incomplete hyperkähler metric away from a neighborhood of the negative poles of V. Then one can glue in multi-Taub-NUT spaces and ALF gravitational instantons of dihedral type (which have cubic volume growth and tangent cone at infinity given by $\mathbb{R}^3/\mathbb{Z}_2$). The appearance of negative poles requires one to use gravitational instantons of negative mass, and these are known to exist, as the Atiyah-Hitchin space and its quotient. See [6] for more details.

Case 2: $\dim X_\infty = 2$.

The known constructions use an elliptic fibered K3 surface, i.e. a holomorphic fibration $\pi : \mathcal{K} \to \mathbb{CP}^1$. One takes a family of Kähler classes on $\mathcal{K}$ so that the area of the fibers tends to zero. These are special cases of Kähler degenerations mentioned in the introduction, but can be turned into complex degenerations via hyperkähler rotation (see [8, 18]). If we fix the diameter, then the hyperkähler metrics collapse to a singular metric on $\mathbb{CP}^1$, which is smooth away from the discriminant locus of π and the Ricci curvature agrees with the pull-back of the Weil-Petersson metric on the moduli space of elliptic curves.

Away from singular fibers the hyperkähler metrics are modeled on *semi-flat metrics* introduced in [7], which are flat when restricted to elliptic fibers and whose horizontal variation is characterized by the variation of the moduli of elliptic curves. Near singular fibers, there are various multi-scale collapsing behaviors. In the most generic case when π has exactly 24 singular fibers of type I_1, and around a neighborhood of each I_1 fiber the metric is modeled on the Ooguri-Vafa metric. A Taub-NUT metric bubbles off as rescaling limit. There are other types of singular fibers, and the bubbles are related to ALG gravitational instantons constructed by Hein [9]. We refer to [8, 21, 3] for details.

Case 3: $\dim X_\infty = 1$.

Then one can show that X_∞ must be the interval $[0,1]$ with the standard metric. There are two known types of collapsing behaviors via the gluing construction

(a). Collapsing along a generic T^3 fibration [2]. At the end points the curvature blows up, and if we rescale the metrics suitably, then the bubbles we obtain are Tian-Yau spaces which are asymptotically cylindrical.

(b). Collapsing along a generic nilmanifold fibration [10]. The collapsing is with uniformly bounded curvature away from the end points and an interior point t_*. At the end points the bubbles are Tian-Yau spaces which are asymptotic to the Calabi model spaces in Section 2.3. In the interior region the metric is modeled by the incomplete metric Y_T discussed in Section 2.2 (constructed using a Green's function with possibly more than one singular point). One can see various multi-scale collapsing phenomenon in this example. The construction in [10] is motivated by studying complex degenerations of K3 surfaces, more precisely, Type II polarized degenerations, but as explained above the proof in [10] uses gluing construction in terms of a triple of symplectic forms, so does not lead to a clear relationship with complex geometry.

Notice when the collapsing limit is of dimension 1, the metric structure is unique [11]. However the limit measure ν_∞ can have various possibilities. In Case (a), ν_∞ is proportional to the Lebesgue measure, but in Case (b), ν_∞ has only Hölder continuous density, and the singular points of the density function agrees with the points where curvature blows up.

It seems reasonable to expect at least topologically all the possible collapsing limits of $g_j \in \mathcal{M}$ belong to one of $T^3/\mathbb{Z}_2, T^2/\mathbb{Z}_2, T^1/\mathbb{Z}_2$. In the case when the limit is of dimension 1, one also obtains an interesting normalized limit measure and it is interesting to classify all the possible limit measures. Some preliminary progress has been made in [11].

It is also natural to connect the Riemannian geometric collapsing limits to compactifications of the locally symmetric space $\mathcal{D}$ (see for example [18]).

The first step would be to understand the behavior of periods under the above construction of examples. We refer to [18] for conjectures in this direction and partial progress.

4 Higher dimensional situation

4.1 Calabi-Yau metrics with T^k symmetry

As mentioned in the introduction, when we study complex structure degenerations of Calabi-Yau metrics it is natural to expect in certain regions they exhibit approximate symmetries. Calabi-Yau metrics with abelian symmetry have a reduction which satisfies a non-linear generalization of the Gibbons-Hawking ansatz. These equations were first derived by Matessi [17] and Zharkov [23].

