Geometry of Calabi-Yau metrics

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Introduction

Calabi-Yau metrics are named after two mathematicians: E. Calabi and S.-T. Yau. They are fundamental objects in geometry and physics.

Let M be a differentiable manifold of dimension m. A Riemannian metric g defines a smoothly varying family of inner products on the tangent spaces of M. We fix a point p in M and choose an identification Ψ between T_pM and the Euclidean space $\mathbb{R}^m$. Then for any piecewise smooth loop γ based at p, parallel transport along γ with respect to the Levi-Civita connection yields an orthogonal transformation $P_\gamma : \mathbb{R}^m \to \mathbb{R}^m$. The *holonomy group* of g is by definition the subgroup of $O(m)$ consisting of all such transformations P_γ. This is up to conjugation independent of the choices of p and Ψ. We say g is a *Calabi-Yau metric* if $m = 2n$ and there is a further identification $\mathbb{R}^m \simeq \mathbb{C}^n$ so that the holonomy group of g is contained in $SU(n) \subset O(m)$. In the literature one often uses a weaker notion of local holonomy group which involves only null-homotopic loops γ. However for our purposes in this article we will take the above more restrictive definition.

A salient feature of Calabi-Yau metrics is that they have vanishing Ricci curvature, hence provide solutions to the Riemannian vacuum Einstein equation. This can be deduced from a consequence of the Ambrose-Singer holonomy theorem and the Bianchi identity. In terms of the Berger classification of Riemannian holonomy groups, Calabi-Yau metrics are examples of Riemannian metrics with *special holonomy*, which play a pivotal role in string theory.

This article is an expanded version of the notes for some recent colloquia and mini-school lectures given by the author. The main goal here is to explain to the readers some constructions and the geometry of Calabi-Yau metrics; in the meantime we aim to selectively discuss several interesting examples in the field and some recent research progress. Obviously, this article is by no means supposed to be a comprehensive historic survey of the subject. The topics are merely chosen according to the personal taste of the author, and there are many other related papers emphasizing different aspects. Also for lack of space it is impossible to provide precise references to all the results mentioned below, but the author hopes that the interested readers can easily look up further details on their own.

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Yau's existence theorem

We first consider the more flexible notion of *Kähler metrics*. A Riemannian metric g on a differentiable manifold M of dimension $2n$ is Kähler if its holonomy group is contained in $U(n)$. Since $U(n) = GL(n;\mathbb{C}) \cap O(2n)$, one can define an almost complex structure on M, i.e., a tensor field $J : TM \to TM$ satisfying $J^2 = -Id$ that is orthogonal with respect to g and is parallel under the Levi-Civita connection. By the Newlander-Nirenberg theorem it follows that M is a *complex* manifold, that is to say, near each point one can find complex-valued coordinates $\{z_\alpha = x_\alpha + \sqrt{-1}y_\alpha\}_{\alpha=1}^n$ such that the transition functions are holomorphic and J is represented by $J\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}$ for $\alpha = 1, \cdots, n$. Furthermore, the associated *Kähler form* ω defined by $\omega(v_1, v_2) \equiv g(Jv_1, v_2)$ satisfies $d\omega = 0$, and it can be locally written as $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ for a real-valued *potential* function ϕ. In terms of the complex coordinates, the latter means that

$$\omega = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta.$$

For convenience we also call ω a Kähler metric. Conversely, any real-valued function ϕ such that the matrix $(\frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta})$ is positive definite defines a local Kähler metric via the above formula. If M is compact, a Kähler metric ω defines a non-trivial cohomology class in $H^2(M;\mathbb{R})$. A standard example is the Fubini-Study metric ω_{FS} on the complex

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projective space $\mathbb{CP}^N$. In local affine coordinates $(z_1, \cdots, z_N)$, one can take the potential function $\phi = \log(1 + |z_1|^2 + \cdots |z_N|^2)$. Clearly ω_{FS} also restricts to a Kähler metric on any complex submanifold of $\mathbb{CP}^N$.

Suppose now ω is a Kähler metric on a complex manifold M. Then by definition it is a Calabi-Yau metric if and only if there exists a complex-valued n-form Ω, locally given as $f dz_1 \wedge \cdots dz_n$, which is parallel with respect to the Levi-Civita connection. This condition is equivalent to saying that Ω is holomorphic and nowhere vanishing, i.e., Ω is a holomorphic volume form, and moreover the following equation holds:

$$\omega^n = c\Omega \wedge \bar{\Omega}, \tag{1}$$

where c is a constant that can be determined by integrating over M. A straightforward but crucial computation in Kähler geometry shows that (1) is indeed equivalent to ω being a Ricci-flat metric on a compact Kähler manifold. This is related to the remarkable feature of Kähler metrics that the Ricci curvature is determined locally by a real-valued function, a huge simplification that does not exist for general Riemannian metrics.

What we have done in the above is to decouple the definition of a Calabi-Yau metric into two ingredients of different flavor. First we need M to be a complex manifold admitting a holomorphic volume form Ω and a Kähler metric. For simplicity we will call such a complex manifold a *Calabi-Yau variety.* The existence of Ω is a purely complex analytic condition which, in the case when M is compact and simply connected, is equivalent to the vanishing of the first Chern class $c_1(M)$. Complex algebraic geometry provides abundant examples of Calabi-Yau varieties.

The second ingredient we need is a Kähler metric ω on M satisfying (1). For this we have Yau's celebrated solution of the Calabi conjecture[1], which is the reason for the name "Calabi-Yau".

Theorem 0.1 ([Yau78]). *Given a compact Calabi-Yau variety M with a holomorphic volume form Ω and a Kähler metric $\underline{\omega}$, there exists a unique Calabi-Yau metric ω on M in the cohomology class of $\underline{\omega}$.*

A classical fact is that since M is compact, any Kähler metric on M cohomologous to $\underline{\omega}$ is of the form $\underline{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ for some globally defined real-valued function φ. Then (1) becomes a non-linear PDE in φ:

$$(\underline{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n = C\Omega \wedge \bar{\Omega}. \tag{2}$$

In terms of local complex coordinates this takes the form of a complex Monge-Ampère equation. Yau's proof of Theorem 0.1 is by solving (2) via a *continuity method.* Write $\underline{\omega}^n = e^f \Omega \wedge \bar{\Omega}$ for a smooth function f, and consider a family of equations parameterized by $t \in [0, 1]$:

$$(\underline{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{(1-t)f + c_t}\Omega \wedge \bar{\Omega}, \tag{$*_t$}$$

where again c_t is a constant determined by integration. Let T be the set of $t \in [0, 1]$ such that $(*_t)$ has a smooth solution φ_t satisfying $\int_M \varphi_t \underline{\omega}^n = 0$. By an integration by parts argument one sees that such a φ_t, if it exists, must be unique. Obviously $0 \in T$ and the goal is to show that $1 \in T$. Therefore it suffices to prove that T is both open and closed. The openness follows from a standard implicit function theorem on Banach spaces, using the fact that the linearization of (2) is the Laplace equation. The closedness will follow if one can obtain *a priori* estimates, namely, if one can show that for any $t \in T$, the unique solution φ_t satisfies the bound $|\varphi_t|_{C^3(M)} \leq A$ for a constant $A > 0$ independent of t, where the norm is defined with respect to the fixed metric $\underline{\omega}$. This is the heart of the proof and we refer the interested readers to Yau's original paper [Yau78].

