Collapsing Geometry of hyperkähler 4-manifolds

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In this talk I will give an overview of some recent progress on 4d hyperkähler metrics.

I will explain some applications of our analysis on the “collapsing" geometry of these objects (joint work with R. Zhang). There will be two main theorems, one for compact manifolds, and one for open manifolds. In the meantime I will discuss many examples and related previous results.
Plan:

(1) Basics

(2) Compact setting: K3 manifold

(3) Open setting: gravitational instantons
Part 1: Basics

A Riemannian 4-manifold \((X, g)\) is hyperkähler if its holonomy group is contained in \(SU(2) \subset SO(4)\).

This means that there are 3 parallel orthogonal complex structures \(J_1, J_2, J_3\) satisfying \(J_1 J_2 = J_3\) etc, and hence an \(S^2\) family of these

\[
J = a_1 J_1 + a_2 J_2 + a_3 J_3,
\]

where

\[
a_1^2 + a_2^2 + a_3^2 = 1.
\]
Properties:

\[ \text{Ric}(g) = 0; \]
\[ W^+(g) = 0; \]

If \( X \) is compact, then by Chern-Gauß-Bonnet,

\[
\int_X |Rm(g)|^2 = 8\pi^2 \chi(X).
\]

This is the simplest non-trivial case of an Einstein metric, and of a metric with special holonomy. So understanding the analysis of hyperkähler metrics is the first step towards the more general setting, where many fundamental questions remain open.
A hyperkähler structure can be alternatively characterized by the existence of a triple of closed 2-forms \( \{\omega_1, \omega_2, \omega_3\} \) satisfying

\[
\frac{1}{2} \omega_i \wedge \omega_j = \delta_{ij} \, d\text{vol}_g. \tag{1}
\]

For example, \( \omega_1 \) determines a complex structure \( J_1 \) such that \( \omega := \omega_1 \) is a Kähler form, and \( \Omega := \omega_2 + \sqrt{-1}\omega_3 \) is a holomorphic 2-form.

(1) can be re-written as

\[
\omega^2 = \frac{1}{2} \Omega \wedge \bar{\Omega}
\]

This is the complex Monge-Ampère equation for a *Calabi-Yau metric*. 
When $X$ is compact and non-flat, it follows that $(X, J_1)$ must be a complex K3 surface.

Example: One model of a complex K3 surface is given by a smooth quartic surface in $\mathbb{CP}^3$.

The holomorphic 2-form $\Omega$ on a K3 surface can sometimes be explicitly written down. The existence of the Kähler form $\omega$ satisfying the complex Monge-Ampère equation is guaranteed by the Calabi-Yau theorem, but it is far from being explicit.

The $K3$ manifold $\mathcal{K}$ the (unique) oriented differentiable manifold underlying a complex K3 surface.
For non-compact manifolds, we can consider hyperkähler metrics with a continuous symmetry. They are locally given by the Gibbons-Hawking ansatz, which reduces the hyperkähler equation to the Laplace equation.

Take a positive harmonic function $V$ on a domain $Q \subset \mathbb{R}^3$, with $\frac{1}{2\pi}[*dV] \in H^2(Q; \mathbb{Z})$. One can associate a principal $S^1$ bundle $P$ together with a unitary connection $-i\theta$, such that $d\theta = *dV$. Then $P$ is endowed with a hyperkähler metric

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2,$$

with $\omega_1 = Vdx \wedge dy + dz \wedge \theta$ etc.

It is invariant under the free $S^1$ action.
If $V$ has singularities locally of the form $\frac{1}{2r}$, then the metric can be compactified by adding a point fixed by the $S^1$ action.

$$V = \sum_{j=1}^{k} \frac{1}{2|x-x_j|} \text{ on } \mathbb{R}^3 \rightsquigarrow \text{ALE hyperkähler metric.}$$

- $k = 1 \rightsquigarrow$ the flat $\mathbb{R}^4$ with the Hopf fibration $\mathbb{R}^4 \to \mathbb{R}^3$. Replacing $V$ by $V + 1 \rightsquigarrow$ the Taub-NUT metric on $\mathbb{R}^4$.
- $k = 2 \rightsquigarrow$ the Eguchi-Hanson metric on $T^*S^2$. It is asymptotic to the flat cone $\mathbb{R}^4/\mathbb{Z}_2$ at infinity.

