Kähler-Einstein metrics

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The speaker is represented to talk about the paper

Kähler-Einstein metrics on Fano manifolds, I, II, III. Journal of the AMS (28) 183-278, 2015, Xiuxiong Chen, Simon Donaldson and Song Sun

We will also discuss some further development on related topics.

Plan:

- Background
- Main result
- Further developments

Background

Kähler geometry studies triples (g, J, ω), where g is a Riemannian metric, J is an almost complex structure, ω is a 2-form, that satisfy

- (algebraic condition) $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$
- (differential condition)
  - $d\omega = 0$ (ω is symplectic)
  - $N_J = 0$ (J is integrable).

A central theme in Kähler geometry is the connection between

differential geometry (g and ω)

and

(complex) algebraic geometry (J)

Complex geometry

Let X be a complex manifold. In local holomorphic coordinates $\{z_1, \cdots, z_n\}$, a Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

where $(h_{\alpha\bar{\beta}})$ is positive definite Hermitian and $d\omega = 0$.

Locally $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ for ϕ pluri-subharmonic $\rightsquigarrow$ several complex variables theory

If X is compact, a Kähler form is determined by $[\omega] \in H^2(X; \mathbb{R})$ up to variation of a potential function: $\omega + \sqrt{-1}\partial\bar{\partial}\phi$.

Standard example:

- $\mathbb{CP}^N$ is endowed with the Fubini-Study metric

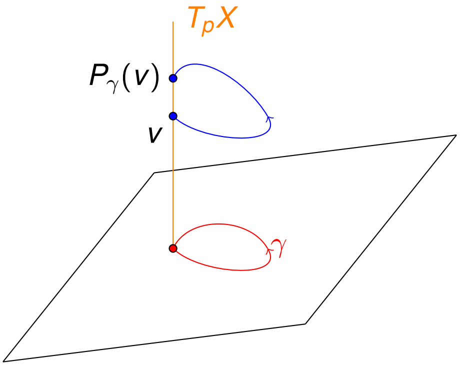
$$\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log|z|^2$$

- a smooth algebraic subvariety $X \subset \mathbb{CP}^N$ is endowed with the induced Kähler metric $\omega_{FS}|_X$.

Riemannian geometry

A Riemannian metric g is Kähler if it has holonomy group contained in $U(n)$.

Note: the complex structure J is (essentially) determined by the metric g.



Ricci curvature

For a Riemannian metric g, the Ricci curvature tensor at p is given in normal coordinates by

$$Ric = \sum_{i,j} R_{ij} dx_i dx_j$$

where

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} \sum_{i,j} R_{ij} x_i x_j + O(|x|^3)$$

For a Kähler metric ω, the corresponding Ricci form is given by the Chern curvature of K_X^{-1}, with respect to the induced hermitian metric ω^n.

Locally

$$Ric = -\sqrt{-1}\partial\bar{\partial}\log\det(h_{\alpha\bar{\beta}})$$

Chern-Weil theory

$$[Ric] = 2\pi c_1(X) \qquad \in H^2(X;\mathbb{R})$$

Canonical Kähler metrics

Let X be a compact complex manifold. A Kähler metric ω is Kähler-Einstein if

$$Ric(\omega) = \lambda\omega, \quad \lambda \in \mathbb{R}$$

Local form:

$$\det\left(\frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}\right) = e^{-\lambda\phi + f}$$

Necessary condition: $c_1(X)$ is definite, i.e., for some Kähler form β and $\lambda \in \mathbb{R}$,

$$c_1(X) = \lambda \cdot [\beta]$$

- More generally, we can study the equation

$$d^* Ric(\omega) = dRic(\omega) = 0$$

which is equivalent to $S(\omega) = \underline{S}$, i.e. ω having constant scalar curvature (CSCK) .

- CSCK metrics are special cases of Calabi's extremal Kähler metrics, which are defined as critical points of the functional

$$\tilde{\omega} \in [\omega] \mapsto \int_X |Riem(\tilde{\omega})|^2 \tilde{\omega}^n$$

Calabi conjecture (1954):

There is a Kähler-Einstein metric on X if and only if $c_1(X)$ is definite.