Let (X, ω, Ω) be an n dimensional Calabi-Yau manifold with a $T^k(k \geq 1)$ action which is holomorphic and Hamiltonian. We first assume the action is free. Let $(z_1, \cdots, z_k) : X \to \mathbb{R}^k$ be the moment map, and let ξ_j be the Hamiltonian vector field generated by z_j. Then the quotient space X/T^k locally splits as $Q = D \times \mathbb{R}^k$, where D is the complex quotient (or symplectic reduction), and $\mathbb{R}^k$ is given the above moment map coordinates. We also obtain a family of Kähler forms $\tilde{\omega}$ on the complex quotient D parametrized by $(z_1, \dots z_k) \in \mathbb{R}^k$, $U(1)$ connection 1-forms $-\sqrt{-1}\Theta_j(j = 1, \cdots, k)$ over Q, and a positive definite $k \times k$ real symmetric matrix valued function $W = (W_{ij})$ on Q. The Kähler form can be written as

$$\omega = \sum_j dz_j \wedge \Theta_j + \tilde{\omega}$$

with

$$\Theta_i(\xi_j) = -\delta_{ij},$$

and

$$Jdz_j = \sum_l W^{jl}\Theta_l,$$

where (W^{ij}) is the inverse matrix of (W_{ij}). The Kähler condition becomes a system of equations

$$\begin{cases} \partial_{z_i}\partial_{z_j}\tilde{\omega} + d_D d_D^c W_{ij} = 0. & (4.1) \\ d_D\tilde{\omega} = 0 & (4.2) \\ \partial_{z_l} W_{ij} = \partial_{z_j} W_{il} & (4.3) \\ d\Theta_j = \partial_{z_j}\tilde{\omega} - \sum_l dz_l \wedge d_D W_{jl}. & (4.4) \end{cases}$$

From the last equation (4.4) we see that up to gauge equivalence and modulo flat connections, the connection Θ_j's are determined by $(\tilde{\omega}, W)$. So the most essential are the first three equations (4.1), (4.2), (4.3) on $(\tilde{\omega}, W)$. The second equation is the Kähler condition on $\tilde{\omega}$, and the third equation can be viewed as a *tropical* analogue of the Kähler condition for Riemannian metrics on $\mathbb{R}^k$.

Now the holomorphic volume form Ω also descends to a holomorphic volume form Ω_D on the local complex quotient, and the Calabi-Yau equation on X becomes

$$\frac{\tilde{\omega}^{n-k}}{(n-k)!} = \frac{(\sqrt{-1})^{(n-k)^2}}{2^{n-k}} \det(W_{ij}) \cdot \Omega_D \wedge \bar{\Omega}_D. \tag{4.5}$$

Conversely, given an $n-k$ dimensional complex manifold D with holomorphic volume form Ω_D, and $(\tilde{\omega}, W)$ satisfying (4.1) and (4.5), we can locally construct an n dimensional Calabi-Yau metric (X, ω, Ω) with a T^k action.

Now we make a few observations. When $k = 1$, we denote $z = z_1$, the matrix W becomes a positive function, and (4.1), (4.2), (4.3), and (4.5) combine to one equation on $\tilde{\omega}$:

$$\partial_z^2 \tilde{\omega} + d_D d_D^c \frac{2^{n-1} \tilde{\omega}^{n-1}}{(\sqrt{-1})^{(n-1)^2} \Omega_D \wedge \bar{\Omega}_D} = 0. \tag{4.6}$$

If $n = 2$ then the equation (4.6) reduces to a linear equation and we get back to the Gibbons-Hawking construction for the function

$$V = \frac{\tilde{\omega}}{\frac{\sqrt{-1}}{2} \Omega_D \wedge \bar{\Omega}_D}.$$

Notice in this case since we make a choice of preferred complex structure, the base $\mathbb{R}^3$ naturally splits as $\mathbb{R} \oplus \mathbb{C}$. When $n > 2$, (4.5) and (4.6) are still non-linear. We call them the *non-linear Gibbons-Hawking ansatz* for Calabi-Yau metrics with T^k(or S^1) symmetry.

As the local algebraic model in the Introduction suggests, in geometrically interesting situation it is important to allow the T^k action to have fixed points. This corresponds to adding distributional terms to the right hand side of (4.5) and (4.6). One can work out the generic models by a local study. For example when $k = 1$, there are situations where the fixed point set is of complex codimension 2 in X, and transverse to the fixed point set the S^1 action on $\mathbb{C}^2$ is modeled by (2.6). Then the discriminant locus (the image of the fixed point set under the quotient map) P is of the form

$H \times \{0\}$ where H is a smooth divisor in D and we assume $z = 0$ on P. The distributional equation is

$$(\partial_z^2 \tilde{\omega} + d_D d_D^c \frac{2^{n-1}\tilde{\omega}^{n-1}}{(\sqrt{-1})^{(n-1)^2}\Omega_D \wedge \bar{\Omega}_D}) \wedge dz = -2\pi \cdot \delta_P, \tag{4.7}$$

where δ_P is the current of integration along P. It is a non-linear analogue of (2.8). Similarly for $k > 1$ in interesting cases one should allow distributional terms on the right hand side of (4.5), where the discriminant locus could be of the form $H_1 \times H_2$, where H_1 is complex hypersurface in D and H_2 is the union of real hyperplanes in $\mathbb{R}^k$.