Theorem 0.1 immediately produces many examples of non-trivial compact Calabi-Yau metrics from algebraic geometry.

Example 0.2. *Let $F = F(Z_0, \cdots, Z_{n+1})$ be a homogeneous polynomial of degree $n+2$ and consider the hypersurface X in $\mathbb{CP}^{n+1}$ defined as the zero set of F. For a generic choice of F, X is smooth. We claim that it is a Calabi-Yau variety. Indeed, one may write down an explicit holomorphic volume form Ω which, in terms of the homogeneous coordinates, is given by*

$$\Omega = \left(\frac{\partial F}{\partial Z_0}\right)^{-1} dZ_1 \wedge \cdots \wedge dZ_{n+1}$$

on the open set $\{\frac{\partial F}{\partial Z_0} \neq 0\}$. Yau's theorem implies the existence of a unique Calabi-Yau metric ω on X in the cohomology class of the restriction of the Fubini-Study metric.

In the above example, when $n = 1$, we know X is an elliptic curve and the metric ω is a flat metric on the 2-torus. In this case one can use elliptic integrals to write down a formula for ω in the projective coordinates $[Z_0 : Z_1 : Z_2]$. When $n \geq 2$, there is no known closed formula for the metric ω.

[1] We remark that the original Calabi conjecture was stated in a more general form, and has lead to a far-reaching program in Kähler geometry for the last few decades, centered around the question of finding canonical Kähler metrics on complex manifolds.

In light of this, it is interesting to understand more precisely the geometry of the Calabi-Yau metrics resulting from Theorem 0.1. This is the topic that we shall discuss in the rest of this article.

Calabi-Yau metrics with symmetry

By the Bochner technique, having vanishing Ricci curvature implies that a compact Calabi-Yau metric cannot admit any non-trivial continuous symmetry, i.e., any Killing vector field must be parallel. But this does not have to be the case for non-compact manifolds. Indeed, there are explicit constructions of non-compact Calabi-Yau metrics using symmetry which, as we shall see later, often provide models and intuition for understanding the geometry of compact Calabi-Yau metrics near the degeneration limit.

In the simplest setting when the complex dimension is 2, we recall the well-known *Gibbons-Hawking ansatz* [GH]. Choose a positive harmonic function V defined on a contractible open set Q in the Euclidean space $\mathbb{R}^3$. Then one can write $*dV = d\theta$ for a 1-form θ, where $*$ denotes the Hodge star operator. On the product space $P = Q \times S^1$, we consider a Riemannian metric given by the formula

$$g = V \sum_{i=1}^{3} dx_i^2 + V^{-1}(dt + \theta)^2,$$

where x_i's are standard coordinates on $\mathbb{R}^3$, and t denotes the standard coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$. This metric is invariant under the obvious free S^1 rotation on the second factor.

We claim that g is a Calabi-Yau metric. One way of seeing this is to define 3 orthogonal almost complex structures J_i $(i = 1, 2, 3)$ by setting $J_i dx_i = dx_{i+1}$ and $J_i dx_{i+2} = V^{-1}(dt + \theta)$, where we make the convention to identify the subscripts modulo 3. They satisfy the quaternionionic relations $J_i J_{i+1} = J_{i+2}$, and it is an exercise to check that they are all parallel with respect to the Levi-Civita connection. This means that the holonomy group is given by the unit quaternions $Sp(1)$, which is isomorphic to $SU(2)$ if we identify further $\mathbb{C}^2$ with the quaternions $\mathbb{H}$. Notice there are indeed a whole family of parallel orthogonal almost complex structures given by $\sum_{i=1}^{3} a_i J_i$, where (a_1, a_2, a_3) lies in the unit sphere in $\mathbb{R}^3$. In terms of usual terminology, the metric g is *hyperkähler*.

Conversely, any Calabi-Yau metric in 2 complex dimensions with a free S^1 action is locally given in the above form. Indeed, one can recover the coordinates x_i's as the moment maps with respect to the Kähler form ω and the real and imaginary part of the holomorphic volume form Ω, the function $V^{-\frac{1}{2}}$ as the length of the corresponding Killing field $\frac{\partial}{\partial t}$, and θ as the dual 1-form of $\frac{\partial}{\partial t}$. Most strikingly, the S^1 symmetry allows us to reduce the non-linear PDE for Calabi-Yau metrics to the Laplace equation on $\mathbb{R}^3$ which is a linear PDE.

The Calabi-Yau metrics constructed this way have little topology. The situation becomes more appealing if one makes certain variants of the construction. First we can let Q be a domain in a general flat 3 manifold with a global orthonormal frame, which we also allow to have non-trivial topology. The above procedure goes through if V satisfies an integrality condition that $\frac{1}{2\pi}\int_C *dV \in \mathbb{Z}$ for all 2-cycles $C \in H_2(U; \mathbb{Z})$. Then P is replaced by a principal S^1 bundle $\pi : P \to Q$, and $dt + \theta$ is replaced by a connection 1-form on P whose curvature is $*dV$. Next we want to include the case when the S^1 action has fixed points. This corresponds to allowing V to be ∞ on a discrete subset of Q. The condition for the total space P to be smoothly compactified is that locally the singular term of V is given by the Green's function $\frac{1}{2r}$ on $\mathbb{R}^3$.

At this point we can build various examples. To start with, we can take $Q = \mathbb{R}^3$ and $V = \frac{1}{2r}$. Then we recover the flat metric on $\mathbb{C}^2$ with the standard holomorphic volume form, and with the S^1 action given by $\lambda.(z_1, z_2) = (\lambda^{-1} z_1, \lambda z_2)$. The projection map $\pi : \mathbb{C}^2 \to \mathbb{R}^3$ in this case is the *Hopf fibration*, explicitly expressed as

$$(z_1, z_2) \mapsto \big(Re(z_1 z_2), Im(z_1 z_2), \frac{1}{2}(|z_1|^2 - |z_2|^2)\big).$$

If we instead take $V = \frac{k}{2r}$ for a positive integer $k > 1$, then we get the flat orbifold given by the quotient of $\mathbb{C}^2$ by a cyclic subgroup of order k in $SU(2)$.