![Diagram](attachment:diagram.png)
Part 2: K3 manifold

Even though a single hyperkähler metric on the K3 manifold does not have explicit form, the moduli space

\[ \mathcal{M} = \{ \text{hyperkähler metrics on } \mathcal{K} \text{ with diameter } 1 \} / \text{Diff}_0(\mathcal{K}) \]

can be globally described using the Torelli theorem.

We know \( H^2(\mathcal{K}; \mathbb{R}) \cong \mathbb{R}^{3,19} \). Any hyperkähler metric \( g \) determines a 3d positive subspace \( \mathbb{H}^+_g \) spanned by the \( \omega_i \)'s.
We define the period map:

\[ P : \mathcal{M} \to \mathcal{D} := Gr^+ (\mathbb{R}^{3,19}); \quad [g] \mapsto \mathbb{H}_g^+. \]

Torelli theorem: \( P \) is injective, and is surjective onto

\[ \mathcal{D}^* = \mathcal{D} \setminus \bigcup_{\delta \in H_2(K;\mathbb{Z}), \delta.\delta = -2} \delta^\perp, \]

where \( \delta^\perp \) consists of the set of all 3 dimensional positive subspaces \( H \subset H^2(K;\mathbb{R}) \) so that for all \( \alpha \in H, \int_\delta \alpha = 0. \)

In particular, \( \dim_{\mathbb{R}} \mathcal{M} = 57. \)
It is an interesting question to understand the compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$, under the Gromov-Hausdorff topology.

Since a hyperkähler metric is Ricci-flat, one may appeal to the general convergence theory for Riemannian metrics with Ricci bounded (from below).

The latter is relatively well-established in the volume non-collapsing case, namely, when $\text{Vol}(\mathcal{K}, g) \geq \kappa$ for a fixed $\kappa > 0$. But the collapsing case is much more challenging (and hence intriguing!)
Specific to dimension 4, the results of Anderson, Bando-Kasue-Nakajima, Tian around 1990 assert that given a sequence $g_i \in \mathcal{M}$ satisfying the volume non-collapsing condition, we have

(1) any Gromov-Hausdorff limit is a compact hyperkähler orbifold;

(2) suitable rescaled limits give rise to ALE hyperkähler spaces, which are asymptotic to flat cones $\mathbb{R}^4 / \Gamma (\Gamma \subset SU(2))$ at infinity. Collecting all the rescaled limits gives rise to a finite “bubble tree”.

Crucial: $Vol(B_g(p, r))/r^4$ is decreasing in $r$ (Bishop-Gromov monotonicity). This gives rise to conical structures around singularities (Cheeger-Colding).
Conversely, any compact hyperkähler orbifold arises as such a limit, by a *glueing* construction.

An an example, we consider the Kummer construction of K3. Given a flat Kähler orbifold $T^4/\mathbb{Z}_2$, one can glue to it 16 Eguchi-Hanson spaces, and form an approximate solution to the hyperkähler equation. Then one uses the implicit function theorem to perturb it to a genuine hyperkähler metric.
Denote by $\overline{\mathcal{M}}^\partial$ the partial compactification of $\mathcal{M}$ by adding all compact hyperkähler orbifolds. Then $P$ extends to a bijection $P : \overline{\mathcal{M}}^\partial \to \mathcal{D}$.

A sequence $g_j \in \mathcal{M}$ is volume collapsing if and only if $P(g_j)$ diverges to infinity in $\mathcal{D}$.

Odaka and Oshima (2018) has a conjecture relating $\overline{\mathcal{M}}$ (modulo the mapping class group) to certain compactifications of $\mathcal{D}/\Gamma$ in representation theory.

In this talk we are interested in the metric collapsing geometry.
Known general theory in our setting

(1) The bound on $\int_K |Rm|^2$ implies that the collapsing is with uniformly bounded curvature away from finitely many points (singularities). This follows from the $\epsilon$-regularity theorem of Cheeger-Tian 2004 (specific to 4d Einstein metrics).

