Bubbling of Kähler-Einstein metrics

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We will discuss Kähler-Einstein metrics. For simplicity we focus on Calabi-Yau metrics.

We want to understand bubbling structure associated to singularity formations.

This is related to both differential geometry and algebraic geometry.

My goal is to survey some results and discuss some open questions.
Plan:

- Background, volume collapsing/non-collapsing
- Non-collapsing in complex dimension 2 (classical results)
- Collapsing in complex dimension 2
- Non-collapsing in higher dimensions
Part 1: Background
A Calabi-Yau manifold is a Kähler manifold $X$ together with a holomorphic volume form $\Omega$.

A Calabi-Yau metric on $(X, \Omega)$ is a Kähler metric $\omega$ on $X$ such that

$$\omega^n = C \Omega \wedge \bar{\Omega}$$

Alternatively: a Calabi-Yau metric is a Riemannian metric whose holonomy group is contained in $SU(n)$ (In particular, $Ric \equiv 0$)
Theorem (Yau, Calabi conjecture)

Given a compact Calabi-Yau manifold \((X, \Omega)\) with a Kähler metric \(\tilde{\omega}\), there is a unique Calabi-Yau metric \(\omega\) which is co-homologous to \(\tilde{\omega}\).

\(n = 1\): elliptic curves with flat metrics

\(n = 2\): flat tori, K3 surfaces (non-flat)
The set of all Calabi-Yau metrics on a compact manifold can be characterised in terms of two pieces of algebro/complex geometric data:

- A complex structure, determined by $\Omega$

- A Kähler class, given by an element in the Kähler cone. The latter is an open convex cone in $H^2(X; \mathbb{R}) \cap H^{1,1}(X; \mathbb{C})$. 
Example (1)

\(X: \text{a hypersurface } \{f(x) = 0\} \text{ in } \mathbb{CP}^{n+1} \text{ of degree } n+2.\)

\(\Omega\) can be written down explicitly in terms of \(f\).

The Kähler class is given by \(c_1(O(1))\).

Varying \(f\) yields deformation of complex structures.
Example (2)

Consider a family of elliptic curves

\[ \{ f_u(x) = 0 \} \subset \mathbb{CP}^2, \quad u \in B^\circ \]

Under suitable assumptions the total space may be compactified to a Calabi-Yau manifold \( X \) by adding certain singular fibers, and there is a holomorphic fibration \( \pi : X \to B \).

\( \beta \): Kähler class on \( X \)
\( \gamma \): Kähler class on \( B \) \quad \leadsto \quad t\beta + \pi^*\gamma \) is Kähler for \( t > 0 \)
Degenerations of Calabi-Yau metrics can arise from algebraic geometry

- Complex structure degenerations:

  In Example (1), the defining equation $f$ may become singular.

- Kähler class degenerations: the Kähler class may reach the boundary of the Kähler cone.

  In Example (2), as $t \to 0$, the limit class $\gamma$ is not Kähler.
Well-known problems regarding Calabi-Yau manifolds are potentially related to the study of Gromov-Hausdorff limits of Calabi-Yau metrics. The central issue is singularity formation.

- Topological finiteness (Yau’s conjecture)

- SYZ mirror symmetry: complex structure degenerations

- Abundance conjecture (for Calabi-Yau): Kähler class degenerations
We consider a sequence of $n$ dimensional compact Calabi-Yau metrics $(X_j, \omega_j, \Omega_j)$ with $\text{diam}(X_j, \omega_j) \leq D$. Fix a base point $p_j \in X_j$.

Passing to a subsequence there is a Gromov-Hausdorff limit $X_\infty$ with a metric $d_\infty$, and $p_j$ converges to a limit point $p_\infty$. We assume $X_\infty$ is not a point.

Given $\lambda_j \rightarrow \infty$, passing to a subsequence there is a (pointed) Gromov-Hausdorff limit $(Z, p_Z)$ of $(X_j, \lambda_j \omega_j, p_j)$. We refer to this as a **bubble limit**.
When \( n = 1 \), \( X_\infty \) is either an elliptic curve or a circle \( S^1 \).

Possible bubble limits are \( \mathbb{R}, S^1 \times \mathbb{R} \) and \( \mathbb{R}^2 \).
In general we divide the discussion into two cases:

(1). (Volume) non-collapsing:

\[ \text{Vol}(X_j, \omega_j) \geq \kappa > 0 \text{ for all } j. \]

(2). (Volume) collapsing:

\[ \text{Vol}(X_j, \omega_j) \to 0. \]
Part 2: Non-collapsing in dimension 2
We now assume \( n = 2 \) and study the volume non-collapsing situation.