Next we make a small change, and take $V = \frac{1}{2r} + 1$ on $Q = \mathbb{R}^3$. Then one obtains the *Taub-NUT* metric. It is a complete Calabi-Yau metric on $\mathbb{R}^4$. Since adding the constant 1 does not change dV, the projection map π is still the Hopf fibration. As discovered by LeBrun [LeB91], for any choice of complex structure, one can always identify the underlying Calabi-Yau variety with $\mathbb{C}^2$ equipped with the standard holomorphic volume form. However, the geometry of the Kähler metric ω is quite different from that of the flat metric. For example, near infinity, the Hopf fibers now have uniformly bounded diameter, and the volume of a ball of radius R around a fixed point grows at the rate R^3 as $R \to \infty$.

Now we choose 2 distinct points $\boldsymbol{x}_1, \boldsymbol{x}_2$ in $Q = \mathbb{R}^3$, and let

$$V = \frac{1}{2|\boldsymbol{x} - \boldsymbol{x}_1|} + \frac{1}{2|\boldsymbol{x} - \boldsymbol{x}_2|}.$$

Then we obtain the well-known *Eguchi-Hanson space*. It contains an embedded 2-sphere S with self-intersection -2, which can be seen as the inverse image under π of the line segment connecting $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The space P is diffeomorphic to the total space T^*S, but the underlying complex manifold depends on choices of the complex structure J. For generic J, it is bi-holomorphic to the affine hypersurface $\{z_1^2 + z_2^2 + z_3^2 = 1\}$; for two special choices of J, S becomes a complex submanifold and the space is bi-holomorphic to the total space of the holomorphic line bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^1$. The Eguchi-Hanson metric is asymptotic to the flat cone given by the quotient of $\mathbb{C}^2$ by the involution $z \mapsto -z$. This follows from the observation that for $\boldsymbol{x}$ large, V is approximately $\frac{2}{2r}$. The Eguchi-Hanson space provides a model for singularity formation of Calabi-Yau metrics, and we will meet it several times later in this article.

As a more general example of Q, we consider the product of a flat 2-torus $\mathbb{T}^2$ and $\mathbb{R}$, where the area of $\mathbb{T}^2$ is normalized to be 2π. Notice that in this case on Q there is no globally positive harmonic function with singularities. But we can take for example $V = bz$ for a positive integer b, where z is the coordinate on $\mathbb{R}$. This is positive on the subset $\{z \geq 1\}$, so the Gibbons-Hawking ansatz yields a Calabi-Yau metric with a boundary and a complete end. The resulting space P exhibits interesting inhomogeneous Riemannian geometry: as z tends to infinity, the size of the S^1 orbits shrinks, whereas the size of the base $\mathbb{T}^2$ expands. The volume of a ball of radius R around a fixed point grows at the fractional rate $R^{4/3}$. See Figure 1.

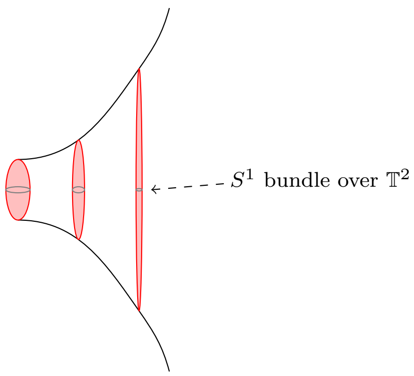

Figure 1: The Calabi model space

In terms of complex geometry, we can choose a complex structure so that the obvious projection map to the Riemannian surface $\mathbb{T}^2$ is holomorphic. Then P is bi-holomorphic to a neighborhood of the zero section in a holomorphic line bundle over $\mathbb{T}^2$, minus the zero section, and the S^1 action acts by the standard multiplication on each fibers. Viewed this way, the Calabi-Yau metric on P agrees with the metric obtained via the *Calabi ansatz*. The latter is a general way of producing a canonical metric on the total space of a holomorphic vector bundle over a given Kähler manifold. In our setting, we consider an $n-1$ dimensional complex manifold endowed with a Kähler metric ω_D. Suppose the cohomology class of $[\frac{1}{2\pi}\omega_D]$ is integral. Then there exists a holomorphic line bundle L_D over D and a hermitian metric $\|\cdot\|$ on L_D whose *Chern connection* has curvature form given by $-\sqrt{-1}\omega_D$. Now if Ω_D is a holomorphic volume form on D and ω_D is Calabi-Yau, then one can write down a holomorphic volume form Ω on the complement of the zero section $\mathbf{0}$ in the total space of L_D: in a local trivialization $U \times \mathbb{C}$ of L_D, Ω is given by $u^{-1}du \wedge \Omega_D$, where u is the coordinate on $\mathbb{C}$. Then one can look for a Calabi-Yau metric on $L_D \setminus \{\mathbf{0}\}$ of the form $\omega = \sqrt{-1}\partial\bar{\partial}f(\|\zeta\|^2)$, where ζ denotes a point in L_D and f is a function of one variable. Equation (1) becomes an ODE in f, and one can find a solution given by $f(t) = (-\log t)^{\frac{n+1}{n}}$. It gives a complete Calabi-Yau metric defined on the manifold with boundary defined by $|t| \leq \frac{1}{2}$. We call this a *Calabi model space*. When $n = 1$ and D is the flat torus $\mathbb{T}^2$, this coincides with the above construction via the Gibbons-Hawking ansatz (see [HSVZ]).

In [TY90], Tian-Yau constructed complete Calabi-Yau metrics asymptotic to the Calabi model space. The underlying complex manifold is the complement of a smooth anti-canonical divisor D in a Fano manifold X. Examples include the complement of a smooth hypersurface in $\mathbb{CP}^{n+1}$ of degree $n+2$. The central point is the fact that D is itself a Calabi-Yau variety, and a punctured neighborhood of D in X can be approximated by the space $L_D \setminus \{\mathbf{0}\}$ as above, where L_D is the holomorphic normal bundle of D in X. The importance of both the Calabi model spaces and the Tian-Yau metrics has been seen recently in the study of the degeneration of Calabi-Yau metrics. We will discuss this later in this article.

Of course one can play with the Gibbons-Hawking ansatz and generate many more examples of non-compact Calabi-Yau metrics. Often they have significance in modeling singularity formations of Calabi-Yau metrics, just like the Eguchi-Hanson spaces.

Gluing construction on K3 surfaces

Now we describe a very different method of constructing compact Calabi-Yau manifolds, via the *gluing* technique. There are many references on this, see for example [Don12].