The Calabi conjecture is known when

- (Yau 1976) $c_1(X) = 0$ $\rightsquigarrow$ Calabi-Yau metric
- (Yau 1976, Aubin 1976) $c_1(X) < 0$.

In both cases, the Kähler-Einstein metric is uniquely determined by the cohomology class.

When $c_1(X) > 0$, X is a Fano manifold.

Kähler-Einstein metrics do not always exist. If exists, then

- Matsushima 1957: $Aut(X)$ must be reductive.
- Futaki 1983: The Futaki invariant on $Lie(Aut(X))$ must vanish.

Example: $Bl_p\mathbb{CP}^2$ does not admit a Kähler-Einstein metric.

Uniqueness (Bando-Mabuchi 1987): a Kähler-Einstein metric, if exists, is unique modulo the action of $K^{\mathbb{C}}$, where K is the isometry group.

There is a heuristic reason why NOT all Fano manifolds can admit Kähler-Einstein metrics:

Let $\mathcal{J}$ be the space of all Fano complex structures on X, and $\mathcal{K}$ be the space of Kähler-Einstein metrics with positive Ricci curvature on X.

The natural map $\Phi : \mathcal{K}/\mathit{Diff}(X) \to \mathcal{J}/\mathit{Diff}(X)$ is injective by uniqueness.

If Φ were surjective, then the fact that a Kähler-Einstein metric is a canonical metric would suggest Φ be a homeomorphism.

However, $\mathcal{J}/\mathit{Diff}(X)$ is well-known to be non-Hausdorff in general, but $\mathcal{K}/\mathit{Diff}(X)$ is always Hausdorff. Contradiction!

Main result

Theorem (Chen-Donaldson-S., 2012):

X admits a Kähler-Einstein metric $\Longleftrightarrow$ *X is K-stable*.

The “only if" direction was due to Tian, Donaldson, Stoppa, Mabuchi, Berman...

Solvability of a nonlinear PDE reduces to (essentially) finite dimensional algebra.

Compare Donaldson-Uhlenbeck-Yau theorem:

Existence of Hermitian-Yang-Mills connections $\Longleftrightarrow$ stable holomorphic bundles.

Theorem proves a conjecture that goes back to Yau for Kähler-Einstein metrics.

Yau-Tian-Donaldson Conjecture:

Let X be a compact complex manifold and L be an ample line bundle over X. Then there is a CSCK metric $\omega \in 2\pi c_1(L)$ if and only if (X, L) is K-stable.

Our result proves the conjecture when X is Fano and $L = K_X^{-1}$.

K-stability

Fix a Kähler metric ω on X. Consider the space of Kähler potentials

$$\mathcal{H} := \{\phi \in C^\infty(X) | \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Riemannian metric on $\mathcal{H}$: for $\delta\phi_1, \delta\phi_2 \in T_\phi\mathcal{H}$,

$$\langle \delta\phi_1, \delta\phi_2 \rangle_\phi = \int_X \delta\phi_1 \delta\phi_2 \omega_\phi^n$$

(Mabuchi, Semmes, Donaldson) $\mathcal{H}$ is formally a symmetric space with non-positive curvature.

Define the 1-form α on $\mathcal{H}$

$$\alpha(\delta\phi) = -\int_X \delta\phi(S(\omega_\phi) - \underline{S})\omega_\phi^n$$

α is closed, so $\alpha = dE$, where E is the Mabuchi functional.

- Critical points of E are CSCK metrics;
- E is geodesically convex on $\mathcal{H}$.
- $Aut_0(X)$ acts naturally on $\mathcal{H}$ and preserves α, so any $V \in Lie(Aut(X))$ generates a vector field v on $\mathcal{H}$ with $\mathcal{L}_v\alpha = d(\alpha(v)) = 0$
- Futaki invariant $Fut(V) = \alpha(v)$.