We only consider the non-trivial case of K3 surfaces.
We have the well-established classical results in the 1990s

- $X_\infty$ is a Calabi-Yau orbifold with isolated singularities locally modeled on $\mathbb{C}^2/\Gamma$

- Conversely, any compact Calabi-Yau orbifold can arise as such a limit

- Bubble limits are complete Calabi-Yau orbifolds that are asymptotic to a flat cone $\mathbb{C}^2/\Gamma$ at infinity (ALE).
In particular $T^4/\mathbb{Z}_2$ is a Gromov-Hausdorff limit (Kummer construction).

The nontrivial bubble limit is given by the Calabi-Eguchi-Hanson metric on $T^* S^2$. 
Corollary

*Bubbling terminates in finitely many steps.*
This depends on the following

- **Bishop-Gromov**: \( \text{Vol}(B(p, r))/r^4 \) is decreasing in \( r \) when \( \text{Ric} \geq 0 \).

  In particular, for a bubble limit \((Z, \rho_Z)\) which is asymptotic to \( \mathbb{C}^2/\Gamma_1 \) at infinity and is modeled on \( \mathbb{C}^2/\Gamma_2 \) at \( \rho_Z \), we have

  \[ |\Gamma_2| \leq |\Gamma_1|. \]

  The equality holds if and only if \( Z \) is a flat cone.

- **(Volume rigidity)** The set \( \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_{>0} \right\} \) is a discrete set.

- **(Cone rigidity)** \( \mathbb{C}^2/\Gamma \) is rigid as a flat cone.
Part 3: Collapsing in dimension 2
Now we assume $n = 2$ and volume collapsing. Again restrict to the case of K3 surfaces.

From the Kummer construction, we know that $T^k/\mathbb{Z}_2$ ($k = 1, 2, 3$) can be a possible collapsing Gromov-Hausdorff limit.

Locally, bubbling may not terminate in finitely many steps.
Consider the flat cone $\mathbb{C}^2/\mathbb{Z}_j$ ($\mathbb{Z}_j \subset SU(2)$ is cyclic of order $j$), as the asymptotic cone of an ALE Calabi-Yau manifold $(X_j, \omega_j)$.

Let $j \to \infty$, $\mathbb{C}^2/\mathbb{Z}_j$ collapses to the Riemannian quotient $\mathbb{C}^2/S^1$, which is the cone over the round $S^2$ with radius $\frac{1}{2}$.

For suitable choices of $p_j \in X_j$ and $\lambda_j \to \infty$, $(X_j, \lambda_j \omega_j, p_j)$ collapses to the same cone. It is possible to arrange so that each $\mathbb{C}^2/\mathbb{Z}_j$ arises as a bubble limit.
General collapsing theory:

- **Cheeger, Fukaya, Gromov 1980s**: When Riemannian metrics collapse with uniformly bounded curvature, locally the collapsing is along certain nilpotent fibrations.

- **Cheeger-Tian 2004**: For a sequence of 4 dimensional Riemannian Einstein metrics \((M_j, g_j)\) with \(\int_{M_j} |Rm|^2 \leq C\), the curvature stays bounded away from finitely many points.
For Calabi-Yau metrics on K3 surfaces, we have $\int_{X_j} |Rm|^2 = 192\pi^2$.

By Cheeger-Tian, the limit space $X_\infty$ is smooth away from finitely many singularities.

The issue is to understand the singularity structure. A priori we can not say it is conical.
Theorem (S.-Zhang 2021)
A collapsing limit of Calabi-Yau metrics on the K3 surfaces is isometric to one of the following

(1). \((\text{dim} = 1)\) The unit interval \([0, 1] = T^1/\mathbb{Z}_2\).

(2). \((\text{dim} = 2)\) A singular special Kähler metric on \(S^2\);

(3). \((\text{dim} = 3)\) A flat \(T^3/\mathbb{Z}_2\);

Remark: we also have similar classification results for collapsed bubble limits.

Key: regularity region inherits an affine structure from the Calabi-Yau structure.
Application:

Hongyi Liu 2023: a new proof of the surjectivity of the period map for K3 surfaces, without using Yau’s proof of Calabi conjecture.

\( \mathcal{M} \): space of all Calabi-Yau metrics on K3 surfaces with diameter 1 (modulo isometry)

Period map

\[
P : \mathcal{M} \to \mathcal{D} := \text{Gr}^+(\mathbb{R}^{3,19}); \quad g \mapsto \mathbb{H}^+_g
\]

Torelli theorem: \( P \) extends to a bijective map from \( \mathcal{M}^\partial \) to \( \mathcal{D} \), where \( \mathcal{M}^\partial \) is the partial Gromov-Hausdorff compactification by including all the non-collapsing limits.
Open question: Relate the full Gromov-Hausdorff compactification $\overline{\mathcal{M}}$ to (certain) compactification of $\mathcal{D}$ as locally symmetric spaces.