Let $\mathbb{C}^2$ be endowed with a standard flat Kähler metric ω_0 and holomorphic volume form Ω_0. Fix a lattice Γ in $\mathbb{C}^2$ and denote by $\mathbb{T} = \mathbb{C}^2/\Gamma$ the corresponding flat torus. The map $\iota : (z_1, z_2) \mapsto -(z_1, z_2)$ on $\mathbb{C}^2$ induces an involution on $\mathbb{T}$ with $2^4 = 16$ fixed points. The quotient space $Y = \mathbb{T}/\langle\iota\rangle$ is an orbifold with 16 singular points, each of which is modeled on $\mathbb{C}^2/\langle\iota\rangle$. Denote by Y^0 the smooth locus of Y. Then both ω_0 and Ω_0 descend to Y^0, making it an (incomplete) Calabi-Yau manifold.

One can easily resolve the singularities of Y in the sense of complex geometry, by the well-known *Kummer construction.* To see this we consider the local model near each singularity. Let $\widehat{\mathbb{C}^2}$ be the blow-up of $\mathbb{C}^2$ at the origin. Then ι has a natural lift to $\widehat{\mathbb{C}^2}$, and the quotient $Q = \widehat{\mathbb{C}^2}/\langle\iota\rangle$ is a smooth complex surface which can be viewed as a resolution of singularity of $\mathbb{C}^2/\langle\iota\rangle$. One can show that Ω_0 induces a holomorphic volume form on Q, so that Q is a non-compact Calabi-Yau variety. Notice Q can be identified with the total space of the line bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^1$, so in particular it contains a holomorphically embedded $\mathbb{CP}^1$ with self-intersection -2. One can make this construction at each of the 16 orbifold points of Y and obtain a compact Calabi-Yau variety X which contains 16 disjoint exceptional spheres. This is an example of a *K3 surface.*

We want to construct Calabi-Yau metrics on X, by perturbing the flat metric on Y^0. For this purpose we need to graft to Y^0 certain Calabi-Yau metrics on Q. Notice that unlike the holomorphic volume form, the Kähler metric ω_0 cannot extend smoothly across the exceptional sphere on Q. Conceptually this is because the induced metric space structure on Y^0 is intrinsic so its singular behavior does not depend on the choice of coordinates. Instead one can explicitly solve the equation (1) again via the Calabi ansatz. Namely, take a function of one variable $F = F(\eta)$ for $\rho > 0$ and consider a Kähler metric of the form $\omega = \sqrt{-1}\partial\bar{\partial}F(|z|^2)$ on $\mathbb{C}^2 \setminus \{0\}$. For ω to define a smooth and complete Calabi-Yau metric on Q, we may take

$$F(\eta) = \sqrt{\eta^2+1} + \log \frac{\eta}{\sqrt{\eta^2+1}+1}.$$

This is indeed the Eguchi-Hanson metric in disguise. For $\delta > 0$, the function $F_\delta(\eta) = \delta^2 F(\delta^{-2}\eta)$ defines a Kähler metric isometric to $\delta^2\omega$.

Now given $\underline{\epsilon} = (\epsilon_1, \cdots, \epsilon_{16})$ such that each ϵ_j is small and positive, we choose 16 Eguchi-Hanson spaces, suitably rescaled so that the exceptional spheres have area ϵ_j^2 $(j = 1, \cdots, 16)$, and attempt to glue them to a neighborhood of the 16 singular points on Y. More precisely, fix a smooth cut-off function $\chi : (0, \infty) \to [0, \infty)$ which vanishes for $t \leq \frac{1}{2}$ and equals 1 for $t \geq 1$. Then we define a Kähler metric $\omega_{\underline{\epsilon}}$ on Y^0, which near each orbifold point p_j is given by the interpolation $\sqrt{-1}\partial\bar{\partial}(\chi(\epsilon_j^{-1}|z|^2)|z|^2 + (1-\chi(\epsilon_j^{-1}|z|^2))F_{\epsilon_j}(|z|^2))$, and agrees with ω_0 outside a neighborhood of the orbifold points. By definition it can be viewed as a smooth Kähler metric on X, and it solves Equation (1) except in a small "transition region" around the orbifold points.

Now one wants to correct the error and deform $\omega_{\underline{\epsilon}}$ to a genuine Calabi-Yau metric of the form $\widetilde{\omega_{\underline{\epsilon}}} = \omega_{\underline{\epsilon}} + \sqrt{-1}\partial\bar{\partial}\phi_{\underline{\epsilon}}$. For this we appeal to a quantitative version of the implicit function theorem on Banach spaces. Notice that to make the error small, we need to choose $\underline{\epsilon}$ very small, but then the metric is very degenerate (for example, the curvature at the points on the exceptional spheres is comparable to ϵ_j^{-2}), and the usual elliptic estimate for the linearized operator, i.e., the Laplace operator, fails to be uniform in $\underline{\epsilon}$. However, to compensate for this, we can introduce certain *weights* into the Banach spaces. This captures the degenerate geometry and results in uniform weighted elliptic estimates. This is the crucial technical point of the construction, and the upshot is that the correction is possible if $\underline{\epsilon}$ is sufficiently small.

Compared with Yau's existence theorem, an obvious drawback of the gluing construction is that it can only describe a small open set of the space of all Calabi-Yau metrics. On the other hand, the benefit is that it provides a more precise geometric description of the metrics. More importantly, the gluing technique is also a general construction useful for many other geometric PDEs, where it is impossible to have an analogue of Yau's existence theorem. For example, it was used by Joyce to construct compact Riemannian manifolds with holonomy group G_2 and $Spin(7)$, which are other examples of special holonomy metrics.

The gluing construction, viewed in reverse, also yields examples of geometric degenerations of Calabi-Yau metrics, fitting into the general convergence theory of Riemannian metrics with bounded Ricci curvature. The latter, when applied to our setting, states that given a sequence of complete Calabi-Yau metrics (M_j, g_j) of fixed dimension with a choice of base point $p_j \in M_j$, after passing to a subsequence, one may always obtain a *Gromov-Hausdorff limit*, which is a complete metric space M_∞ with a base point p_∞. The notion of Gromov-Hausdorff convergence is a convenient one when talking about the convergence of metric spaces. The convergence above is, roughly speaking, obtained by discretizing M_j using an approximation by a locally finite set, and controlling uniformly the size of the latter using the non-negative Ricci curvature together with the Bishop-Gromov inequality. The main question is to understand the formation of *singularities* of M_∞.