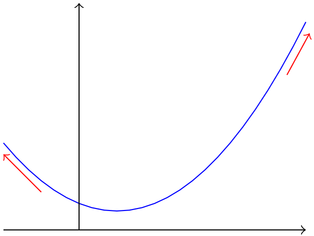
Formally, there exists a CSCK metric in $[\omega]$ if and only if

$$lim_{t\to\infty}\frac{dE}{dt}(\phi(t)) > 0$$

along any geodesic ray $\phi(t)$.

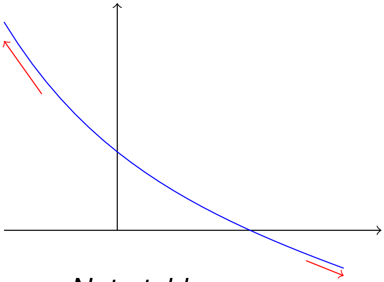
K-stability is an algebraization of this, by replacing geodesic rays in $\mathcal{H}$ by 1-parameter subgroups in $PGL(N+1;\mathbb{C})$.

Heuristic picture: convex functions on $\mathbb{R}$



Stable

Critical point exists



Not stable

No critical points

K-stability

K-stability is an algebro-geometric notion (Tian 1997, Donaldson 2002).

It generalizes the Hilbert-Mumford criterion for stability in geometric invariant theory.

It involves two key concepts:

Test Configurations and the Futaki invariant.

A Test Configuration $\mathcal{X}$ is a degeneration of (X, L) given by

- a projective embedding of $X \subset \mathbb{CP}^N$ using sections of L^r for some r, and
- a one parameter subgroup

$$\lambda : \mathbb{C}^* \to PGL(N+1; \mathbb{C})$$

Taking the limit of $\lambda(t).X$ as $t \to 0$, we obtain an algebraic scheme (X_0, L_0).

The Futaki invariant $Fut(\mathcal{X})$ is a number associated to $\mathcal{X}$.

It is defined in terms of the weights of the $\mathbb{C}^*$ action on the vector spaces $H^0(X_0, L_0^k)$ for $k \gg 1$.

It agrees with the original Futaki invariant when X_0 is smooth.

One can re-interprete $Fut(\mathcal{X})$ in terms intersection numbers on the natural compactification of $\mathcal{X}$ (X. Wang, Y. Odaka).

Definition: (X, L) is K-stable if $Fut(\mathcal{X}) \geq 0$ for all test configurations and "$=$" occurs only in trivial cases.

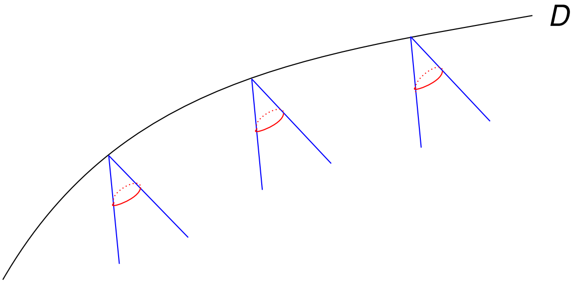
There are many variations of the notion of K-stability, on different levels of strength.

Ideas in the proof

Main strategy is via a continuous deformation.

- Start with a chosen initial Kähler metric
- Deform it towards a Kähler-Einstein metric via a prescribed process
- If the deformation breaks down at some point, then we produce a de-stabilizing test configuration.

In our case, we consider Kähler-Einstein metrics with cone singularities along a smooth divisor D, cut out by a holomorphic section of K_X^{-a} for some $a > 1$.



Families of equations:

$$Ric(\omega_\beta) = \mu_\beta \omega_\beta + 2\pi(1-\beta)[D], \quad \beta \in (0,1]$$

• The set of $\beta \in (0,1]$ such that the above equation is solvable is open (Donaldson 2011)

• $\mu_\beta < 0$ for $0 < \beta = 1/p \ll 1$, so one can solve the equation and get a starting point. Then we increase β.

• $\mu_\beta = 1$ when $\beta = 1$, which yields the desired Kähler-Einstein metric

If the deformation breaks down at some point β, then we need to construct a de-stabilizing test configuration

- Construct X_0;
- Construct a one parameter subgroup $\lambda(t)$;
- Show that the Futaki invariant is non-positive.