Odaka-Oshima 2018 formulated more precise conjectures and made progress.

To study the question requires deeper investigation of the behavior of periods along a collapsing sequence.
Corollary

Bubbling terminates in finitely many steps.
There is a sequence $\lambda_j \to \infty$, such that the bubble limits of $(X_j, \lambda_j \omega_j, p_j)$ are complete Calabi-Yau orbifolds $(Z, p_Z)$ with $\text{Vol}(B(p_Z, R)) = o(R^4)$ as $R \to \infty$.

These are local models for collapsing singularities.
A byproduct of the analysis is

**Theorem**

A 2 dimensional complete Calabi-Yau metric with finite energy (i.e. $\int |Rm|^2 < \infty$) is asymptotic to a model end (ALE, ALF, ....).

Each AL family is essentially classified, by Kronheimer, Minerbe, Chen-Chen, Chen-Viaclovsky-Zhang, Collins-Jacob-Lin, Hein-S.-Viaclovsky-Zhang etc.
Conjecture (Yau’s compactification conjecture)

A complete Calabi-Yau manifold is bi-holomorphic to the complement of a divisor in a compact complex manifold.

The above results give an affirmative answer in complex dimension 2, under the finite energy assumption and the hyperkähler rotation of complex structure.

The conjecture fails without the finite energy assumption (Anderson-Kronheimer-LeBrun 1989)
We propose the following

**Conjecture (S.-Zhang 2021)**

A complete 4 dimensional Ricci-flat manifold \((X, g)\) has finite energy (i.e. \(\int_X |Rm|^2 < \infty\)) if and only if it has finite topology.

An affirmative answer to this conjecture would have interesting consequences. For example, it would give non-existence results for complete Calabi-Yau metrics on certain quasi-projective varieties.
ALE Calabi-Yau metrics in complex dimension 2 were classified by Kronheimer. Their quotients yield all ALE Ricci-flat Kähler metrics.

A folklore open question is

*Are all ALE Ricci-flat metrics in real dimension 4 Kähler?*

**Mingyang Li 2022:** if $W^+$ has repeated eigenvalues and the asymptotic cone is $\mathbb{C}^2/\Gamma$ for $\Gamma \subset SU(2)$ then it is Kähler.
Part 4: Non-collapsing in dimension $> 2$
In higher dimensions we consider a sequence of $n$ dimensional Calabi-Yau metrics $(X_j, \Omega_j, \omega_j, p_j)$ with

- $[\omega_j] \in H^2(X_j; \mathbb{Z})$
- $\text{diam}(X_j, \omega_j) \leq D$.

This is in the non-collapsing situation.

Suppose it has a Gromov-Hausdorff limit $(X_\infty, d_\infty, p_\infty)$. We assume $p_\infty$ is a singular point.

**Cheeger-Colding** theory gives a partial regularity of $X_\infty$. Roughly speaking, singularities are **conical**. We know very little about the structure around the singular set.
Donaldson-S. 2012: after passing to a subsequence

- $X_\infty$ is naturally a projective variety with mild singularities

- $d_\infty$ is the metric completion of a Calabi-Yau metric on the regular part of $X_\infty$

- The convergence can be realized in a flat family over a fixed Hilbert scheme.
Donaldson-S. 2015:

- A bubble limit $Z$ is naturally an affine algebraic variety with mild singularities.

The coordinate ring $R(Z)$ is the space of polynomial growth holomorphic functions.
Definition
A cone limit is a bubble limit that is also a metric cone.

Examples of cone limits (Cheeger-Colding)

- Limits of \((X_\infty, \lambda_j d_\infty, p_\infty)\) for \(\lambda_j \to \infty\).

- Limits of \((Z, \lambda_j^{-1} d_Z, p_Z)\) for \(\lambda_j \to 0\) (or \(\infty\)), where \(Z\) is a bubble limit.

Donaldson-S. 2015: The above limits are independent of the choice of \(\lambda_j\).

We obtain the tangent cones \(C(p_\infty), C(p_Z)\) and the asymptotic cone \(C_\infty(Z)\).
A cone limit $\mathcal{C}$ is an affine algebraic variety in $\mathbb{C}^N$ preserved by a Reeb vector field

$$\xi = \text{Im}(\sum_j a_j z_j \partial z_j), \quad a_j > 0$$

Algebro-geometrically this corresponds to an $\mathbb{R}$-grading on $R(\mathcal{C})$. 
A cone limit is of the form $C = C(Y)$ for a cross section $Y$.