The theory is well-developed when one imposes a further *volume non-collapsing condition*. This means that there exists $\epsilon > 0$ such that the volume of the unit geodesic ball in M_j centered at p_j has volume bounded below uniformly by ϵ. Assuming this, Cheeger-Colding theory implies that M_∞ decomposes into a disjoint union $\mathcal{R} \cup \mathcal{S}$. The *regular* set $\mathcal{R}$ is a smooth manifold endowed with a Calabi-Yau metric, and the *singular* set $\mathcal{S}$ is a closed set which is small in terms of Hausdorff dimension. Near a singular point of M_∞, one can dilate the metric and obtain metric tangent cones. It is expected that there is always a unique tangent cone at each singularity, but this has not been proven in general. Furthermore, one can take a sequence $\lambda_j \to \infty$ and obtain rescaled Gromov-Hausdorff limits of $(M_j, \lambda_j \omega_j)$. The set of all rescaled limits encodes more refined information of the singularity formation of the convergence. In general one can attempt to associate to this a *bubble tree structure*. Heuristically, this appears in many other areas of geometric analysis, such as harmonic maps and Yang-Mills connections.

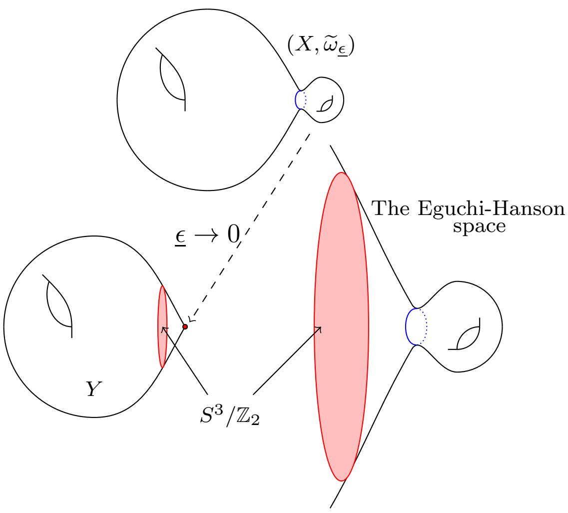

Figure 2: Gluing construction of Calabi-Yau metrics

In the above gluing construction, we are in the non-collapsing situation, and as $\underline{\epsilon} \to 0$, the Gromov-Hausdorff limit of the Calabi-Yau metrics $(X, \widetilde{\omega}_{\underline{\epsilon}})$ is exactly the flat orbifold we start with. There is one non-trivial rescaled limit at each singular point, given by the Eguchi-Hanson space (Figure 2 illustrates the picture at one singular point). This manifests the fact that the Eguchi-Hanson space is a local *model* for the singularity formation of Calabi-Yau metrics.

Moduli spaces

Let (M, g) be a compact Calabi-Yau manifold of complex dimension n. By Theorem 0.1 we know that the space of all Calabi-Yau metrics on M can be characterized in terms of essentially two pieces of algebro-geometric data: the complex structure and the cohomology class of a Kähler metric. Consequently the local deformations of the Calabi-Yau metric are well-understood. First, it is a result of Todorov and Tian that there is a *versal* deformation space of complex structures, which is itself an open set in the complex vector space $H^{n-1,1}(M; \mathbb{C})$. Then once we fix the complex structure, the set of cohomology classes of Kähler metrics form the *Kähler cone*, which is an open cone in the real vector space $H^{1,1}(M; \mathbb{C}) \cap H^2(M; \mathbb{R})$.

It is then natural to study the global structure of the moduli space of Calabi-Yau metrics. In low dimensions, we have the classical results. Namely, when $n = 1$, a compact Calabi-Yau variety is simply an elliptic curve. The Kähler cone is a one dimensional ray, and the moduli space of complex structures modulo diffeomorphisms is the usual modular curve, given by the quotient of the upper half plane by $SL(2; \mathbb{Z})$.

When $n = 2$, a compact Calabi-Yau variety is either a complex torus or a K3 surface. Calabi-Yau metrics on a torus are always flat and their moduli spaces are higher dimensional generalizations of the case $n = 1$. Calabi-Yau metrics on a K3 surface are never flat. It is a fact that the underlying oriented differentiable manifold of all K3 surfaces is unique, which we denote by $\mathcal{K}$. It is simply-connected and the cup product on $H^2(\mathcal{K}; \mathbb{Z})$ has signature $(3, 19)$. Denote by $\mathcal{M}$ the set of all Calabi-Yau metrics on $\mathcal{K}$, normalized to have unit diameter, modulo the action of the diffeomorphism group $\mathrm{Diff}(\mathcal{K})$. Even though a single Calabi-Yau metric on $\mathcal{K}$ does not have an explicit formula, the space $\mathcal{M}$ can be globally understood using the Torelli theorem as follows (see for example [KT87]). Given a Calabi-Yau metric g on $\mathcal{K}$, after choosing a complex structure we have a holomorphic volume form

Ω and a Kähler metric ω, we then associate to this data a 3 dimensional subspace in $H^2(\mathcal{K};\mathbb{R})$ given by the span of $\omega, Re(\Omega), Im(\Omega)$. One can see that this subspace does not depend on the choice of the complex structure. Furthermore, it is positive definite with respect to the intersection form, so it gives rise to a point in the positive Grassmannian $Gr^+(3, H^2(\mathcal{K};\mathbb{R})) \simeq O(3,19)/(O(3)\times O(19))$. On the other hand, the action of Diff(M) is via the homomorphism Diff$(M) \to Aut(H^2(\mathcal{K};\mathbb{Z})))$, and the image is the index 2 subgroup Γ preserving the orientation of the positive part of $H^2(\mathcal{K};\mathbb{Z})$. In this way we produce a period map

$$\mathcal{P}:\mathcal{M}\to\mathcal{D}\equiv\Gamma\setminus Gr^+(3, H^2(\mathcal{K};\mathbb{R})).$$

The Torelli theorem guarantees that $\mathcal{P}$ is injective and furthermore, the image is surjective onto the complement of the set of positive 3 dimensional subspaces $V\subset H^2(\mathcal{K};\mathbb{R})$ such that there exists $\delta\in H_2(\mathcal{K};\mathbb{Z})$ with $\delta^2=-2$ and $\int_\delta\alpha\neq 0$ for all $\alpha\in V$. This exceptional set can be filled in if one includes the non-collapsing Gromov-Hausdorff limits of elements in $\mathcal{M}$, i.e, the Calabi-Yau orbifolds in complex dimension 2.

The higher dimensional situation is more challenging and there are very few general results. The following is a folklore question

Question 0.3. *Given $n\geq 3$, are there infinitely many topologically distinct compact Calabi-Yau varieties in dimension n?*

When $n=3$, it is known that there exist at least tens of thousands of topologically distinct compact Calabi-Yau varieties. On the other hand, there is *Reid's fantasy* hoping that all compact Calabi-Yau varieties can be connected via geometric transitions, through suitable classes of singular Calabi-Yau varieties.