Take a sequence β_i increasing to β, we obtain a Gromov-Hausdorff limit of (X, ω_{β_i}). A priori the latter is only a metric space. Nevertheless, it is a natural candidate for X_0.

Needs to bridge with algebraic geometry. This is possible because

“Metric determines the complex structure"

For Kähler-Einstein metrics on Fano manifolds without cone singularities (Y_i, ω_i), the study of limit spaces fits into the Riemannian convergence theory developed by Cheeger-Colding in the 1990s.

Donaldson-S. (2012): algebraicity of Gromov-Hausdorff convergence:

- A Gromov-Hausdorff limit space Y_∞ is naturally a $\mathbb{Q}$-Fano variety
- the convergence can be realized in a flat family over a Hilbert scheme
- Y_∞ admits a Kähler-Einstein metric in a suitable sense

The proof involves a combination of the Cheeger-Colding theory and Hörmander L^2 theory.

Construction of X_0 requires an extension to the case with cone singularities.

The main new technical difficulty has to do with the codimension 2 cone singularities, especially the possible merging of singularities.

It also involves PDE theory for complex Monge-Ampere equation with singularities.

The construction of the one-parameter subgroup $\lambda(t)$ is a finite dimensional problem.

Question: Suppose V is a complex representation of $G = SL(N; \mathbb{C})$ and $v' \in \overline{G.v}$ for some $v, v' \in V$, can v' be approachable from v via a 1 PS subgroup of G?

In general the answer is NO, but the answer is YES if $Stab(v')$ is reductive (by the Luna slice theorem, c.f. Donaldson 2011).

In our setting one needs to prove that the Gromov-Hausdorff limit has reductive automorphism group. This is exactly the (generalized) Matsushima theorem.

The point is that the limit X_0 admits a Kähler-Einstein metric with cone singularities in a suitable sense. We then apply the uniqueness theorem of Berndtsson 2011 and Berman-Boucksom-Essydieux-Guedj-Zeriahi 2011.

The sign of Futaki invariant is a consequence of the KE $\Longrightarrow$ K-stability by Berman for $\mathbb{Q}$-Fano varieties.

Other proofs of the Kähler-Einstein result:

- Classical continuity path (Datar-Székelyhidi 2015).
- Ricci-flow (Chen-S.-Wang 2015) [based on the proof of the Hamilton-Tian conjecture (Chen-Wang 2015)].
- Variational approach (Berman-Boucksom-Jonsson 2015). This proves the result under the stronger assumption of uniform K-stability.

Extension and further developments

(1) Spotti-S.-Yao 2014: A K-stable smoothable $\mathbb{Q}$-Fano variety admits a Kähler-Einstein metric.

This was used by Li-Wang Xu 2014 and Odaka 2014 to construct moduli compactification for K-stable Fano manifolds, given by adding K-stable smoothable $\mathbb{Q}$-Fano varieties.

Explicit description of the compact moduli spaces are known in dimension 2 (Mabuchi-Mukai 1993, Odaka-Spotti-S. 2012) and some sporadic higher dimensional examples (Spotti-Sun 2017, Liu-Xu 2017, Liu 2021)

(2) General $\mathbb{Q}$-Fano varieties:

C. Li 2019: uniform K-stability implies existence of a weak Kähler-Einstein potential

It remains an open question to show that a weak Kähler-Einstein *potential* is indeed a Kähler-Einstein *metric*.

There are recent progress following the PDE estimates of Guo-Phong-Tong 2021.

(3) Algebro-geometric study of K-stability has seen major advances through the work of Odaka, Li, Wang, Xu, Fujita, Blum, Liu, Zhuang and others.

Liu-Xu-Zhuang 2021: K-stability of a $\mathbb{Q}$-Fano variety implies uniform K-stability.

Moduli spaces for general K-stable $\mathbb{Q}$-Fano varieties were constructed algebraically by the works of Alper, Blum, Halpern-Leistner, Jiang, Liu, Xu, Zhuang.

The deep work of Birkar in birational geometry is used to as an alternative to the Cheeger-Colding theory.