We define the **volume density**

$$V(C) = \frac{Vol(Y)}{Vol(S^{2n-1})} \in (0, 1].$$

Example:

$$V(\mathbb{C}^2/\Gamma) = \frac{1}{|\Gamma|}$$

**Martelli-Sparks-Yau**: $V(C)$ is always in $\mathbb{Q}$.
Theorem (S. 2023, Volume rigidity)
There is a $\mathcal{V} > 0$ depending only on $n$ and $D$ such that

$$\#\{V(C)\mid C \text{ is a cone limit}\} \leq \mathcal{V}$$

Key: if $C_j$ converges to $C_\infty$ then $V(C_j)$ is constant for $j$ large.

Remark 1: This is related to the conjecture on the volume of klt singularities in algebraic geometry (c.f. Han-Liu-Lu 2022, Zhuang 2023).

Remark 2: if $C_\infty$ has a smooth cross section then this follows from the local analytic structure of Einstein moduli spaces (c.f. Besse).
Theorem (S. 2023, Cone rigidity)

There exists a sequence $\lambda_j \to \infty$ such that

1. the bubble limits $(Z, p_Z)$ of $(X_j, \lambda_j \omega_j, p_j)$ satisfy $C_\infty(Z) = C(p_\infty)$ and $V(C(p_Z)) > V(C_\infty(Z))$.

2. the bubbles limits of $(X_j, \lambda'_j \omega_j, p_j)$ for $\lambda'_j \to \infty$ and $\lambda'_j \lambda_j^{-1} \to 0$ are all given by $C(p_\infty)$.

The bubble limits $Z$ occurring in (1) are referred to as the minimal (essential) bubbles.

Key: rule out possible continuous family of cone limits arising as bubble limits. This is an enhancement of the uniqueness of tangent cones.
Corollary

*Bubbling terminates in finite many steps.*

The proofs of both theorems exploit connections with algebraic geometry. Similar statements are open questions in the general Riemannian setting.
An interesting question is to understand the algebro-geometric meaning of the differential geometric objects that arise from this study.
The tangent cone $C(p_{\infty})$ can be viewed as a first order approximation of the singularity formation. It may be described in terms of a 2-step degeneration picture from $(X_{\infty}, d_{\infty}, p_{\infty})$.

At $p_{\infty} \in X_{\infty}$, for a holomorphic function germ $f \in \mathcal{O}_{p_{\infty}}$, we define

$$\nu(f) \equiv \lim_{r \to 0} \frac{\sup_{B(p, r)} \log |f|}{\log r}$$
Donaldson-S. 2015:

The graded ring associated to \( \nu \) defines an affine variety \( W \) with a Reeb vector field. Geometrically, \( W \) is a weighted tangent cone of \( X_\infty \) at \( p_\infty \) under some holomorphic embedding into some \( \mathbb{C}^N \).

There is an equivariant degeneration from \( W \) to the unique tangent cone \( \mathcal{C}(p_\infty) \).
It is conjectured in Donaldson-S. 2015, and confirmed by Li-Wang-Xu 2018, that $W$ and $C(p_\infty)$ are both algebraic invariants of the singularity germ at $p_\infty$.

In particular there are examples where $W \neq C(p_\infty)$.

There have been extensive results on the algebro-geometric aspects, by Li, Xu, Liu, Blum, Zhuang, ...
Donaldson-S. 2015: for a bubble limit $(Z, p_Z)$, there is a similar 2-step degeneration picture for the asymptotic cone $C_\infty(Z)$, using the negative valuation

$$d(f) \equiv - \lim_{r \to \infty} \frac{\sup_{B(p,r)} \log |f|}{\log r}$$

There is an also intermediate $W_\infty$. 
However there are sharp contrasts with the local situation

- Neither $W_\infty$ nor $C_\infty(Z)$ is an algebraic invariant.

On $\mathbb{C}^3$ there are complete Ricci-flat metrics with asymptotic cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ (Li, Conlon-Rochon, Szekelyhidi 2017)

- No semistability at infinity: assuming $C_\infty(Z)$ is smooth, then $W_\infty = C_\infty(Z)$ (S.-Zhang 2022). In general this is also conjectured to be true.

This is closely related to Yau’s compactification conjecture in the case of Euclidean volume growth.
The study of bubbling structure provides a more dynamical and deeper picture of singularity formations.

**Question**

*Give an algebro-geometric meaning of minimal bubbles.*
Consider a projective family $\pi : \mathcal{X} \to \Delta$ of Calabi-Yau varieties. Let $\sigma$ be a section of $\pi$. Then the question is whether the minimal bubbles associated to the family can be characterized algebro-geometrically in terms $\sigma$ and $\pi$.

There is work of de Borbon-Spotti which studies low dimensional examples.

In higher dimensions the situation is subtle, and needs to be explored further.
Thank you!