A closely related question is to study the compactification of a fixed connected component of the moduli space of all Calabi-Yau metrics. Namely, we want to know the behavior of a sequence of Calabi-Yau metrics (M,ω_i,J_i) when either ω_i or J_i do not converge smoothly. The Gromov compactness theorem provides a rough compactification by adding certain limit metric spaces. But the latter do not carry much geometric information since we do not yet have a satisfactory understanding of the singularity formation process in general. Nevertheless this question has already generated many new questions, which are very much related to complex/algebraic geometry. There are two main directions, which we distinguish as *Kähler degenerations* and *complex structure degenerations*, and we will discuss these in the following sections.

Kähler degenerations

Fix a compact Calabi-Yau variety X of complex dimension n, and denote by $\mathcal{K}$ its Kähler cone. By a Kähler degeneration we mean a sequence of Calabi-Yau metrics ω_i on X whose cohomology classes $\beta_i\in\mathcal{K}$ converge to a non-zero limit $\beta_\infty\in\partial\mathcal{K}$. The geometric question is to understand the metric behavior of ω_i as $i\to\infty$. There have been extensive recent results in this direction, most of which make use of known results from algebraic geometry. Below we will only loosely describe the expected picture, and refer to the excellent survey paper of Tosatti [Tos] for references on the precise results and other recent progress and open problems on this topic.

We divide the discussion in two cases. First we assume $\int_X\beta_\infty^n>0$. This is the volume non-collapsing situation. In this case, the conjectural geometric picture is that as $i\to\infty$, the Calabi-Yau metrics ω_i perform certain birational contraction of the *null-locus* of β_∞, which is by definition the union of all subvarieties V with $\int_V\beta_\infty^{\dim_{\mathbb{C}}V}=0$, and the Gromov-Hausdorff limit should be a generalized Calabi-Yau metric on a singular Calabi-Yau variety Y. For example, the gluing construction discussed above fits into this picture - in that case Y is the flat orbifold. This conjecture is known to be true when the class β_∞ is rational. In general the question is related to understanding the singularity structure of non-collapsed Gromov-Hausdorff limits of Calabi-Yau metrics.

Next we assume $\int_X\beta_\infty^n=0$. In this case the Gromov-Hausdorff limits have lower dimensions. A naive hope is that under suitable conditions the collapsing is along certain holomorphic directions. The best scenario is when X admits a holomorphic fibration $p:X\to Y$ onto a possibly singular algebraic variety Y, with generic fiber a lower dimensional Calabi-Yau variety. Then one expects that as $i\to\infty$, the Calabi-Yau metrics ω_i contract the fibers of p and the Gromov-Hausdorff limit should be a generalized Kähler metric on Y whose Ricci curvature characterizes the variation of complex structures on the fibers of p. The existence of the algebraic fibration when β_∞ is rational is related to a version of the *abundance conjecture* in algebraic geometry. Differential geometrically it is an intriguing question to investigate what happens to ω_i near the singular fibers of p. The first work in this direction was by Gross-Wilson [GW00], who studied the case of a K3 surface fibered over $\mathbb{CP}^1$ with 24 nodal fibers. Near a singular point p of a singular fiber F, the holomorphic fibration is modeled on the map $\mathbb{C}^2\to\mathbb{C};(z_1,z_2)\mapsto z_1z_2$. Geometrically, in a neighborhood of F the collapsing Calabi-Yau metric is modeled on an incomplete *Ooguri-Vafa* metric, constructed by applying the Gibbons-Hawking ansatz to an open set in the flat manifold $S^1 \times \mathbb{R}^2$. Moreover, when further restricted to a small neighborhood of p, it is close to a rescaling of the Taub-NUT metric. This picture also generalizes to higher dimensions; see for example [Li19] in the case of a holomorphic K3 fibered Calabi-Yau 3-fold with only nodal singular fibers.

When β_∞ is irrational, wild phenomenon can occur. For example, even on a flat torus, one may see the degeneration along a holomorphic foliation with a dense leaf, whereas the Gromov-Hausdorff limit can be a point.

Complex structure degenerations

Discussion of general complex structure degenerations will necessarily involve some sophisticated terminologies in algebraic geometry, so to avoid introducing unnecessary technical jargon, we will focus on the examples of Example 0.2 as these exhibit the main characteristics of the theory. Consider a family of degree $n+2$ homogeneous polynomials $F_t(Z_0, \cdots, Z_{n+1})$ parameterized by $t \in \mathbb{C}$, and denote by X_t the corresponding family of hypersurfaces in $\mathbb{CP}^{n+1}$. If X_t is smooth for $0 < |t| \ll 1$ and X_0 is singular, then this family can often be viewed as a degeneration of complex structures. We would like to understand the behavior of the Calabi-Yau metrics ω_t as $t \to 0$. Compared to the case of Kähler degenerations, the link with algebraic geometry is less obvious and more transcendental geometric objects may naturally arise. For example, the algebro-geometric limit X_0 is in general not canonical. Such a situation occurs when there is a family of linear transformation $A_t \in GL(n+2; \mathbb{C})$ defined for $t \neq 0$, such that as $t \to 0$ the new family of polynomials $\widetilde{F}_t \equiv A_t^* F_t$ converge to a limit $\widetilde{F}_0$ but A_t itself diverges. Then as algebraic varieties the corresponding family of hypersurfaces X'_t are isomorphic to X_t for $t \neq 0$ but in general X'_0 may not be isomorphic to X_0. On the other hand, the Calabi-Yau metrics are more intrinsic objects and one may hope that they would yield canonical Gromov-Hausdorff limits. Understanding the algebro-geometric meaning of these limits could lead to canonical algebro-geometric invariants of the degeneration. Again there is little general theory known so far, but there are some examples through which one can already see interesting geometric phenomenon and gain insight for further development.

Singular Calabi-Yau metrics

Suppose first that we are in the most generic situation, namely, when X_0 has only conifold singularities. The latter are by definition isolated singularities locally modeled on the complex hypersurface

$$\mathbb{S} = \{w_1^2 + \cdots w_{n+1}^2 = 0\}$$

in $\mathbb{C}^{n+1}$. We know there is an explicit "Stenzel metric" on $\mathbb{S}$ given by

$$\omega_{\mathbb{S}} = \sqrt{-1}\partial\bar{\partial}\big(\sum_{i=1}^{n+1} |w_i|^2\big)^{\frac{n-1}{n}}.$$

Geometrically this space is a metric cone whose cross section is a homogeneous space $SO(n+1)/SO(n-1)$. When $n=2$, one can identify the cone with the flat orbifold $\mathbb{C}^2/\langle \pm 1 \rangle$, but when $n \geq 3$ the metric has unbounded curvature near the cone vertex.

The following theorem confirms a folklore conjecture that goes back to Candelas and de la Ossa [CdlO90] in 1990.