(2) When the volume collapses with bounded curvature, results of Cheeger, Fukaya and Gromov, tells us that (rougly speaking) the collapsing is locally along certain nilpotent fibrations.

However, little information is known around the singularities. The monotonicity formula for volume is not very useful in the collapsing situation. In particular, it does not a priori lead to a conical structure.
In the Kummer construction, one can let $T^4/\mathbb{Z}_2$ collapses to $T^k/\mathbb{Z}_2$ for $k = 1, 2, 3$, it follows that the flat $T^k/\mathbb{Z}_2$ can be a collapsing limit of elements $g_j \in \mathcal{M}$.

More generally, initiated by Gross-Wilson 2000, one may obtain as limit a (singular) special Kähler metric on $S^2$. These are obtained on an elliptically fibered complex K3 surface, by fixing the complex structure and collapsing the area of the fibers.
Theorem (S.-Zhang 2021)

An element in $\overline{\mathcal{M}} \setminus \overline{\mathcal{M}}^\partial$ is one of the following

1. (dim = 1) The unit interval $[0, 1]$.
2. (dim = 2) A singular special Kähler metric on $S^2$;
3. (dim = 3) A flat $T^3/\mathbb{Z}_2$;

Remark 1: This follows from a local result studying isolated singularities of collapsing limit of hyperkähler metrics. In particular, around each singularity there is a unique flat tangent cone.

Remark 2: In case (1), there is an extra information of normalized limit measure, which is given by a concave piecewise linear function, and encodes more refined collapsing geometry (Honda-S-Zhang 2019).
Remark 3: without a uniform bound on the energy $\int |Rm|^2$, a local isolated singularity may not be modeled on flat cones.

Consider the quotient space $\mathbb{C}^2/\mathbb{Z}_k$ ($\mathbb{Z}_k \subset SU(2)$), as the asymptotic cone of an ALE gravitational instanton $X_k$.

Let $k \to \infty$, $\mathbb{C}^2/\mathbb{Z}_k$ collapses to the Riemannian quotient $\mathbb{C}^2/S^1$, which is the cone over a round $S^2$ with radius $\frac{1}{2}$. So suitable rescalings of $X_k$ collapse to the same cone.
Part 3: Gravitational instantons

A gravitational instanton is a complete non-compact hyperkähler manifold $(X, g, p)$ with $\int_X |Rm|^2 < \infty$.

Gravitational instantons with non-maximal volume growth (i.e., $\text{Vol}(B_g(p, r)) = o(r^4)$ as $r \to \infty$) provide local models for collapsing of hyperkähler metrics.

Gravitational instantons exhibit very rich geometry and analysis. There are a variety of constructions using techniques in mathematics and physics.
Gravitational instantons can have diverse asymptotic structures at infinity.

However, one can observe that all the known examples do possess approximate *local* symmetries at infinity.

We say a gravitational instanton is AL if it is asymptotic to an AL end at a polynomial rate.
(1). ALE end: $\mathbb{C}^2/\Gamma$ for a finite subgroup of $\Gamma \subset SU(2)$. It is conical and admits local $SU(2)$ symmetry.

ALE gravitational instantons were constructed by Kronheimer 1989 using hyperkähler quotients. The underlying manifolds are diffeomorphic to minimal resolutions of $\mathbb{C}^2/\Gamma$.

$$\text{Vol}(B(p, r)) \sim r^4.$$
(2). ALF end:

An ALF $A_k$ end is given by applying the Gibbons-Hawking ansatz to the function $V = \frac{k}{2r} + C$ for $k \in \mathbb{Z}$ and $C > 0$. The asymptotic cone is $\mathbb{R}^3$. If $k = 1$, this is the Taub-NUT metric on $\mathbb{R}^4$.

An ALF $D_k$ end is the $\mathbb{Z}_2$ quotient of ALF $A_{2k}$ end. It only admits local $S^1$ symmetry. The asymptotic cone is $\mathbb{R}^3/\mathbb{Z}_2$. ALF $D_k$ gravitational instantons exist for $k \geq -2$. There are constructions due to Atiyah-Hitchin, Cherkis-Kapustin and others.