(4) General CSCK setting

The formal picture of $\mathcal{H}$ is now made precise

- Chen 2000: existence of weak geodesics in $\mathcal{H}$ connecting any two points.
- T. Darvas 2015: metric completion $\overline{\mathcal{H}}$ of $\mathcal{H}$ is a CAT(0) space, which can identified with the space of finite energy classes in pluripotential theory.
- Berman-Berndtsson 2015: geodesic convexity of the Mabuchi functional and uniqueness of CSCK metrics (modulo $Aut_0(X)$)

- Darvas-Rubinstein 2015: properness of E implies the existence of a minimizer in $\overline{\mathcal{H}}$.

- Chen-Cheng 2018: PDE regularity theory for CSCK metrics: weak minimizers are smooth solutions; analytic criterion for existence in terms of geodesic stability

Boucksom-Jonsson, C. Li: connections with stronger version of K-stability.

In a related context, G. Chen 2019 proved algebro-geometric criteria for the solving the J-equation and deformed Hermitian-Yang-Mills equation.

Philosophy:

"Natural objects from canonical Kähler metrics have algebro-geometric meaning"

New directions can arise from further exploration of Kähler-Einstein metrics

In the volume non-collapsing setting we have

- Local stability theory
- optimal degeneration theory
- Stability at infinity
- Bubbling theory

Local stability theory

Consider a Gromov-Hausdorff limit X_∞ of Kähler-Einstein Fano manifolds and a singular point $p \in X_\infty$.

Cheeger-Colding: Singularities are conical. Rescale the limit metric around $p \rightsquigarrow$ metric tangent cones.

A priori uniqueness is a difficult question in geometric analysis.

The cross section of such a cone, if smooth, is a Sasaki-Einstein manifold.

Sasaki-Einstein geometry has close connections with AdS/CFT correspondence.

Martelli-Sparks-Yau 2006: volume minimization for Sasaki-Einstein metrics. In particular, the volume of a Sasaki-Einstein manifold is an algebraic number.

Collins-Szekeleyhidi 2015: Existence of Sasaki-Einstein metrics $\leftrightarrow$ K-stable Fano cones.

Donaldson-S. 2015: There is a unique metric tangent cone $\mathcal{C}_p$, which admits algebro-geometric description.

metric rescaling $\leftrightarrow$ algebraic rescaling

2-step degeneration theory via Kähler-Einstein metrics:

- Valuation on the local ring at p, sending $\mathcal{O}_p$ to a Fano cone W.
- An equivariant degeneration from W to $\mathcal{C}_p$.

$\mathcal{C}_p$ is K-stable and the intermediate object W is K-semistable.

Compare: Harder-Narasimhan and Seshadri filtration of holomorphic bundles.

Conjecture (Donaldson-S. 2015): Both W and $\mathcal{C}_p$ are local algebro-geometric invariants of $\mathcal{O}_p$ (hence leads to local stability theory for KLT singularities).

C. Li 2015: reformulated the DS conjecture using a generalized volume minimization principle.

The conjecture was proved by Li-Liu 2016, Li-Xu 2017, Li-Wang-Xu 2018, Xu-Zhuang 2022 in greater generality.

Optimal degeneration theory

Ricci flow on Fano manifolds

$$\begin{cases} \frac{\partial}{\partial t}\omega_t = \omega_t - Ric(\omega_t), & t \in [0,\infty); \\ \omega_0 \in 2\pi c_1(X). \end{cases}$$

Chen-S.-Wang 2015: If X is K-unstable, ω_t canonically determines a 2-step degeneration

- A (possibly irrational) de-stabilizing test configuration $X \to W$.
- An equivariant degeneration $W \to X'$.

X' is the unique Ricci-flow limit ($\mathbb{Q}$-Fano variety with a Kähler-Ricci soliton)

Conjecture (Chen-S.-Wang 2015): This 2-step degeneration is independent of ω_0 and gives rise to algebro-geometric optimal degenerations of Fano varieties.

W. He 2012, Dervan-Székelyhidi 2016: This test configuration is optimal in a suitable sense.

Han-Li 2020: Conjecture holds.