Theorem 0.4 ([HS17])**.** *Suppose X_0 has only conifold singularities. Then*

(a) as $t \to 0$, the Gromov-Hausdorff limit of (X_t, ω_t) is a singular Calabi-Yau metric on X_0, which is smooth away from the singularities, and near each singularity it is asymptotic to the Stenzel metric;

(b) for each nodal singularity q of X_0, there is a special Lagrangian n-sphere in X_t for $0 < |t| \ll 1$, which converges to q as $t \to 0$.

The second item requires a little explanation. A half dimensional embedded submanifold L of an n dimensional Calabi-Yau manifold (M, Ω, ω) is *special Lagrangian* if $\omega|_L = Im(e^{\sqrt{-1}\theta}\Omega)|_L = 0$ for some constant θ. Special Lagrangians are examples of calibrated submanifolds in the sense of Harvey-Lawson. In particular they are globally volume minimizing within a given homology class. They also play a crucial role in the Strominger-Yau-Zaslow (SYZ) mirror symmetry [SYZ96]. When $n=2$, special Lagrangians become complex submanifolds for a different choice of complex structure, but in higher dimensions they are genuinely different geometric objects.

The proof of (a) makes use of a variety of techniques. In particular, it depends essentially on previous work of S. Donaldson and the author about the complex geometry of Gromov-Hausdorff limits, which was motivated by different reasons.

The proof of (b) follows from understanding the bubbles associated to each singularity, which is in turn based on a gluing construction similar to the case of K3 surfaces. The point is that there is a complete Calabi-Yau metric on the *smoothing* of $\mathbb{S}$ given by $\mathbb{S}_\epsilon = \{w_1^2 + \cdots + w_{n+1}^2 = \epsilon\} \subset \mathbb{C}^{n+1}$ which is asymptotic to the Stenzel cone at infinity. When $n = 2$ this is again the Eguchi-Hanson metric in disguise.

Collapsing to lower dimensions

We now consider the family of hypersurfaces X_t in $\mathbb{CP}^{n+1}$ defined by the polynomials

$$F_t(Z) = tg(Z) + f_0(Z) \cdots f_k(Z), \quad t \in \mathbb{C},$$

where g has degree $n+2$, each f_i has degree $d_i > 0$ with $\sum_{i=0}^k d_i = n+2$. For a general choice of these polynomials one may ensure that X_t is smooth for $0 < |t| \ll 1$ and X_0 consists of precisely $k+1$ irreducible components intersecting transversally. One can form a *dual intersection complex*, which is a CW complex constructed by associating a point to each component of X_0, and two points are connected by a segment if two components intersect, etc.

As a simple model case we first suppose that $n = 1$. Here X_t is a family of elliptic curves. If $k = 1$, then X_0 is a union of a conic and a line intersecting at two points; if $k = 2$ then X_0 is a cycle of 3 lines. In both cases the dual intersection complex of X_0 is homeomorphic to the circle S^1 and the smooth locus of X_0 comprises a disjoint union of $\mathbb{C}^*$'s.

The Calabi-Yau metrics ω_t are obviously flat. Since the volume is fixed, it is easy to see that any Gromov-Hausdorff limit of ω_t can only be the infinite line $\mathbb{R}$. However we can rescale ω_t differently, and obtain two other limits:

- a circle S^1. This corresponds to fixing the diameter of ω_t;
- a flat cylinder $S^1 \times \mathbb{R} \simeq \mathbb{C}^*$. This arises as a rescaled limit.

One can make some simple observations from this example:

- In general when the volume is collapsing the dimension of Gromov-Hausdorff limits can be an odd integer. So one cannot expect a classical algebro-geometric interpretation.
- With the diameter fixed, the Gromov-Hausdorff limit is related to the dual intersection complex of X_0 at the topological level.
- Suitably rescaled Gromov-Hausdorff limits are quasi-projective algebraic varieties, which are related to the smooth locus of the algebraic limit X_0.

Small complex structure limits

Now we move to higher dimensions so that $n \geq 2$, and focus on the example where

$$F_t(Z) = tg(Z) + f_0(Z) f_1(Z), \quad t \in \mathbb{C},$$

with $d_i = \deg(f_i)$. In this case the dual intersection complex of X_0 is an interval, which has the smallest possible topological complexity. For simplicity we call this a "*small complex structure limit*".

By making g, f_0, f_1 generic, we can ensure that X_0 is the union of two irreducible components Y_0 and Y_1, which are transversally intersecting along $D = \{f_0 = f_1 = 0\} \subset \mathbb{CP}^{n+1}$. It is also helpful to view the total space of the family as a subset in $\mathbb{CP}^{n+1} \times \mathbb{C}$. As such it is singular along $H \times \{0\}$, where $H = \{f_0 = f_1 = g = 0\}$ can be assumed to be smooth in $\mathbb{CP}^{n+1}$. Notice H is contained in each X_t. See Figure 3.

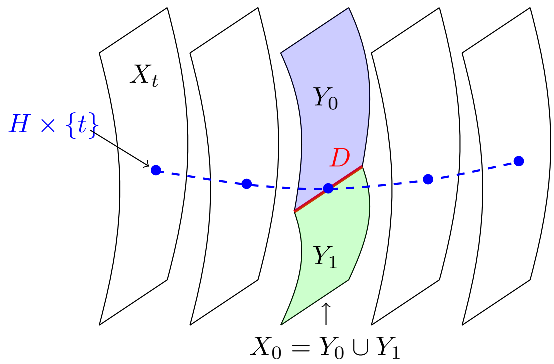

Figure 3: The algebraic picture for a small complex structure degeneration

Let $\widetilde{\omega}_t$ be the rescaling of ω_t to unit diameter. The following theorem characterizes the geometric behavior as $t \to 0$. The intuitive picture for the case $n = 2$, $k = 1$, $d_0 = 1, d_1 = 3$, is described in Figure 4.

Theorem 0.5 ([SZ]). *We have*

- *the Gromov-Hausdorff limit of $\widetilde{\omega}_t$ as $t \to 0$ is the interval $[0, 1]$;*
- *certain rescaled limits around the end points of the interval give rise to some complete Calabi-Yau metrics on $Y_i \setminus D$. Indeed, these metrics are the ones constructed by Tian-Yau* [TY90] *mentioned before.*

In particular, the simple observations we made in the case $n = 1$ continue to hold in this case.

Notice that in Figure 4 there is a neck region $\mathcal{N}$. It admits a singular fibration $\mathcal{F} : \mathcal{N} \to (0, 1)$. For $u \neq u_* \equiv \frac{d_0}{d_0+d_1}$, $\mathcal{F}^{-1}(u)$ is a smooth fiber bundle, and each fiber itself is an S^1 bundle over D. The degree of the S^1 bundle undergoes a wall-crossing as u crosses u_*. The fiber $\mathcal{F}^{-1}(u_*)$ is an S^1 bundle over $D \setminus H$, and the circle fibers pinch to a point over H. In Figure 4, H consist of 12 points, illustrated by the $\times$'s. Appropriately rescaled Gromov-Hausdorff limits around these points in $H \subset X_t$ yield the Taub-NUT metric on $\mathbb{C}^2$.