Notice unlike the classical Gibbons-Hawking construction, the above non-linear equations are still difficult to handle in general. What is important for our applications is that in the adiabatic situation, i.e., when the T^k orbit is very small, one can expect to use a singular perturbation method to obtain solutions. This is done in [16, 19] in special cases relevant to collapsing.

We briefly sketch the case $k = 1$ first, following [19]. Suppose we have a Calabi-Yau metric ω_D on D, then it is clear that $\omega_1(z) := T\omega_D$ satisfies (4.6) for any positive constant T. Suppose we can find a family of real closed $(1, 1)$ forms $\psi(z)$ on D parameterized by z, which satisfy the linear equation

$$(\partial_z^2 + \Delta_D)(\psi(z) \wedge dz) = -2\pi \cdot \delta_P.[1] \tag{4.8}$$

Then $\tilde{\omega}_T(z) := T\omega_D + \psi(z)$ can be viewed as an approximate solution to (4.7) if ψ is fixed and T is large. Then one tries to argue that for T sufficiently large $\omega_T(z)$ can be perturbed to a genuine solution of (4.7). Various technical difficulties arise, including

- Existence and regularity of ψ. In many cases it is not difficult to show the existence of ψ, and the local behavior of ψ near P is important for our construction. Naively one expects that to the leading order the situation is approximated by the flat model where $Q = \mathbb{C}^{n-1} \oplus \mathbb{R}$, $P = \mathbb{C}^{n-2} \oplus \{0\}$, and ψ is essentially the Green's function on the normal space $\mathbb{C} \oplus \mathbb{R}$. In reality for the purpose of doing analysis more refined information is needed.

- Singular perturbation theory. It seems difficult to directly work with the reduced equation (4.6). Instead one can define $W = Tr_{\omega_D}\tilde{\omega}_T + q(z)$

[1] In [19] the corresponding 3-form $\psi(z) \wedge dz$ is called a *Green's current* for the submanifold P.

(for any function $q(z)$), then the Kähler identities ensure (4.1) holds, so $(\tilde{\omega}_T, W)$ together defines an S^1 invariant Kähler metric ω_T. Due to the complicated regularity of ψ near P, such ω_T does not automatically extend smoothly across P. This is very different from the case of classical Gibbons-Hawking ansatz. But in the end one can show ω_T has sufficient regularity and is *approximately* Calabi-Yau in suitable weighted sense, and these allow for the application of implicit function theorems.

When D is compact this is done in [19], and has direct applications to the results to be explained in Section 4.2. The incomplete metrics can be viewed as analogues of the metric Y_T discussed in Section 3.2.

When $D = (\mathbb{C}^*)^2$ and H is the affine hypersurface given by $\{z_1 + z_2 = 1\}$, this is done in [16] and it gives rise to local models for *negative vertices* arising from large complex structure limits of Calabi-Yau 3-folds. The incomplete metrics can be viewed as analogues of the Ooguri-Vafa metric X_T discussed in Section 3.2.

For $k > 1$ one can perform similar analysis. Again in [16] the case $k = 2$, $n = 3$ is used to produce local models for *positive vertices* arising from large complex structure limits of Calabi-Yau 3-folds. These can be viewed as different analogues of the Ooguri-Vafa metrics. Moreover, [16] also constructed a complete Calabi-Yau metric on $\mathbb{C}^3$ with T^2 symmetry and with quartic volume growth. The T^2 quotient is $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{C}$, and the discriminant locus P is of the form $G \times \{0\}$, where G is a trivalent graph given by the cone over the points $(1,0), (0,1), (-1,-1) \in \mathbb{R}^2$. This metric on $\mathbb{C}^3$ can be viewed as a 3 dimensional generalization of the Taub-NUT metric on $\mathbb{C}^2$, and [16] shows that this metric indeed embeds into the local model for positive vertices, just as the Taub-NUT metric is embedded in the usual Ooguri-Vafa metric. The higher dimensional situation is expected to be similar but more complicated, and we refer to [16] for more detailed discussion and related interesting questions.