Blum-Liu-Xu-Zhuang 2021: Construction of the Chen-S.-Wang degeneration using purely algebro-geometric procedure.

Stability at infinity

In the dual setting one considers complete Calabi-Yau metrics with

$$\lim_{R\to\infty} R^{-2n}\mathrm{Vol}(B(p,R)) > 0.$$

We call these *AC* Calabi-Yau metrics.

Scaling down the metric gives rise to asymptotic cones.

Under suitable assumptions there is a dual 2-step degeneration theory for the unique asymptotic cone $\mathcal{C}_\infty$.

There is a sharp contrast to the local setting:

Sun-Zhang 2022 (no semistability at infinity):
For AC Calabi-Yau metrics with $|Rm| = O(R^{-2})$ we have $W = \mathcal{C}_\infty$.

Combined with works by Conlon-Hein, this gives a complete classification of AC Calabi-Yau metrics with $|Rm| = O(R^{-2})$ in terms of algebraic geometry

Conjecture (Yau's compactification conjecture):
An AC Calabi-Yau manifold is quasi-projective.

This is known when the metric is globally of the form $\sqrt{-1}\partial\bar{\partial}\phi$.

In general is not even known that an AC Calabi-Yau metric has finite topology.

AC metric is not unique/canonical non-compact manifolds, even on $\mathbb{C}^3$ (Y. Li 2017, Conlon-Rochon 2017, Szekelyhidi 2017).

It is more natural to study AC metrics on a pair (X, ν), where X is a quasi-projective variety together with an algebraic scaling ν (negative valuation) that degenerates X to a K-stable Fano cone.

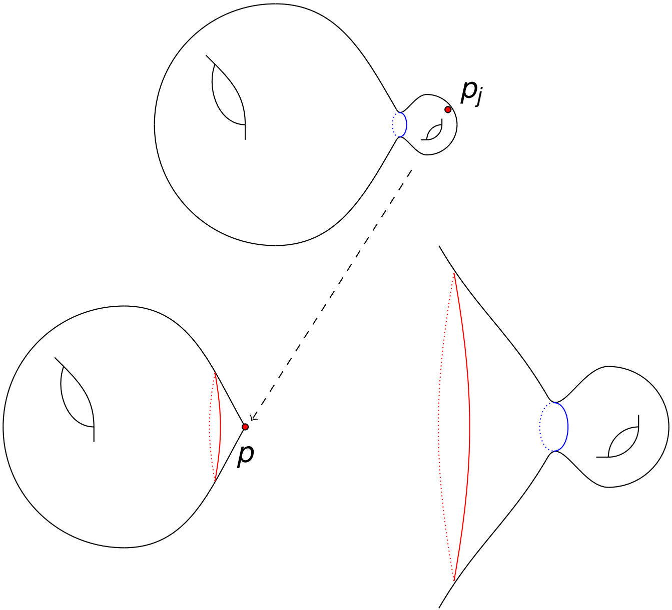
Expect suitable uniqueness statement for a fixed (X, ν).

Bubbling theory

The local theory only deals with the limit space (X_∞, p) itself.

To understand the dynamical singularity formation of the sequence (X_j, p_j), we rescale the metric ω_j based at p_j by $\lambda_j \to \infty$ for all possible $\{\lambda_j\}$.

We obtain complete AC Calabi-Yau metrics as “bubbles".


p_j
p

Iterating the procedure we obtain a bubble tree.

S. 2023: The bubble tree terminates in finitely many steps.

The first bubble at p is $\mathcal{C}_p$.

The next bubble is Z_1, whose asymptotic cone coincides with $\mathcal{C}_p$.

Z_1 is a minimal bubble. It is a polarized Calabi-Yau affine variety.

Question (Algebraic bubble tree)

Give an algebro-geometric description of the bubble tree.

More precisely, given $\pi : \mathcal{X} \to \Delta$ a flat degeneration of KLT singularities with a section σ. How to generate in a canonical way the minimal bubble?

This question is related to recent work of Odaka (and Z. Chen) studying moduli space of Fano cones and singularities.

There are many more interesting questions come in the collapsing setting, but that will be another talk...