It is worth noticing that most of the topology concentrates in the Tian-Yau region, whilst most of the volume is located in $\mathcal{N}$. This is also manifested by the fact that in the Gromov-Hausdorff limit one can define a renormalized limit measure, whose density function with respect to the Lebesgue measure on $[0, 1]$ vanishes at the end points.

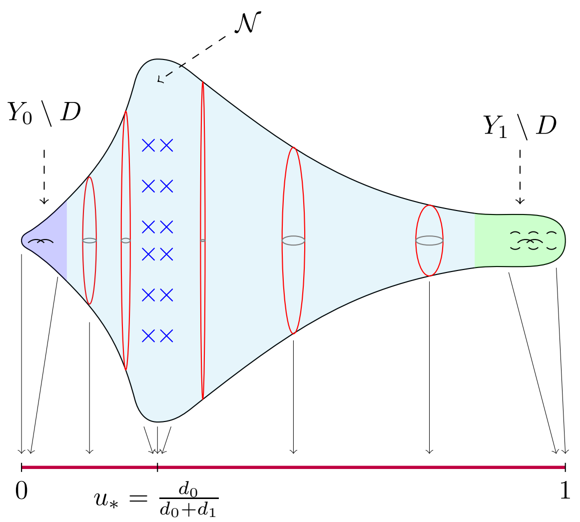

Figure 4: The metric picture for a small complex structure degeneration

The proof of the above results is via a gluing construction. For simplicity first assume $n = 2$. The starting point is that we know each $Y_i \setminus D$ admits a complete Tian-Yau metric, which is asymptotic to a Calabi model space $\mathcal{C}_i$ constructed via the Gibbons-Hawking ansatz. Then the crux of the matter is to match the ends of the two Calabi model spaces. This motivates the construction of the neck region $\mathcal{N}$. This time we apply the Gibbons-Hawking ansatz to the Green's function on $\mathbb{T}^2 \times \mathbb{R}$ with finitely many poles. One can see this function is asymptotic to a piecewise linear function, which tends to $-\infty$ at the two ends. Adding a large positive number we can make it positive over any fixed large interval, such that the resulting space has two boundaries that match well with the two Calabi model spaces. This is the heuristic reason why a gluing construction is possible. The actual proof in [SZ] is substantially more involved. One issue is that in higher dimensions there is a generalization of the Gibbons-Hawking ansatz by D. Matessi, but the resulting equation is still non-linear and one cannot write down explicit precise solutions. Instead one has to appeal to an adiabatic limit method and use a certain linearized equation to obtain approximate solutions. Another issue is that the space $\mathcal{N}$ cannot immediately be seen from the algebraic degeneration picture. To make a connection between them we have to design a birational transformation of the total space of the family.

More general setting

The work of [SZ] leads one to speculate that the trivial observations in the case $n = 1$ may indeed hold in general. In other words, the Gromov-Hausdorff limit of X_t should be closely related to the dual intersection complex of X_0. For example, [SZ] conjectured that they should be homeomorphic as topological spaces, and they have the same dimension (which is well-defined for both). More precise geometric relationship will involve the *non-Archimedean geometry*, which we do not discuss here, but see [Li] for example.

For the family of hypersurfaces we considered above, when $k = n + 1$, the dual intersection complex is an n dimensional sphere. These are examples of *large complex structure limits*, and the expectation reduces to a limiting version of the SYZ conjecture [SYZ96], as formulated by Gross-Wilson [GW00] and Kontsevich-Soibelman [KS06]. In this case, the limit space is conjectured to be a Riemannian metric with singularities on the sphere, and the collapsing is along a fibration by special Lagrangian tori. In the generic region the model is given by a product $U \times \sqrt{-1}(\mathbb{R}^n/\epsilon\mathbb{Z}^n)$, where $U \subset \mathbb{R}^n$ is an open set, and it is endowed with the standard complex structure. Consider a Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ for an $\mathbb{R}^n$-invariant function ϕ. Then ω is a Calabi-Yau metric if and only if ϕ solves a real Monge-Ampère equation $\det(\phi_{ij}) = C$. Given such ϕ, letting $\epsilon \to 0$ yields a collapsing along a special Lagrangian $\mathbb{T}^n$ fibration to a Riemannian metric on U defined by the Hessian of ϕ.

There has been recent progress made by Y. Li in this direction. In particular, for the family of hypersurfaces defined by

$$F_t = Z_0 \cdots Z_{n+1} + t(Z_0^{n+2} + \cdots Z_{n+1}^{n+2}),$$

it is shown in [Lib] that for $|t|$ sufficiently small, X_t has a set of large volume which is locally close to the above model. The proof makes heavy use of pluripotential theory on compact Kähler manifolds (see for example [Kol98]).

Open Calabi-Yau manifolds

An *open* Calabi-Yau manifold is by definition a complete non-compact Calabi-Yau manifold. We have seen that they may appear as rescaled limits of degenerations of compact Calabi-Yau metrics. The study of open Calabi-Yau metrics also has its own interest, in particular in the case of complex dimension 2 when the curvature tensor is square integrable, they are the same as *gravitational instantons* and have long been studied in both mathematics and physics. There is a variety of constructions of gravitational instantons in the literature, which exhibit many non-trivial asymptotic behaviors.

The underlying complex/algebraic geometry of open Calabi-Yau metrics is not well-understood in general. For example, the naive uniqueness as in the compact case can fail - the Taub-NUT metric on $\mathbb{C}^2$ provides an immediate counterexample.

Optimistically we have the following longstanding *compactification conjecture* of Yau [Yau82]:

Conjecture 0.6. *An open Calabi-Yau manifold is bi-holomorphic to $M \setminus D$, where M is a compact Kähler manifold and D is a divisor in M.*

Note this conjecture does not hold on the nose. Using the Gibbons-Hawking ansatz applied to a harmonic function on $\mathbb{R}^3$ with infinitely many poles, Anderson-Kronheimer-LeBrun [AKL89] constructed an open Calabi-Yau manifold with infinite topology, hence provides a counterexample to the above conjecture. However, one can still hope it to be true under suitable extra hypothesis, for example, when we assume a condition of curvature decay or finite topology.

A related question is to develop an existence theory of Calabi-Yau metrics on non-compact Calabi-Yau varieties given in the above form $M \setminus D$. There are partial results in this direction (see for example [TY90]) by constructing a model infinity and adapting Yau's approach in the compact case. There is much in this field that remains to be explored.

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