4.2 Type II degeneration of Calabi-Yau hypersurfaces

Let f_0, f_1, f be homogeneous polynomials in $n+2$ variables of degree $d_0 > 0$, $d_1 > 0$ and $d_0 + d_1 = n+2$ respectively. Let $\mathcal{X} \subset \mathbb{CP}^{n+1} \times \Delta$ be the polarized family of Calabi-Yau hypersurfaces X_t in $\mathbb{CP}^{n+1}$ defined by the equation $F_t(x) = 0$, where

$$F_t(x) \equiv f_0(x) f_1(x) + t f(x), \tag{4.9}$$

and t is the complex parameter on the unit disc $\Delta \subset \mathbb{C}$. The relative ample line bundle $\mathcal{L}$ comes from the natural $\mathcal{O}(1)$ bundle over $\mathbb{CP}^{n+1}$.

We further assume f_0, f_1, f are sufficiently general so that the following hold:

(i) $X_0 = Y_0 \cup Y_1$, where $Y_0 = \{f_0(x) = 0\}$ and $Y_1 = \{f_1(x) = 0\}$ are smooth hypersurfaces in $\mathbb{CP}^{n+1}$;

(ii) X_t is smooth for $t \neq 0$ and small;

(iii) $D = \{f_0(x) = f_1(x) = 0\}$ is a smooth complete intersection in $\mathbb{CP}^{n+1}$;

(iv) $H = \{f_0(x) = f_1(x) = f(x) = 0\}$ is a smooth complete intersection in $\mathbb{CP}^{n+1}$.

We call the above degenerations a *Type II degeneration* of Calabi-Yau hypersurfaces. The terminology is taken from the classification of degenerations of K3 surfaces (notice there is a difference since the total space $\mathcal{X}$ above is not smooth).

Now for $t \neq 0$ small we have the Calabi-Yau metrics $\omega_t \in 2\pi c_1(\mathcal{O}(1)|_{X_t})$. The main theorem in [19] gives a description of the geometric behavior of (X_t, ω_t) as $t \to 0$. It can be summarized as follows.

- If we normalize so that the metrics have diameter 1, then as $t \to 0$, the metrics collapse to the unit interval $([0,1], ds^2)$ in the Gromov-Hausdorff sense. If we renormalize the volume measure of ω_t to be a probability measure, then the limit measure $\nu = C \cdot R(s)ds$ where the density $R(s)$ is given explicitly as

$$R(s) = \begin{cases} (\frac{s}{d_0})^{\frac{n-1}{n+1}}, & s \in [0, \frac{d_0}{d_0+d_1}], \\ (\frac{1-s}{d_1})^{\frac{n-1}{n+1}}, & s \in [\frac{d_0}{d_0+d_1}, 1]. \end{cases} \tag{4.10}$$

- The collapsing is along a smooth fibration except $s = 0, 1, \frac{d_0}{d_0+d_1}$. If we rescale suitably at $s = 0$ ($s = 1$), then we obtain a bubble which is the Tian-Yau complete Ricci-flat metric [20] on $Y_0 \setminus D$ ($Y_1 \setminus D$).

- Suitable rescalings around points on $H \times \{t\} \subset X_t$ yield a bubble which is the product of the Taub-NUT space $\mathbb{C}^2$ with a flat $\mathbb{C}^{n-2}$.

The relationship between the metric collapsing and algebraic geometry can be seen at different scales. First in the maximal scale the collapsing limit is topologically identified with the dual intersectional complex of the algebraic limit X_0. Secondly the smooth locus of the algebraic limit X_0 can be recovered from suitable rescaling limits. The above result suggests a possible general connection between metric collapsing limits and algebraic limits, which generalizes the well-known conjectures in the case of maximal degenerations (see [8, 13]).

The above result is proved by a gluing construction. One key aspect is to obtain a neck region connecting the two ends of the Tian-Yau spaces on $Y_0 \setminus D$ and $Y_1 \setminus D$. Algebraically in a neighborhood of $D \subset X_0$, the family X_t does not admit symmetry, but to leading order it can be approximated by the model situation considered in the Introduction, with L_i given by the normal bundle of D in Y_i, and the section f of $L_0 \otimes L_1 = \mathcal{O}(n+2)|_D$ given by exactly by the polynomial f. Hence one can use the idea of singular perturbation suggested in Section 4.1 to construct the desired neck regions.

The actual gluing construction is more involved, since we need to work on the fixed algebraic family $\mathcal{X}$. One key point is that we need to perform certain birational modifications to $\mathcal{X}$ such that the central fiber consists of 3 components, two of them coming from proper transforms of Y_0 and Y_1, and a new component given by a projective compactification of the total space of $L_0 \otimes L_1$, on which the above neck metric lives.

From the analytic side, an important ingredient is the analysis on the Tian-Yau spaces, and this in turn relies on the linear analysis on the incomplete Calabi model spaces. In [10, 19] a Liouville theorem for harmonic functions is proved, which involves uniform asymptotics of special functions. It is an interesting question to make a more systematic study of the linear analysis on complete spaces with multi-scale asymptotic behaviors.

As a final remark, for more general degenerating family of Calabi-Yau manifolds, one can imagine a similar gluing construction using more complicated local models, for example the ones constructed in [16] (as briefly discussed in Section 4.1). However, when the Gromov-Hausdorff limit has dimension bigger than 1, its generic part is expected to carry a non-trivial metric. This is the main missing building block for the gluing construction, and it is an interesting question to determine it a priori from the algebro-geometric data. What makes the situation simpler in the above special case of Type II degenerations is exactly the fact that a posteriori the (measured) Gromov-Hausdorff limit has a relatively explicit form.

References

[1] J. Cheeger, T. Colding. *On the structure of spaces with Ricci curvature bounded below.* I, J. Differential Geom. 46 (1997), no. 3, 406–480.

[2] G. Chen, X. Chen. *Gravitational instantons with faster than quadratic curvature decay (II)*, arXiv.org:1508.07908, 2015

[3] G. Chen, J. Viaclovsky, R. Zhang. *Collapsing Ricci-flat metrics on elliptic K3 surfaces* . arXiv:1910.11321

[4] J. Cheeger, G. Tian. *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. 19 (2006), no. 2, 487–525.

[5] S. Donaldson. *Two-forms on four-manifolds and elliptic equations*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 153–172.

[6] L. Foscolo. *ALF gravitational instantons and collapsing Ricci-flat metrics on the K3 surface*, arXiv.org: 1603.06315, 2016.

[7] B. Greens, A. Shapere, C. Vafa, S-T. Yau. *Stringy cosmic strings and non- compact Calabi-Yau manifolds*, Nuclear Phys. B 337 (1990), no. 1, 1–36.

[8] M. Gross, P. Wilson. *Large complex structure limits of K3 surfaces*, J. Differential Geom. 55 (2000), no. 3, 475–546.

[9] H. Hein. *Gravitational instantons from rational elliptic surfaces*, J. Amer. Math. Soc. 25 (2012), no. 2, 355–393

[10] H. Hein, S. Sun, J. Viaclovsky, and R. Zhang, *Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces*, arXiv:1807.09367, 2018.

[11] S. Honda, S. Sun, R. Zhang. *A note on the collapsing geometry of hyperkähler four manifolds.* Science China Mathematics.

[12] H. Hein, V. Tosatti. *Higher-order estimates for collapsing Calabi-Yau metrics.* arXiv:1803.06697v1

[13] M. Kontsevich, Y. Soibelman. *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263

[14] C. LeBrun. *Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 297–304.

[15] Y. Li, *A gluing construction of collapsing Calabi-Yau metrics on K3 fibred 3-folds* , arXiv:1809.08120

[16] Y. Li, *SYZ geometry for Calabi-Yau 3-folds: Taub-NUT and Ooguri-Vafa type metrics*, arXiv:1902.08770

[17] D. Matessi. *Constructions of Calabi-Yau metrics and of special Lagrangian submanifolds* (PhD Thesis), Coventry (United Kingdom) : University of Warwick, 2001

[18] Y. Odaka, Y. Oshima *Collapsing K3 surfaces, tropical geometry and moduli compactifications of Satake, Morgan-Shalen type*, arXiv:1810.07685, 2018.

[19] S. Sun, R. Zhang, *Complex structure degenerations and collapsing of Calabi-Yau metrics*,arXiv:1906.03368

[20] G. Tian, S. Yau. *Complete Kähler manifolds with zero Ricci curvature.* I. J. Amer. Math. Soc. 3 (1990), no. 3, 579–609

[21] V. Tosatti. *Adiabatic limits of Ricci-flat Kahler metrics.* J. Differential Geom. 84 (2010), no.2, 427-453.

[22] S-T. Yau. *On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampere equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.

[23] I. Zharkov. *Limiting behavior of local Calabi-Yau metrics*, Adv. Theor. Math. Phys. 8 (2004), no. 3, 395–420.