$\text{Vol}(B(p, r)) \sim r^3$. 
For $\beta \in (0, 1)$, take a wedge $C_\beta$ of angle $2\pi \beta$ in $\mathbb{R}^2$ and pick a flat torus $T^2$, then we can form a flat end by glueing the boundaries of $C_\beta \times T^2$.

In order to be hyperkähler we require the flat $T^2$ to be invariant under rotation by $2\pi \beta$. This implies

$$\beta \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\}.$$ 

It admits local $T^2$ symmetry. The asymptotic cone is $C_\beta$.

$\text{Vol}(B(p, r)) \sim r^2$. 
(4). ALH end: a flat product $T^3 \times [0, \infty)$. It admits $T^3$ symmetry. The asymptotic cone is $\mathbb{R}^+$. This is the familiar cylindrical end in geometry.

$$\text{Vol}(B(p, r)) \sim r.$$ 

Existence of ALG and ALH gravitational instantons can be derived by solving the complex Monge-Ampère equation on a rational elliptic surface removing a fiber with at worst finite monodromy (Tian-Yau, Hein).
There are two exceptional model ends.

The Gibbons-Hawking ansatz can be generalized to

(1) $\mathbb{T}^2 \times \mathbb{R}$, $V = bz$ for $b \in \mathbb{Z}_+$, where $z$ is the coordinate on $\mathbb{R}$;

(2) $S^1 \times \mathbb{R}^2$, $V = \frac{b}{\pi} \log r$ for $b \in \mathbb{Z}_+$, where $r$ is the radial function on $\mathbb{R}^2$.

In the first case we get an ALH* end. $\text{Vol}(B(p, r)) \sim r^{\frac{4}{3}}$. Curvature decay $|Rm| = O(r^{-2})$.

In the second case by taking a $\mathbb{Z}_2$ quotient we get an ALG* end. $\text{Vol}(B(p, r)) \sim r^2$. Curvature decay $|Rm| = O(r^{-2})$.

Examples of ALH* (for $b \leq 9$) and ALG* (for $b \leq 4$) gravitational instantons were constructed by Hein, on a rational elliptic surface removing a fiber with infinite monodromy.
It is known that all the above 6 families of gravitational
instantons (for suitable parameters) can arise as rescaled limits of collapsing of hyperkähler metrics on the K3 manifold.

Heuristically from the Kummer construction one expects to see ALE, ALF, ALG, ALH gravitational instantons as rescaled limits. ALE case is now well-understood. For the remaining cases, rigorous (and more general) results were proved by Foscolo 2016, Chen-Chen 2016, Chen-Viaclovsky-Zhang 2019, etc.
ALH* case is proved by Hein-S-Viaclovsky-Zhang 2018. This has connections with “small” complex structure limits of Calabi-Yau varieties in algebraic geometry (S-Zhang 2019).

The idea is to connect two ALH* gravitational instantons via a neck region, defined using the Gibbons-Hawking ansatz on $T^2 \times \mathbb{R}$ for a harmonic function with finitely many Dirac poles.

ALG* case is proved by Chen-Viaclovsky-Zhang (in progress), by adapting similar idea to the ALH* case.
Example of K3 collapsing to [0, 1]:

\[
\begin{array}{cccccccc}
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\end{array}
\]
Classification of gravitational instantons is itself a long-standing question.

There has been extensive work on the geometry of gravitational instantons with given asymptotics. Torelli-Type theorems have been proved for ALE, ALF, ALG, ALG* and ALH (Kronheimer, Chen-Chen, Chen-Viaclovsky-Zhang, Hein-S.-Viaclovsky-Zhang). The central point is that one can perform “weighted" analysis adapted to each model at infinity.

Our next result bridges the gap between these results and the general collapsing theory, which yields only a topological infranil fibration at infinity.
Theorem (S.-Zhang 2021)

Any non-flat gravitational instanton must be of type ALE, ALF, ALG, ALH, ALG*, ALH*.

Previous partial result: under the extra assumption $|Rm| = O(r^{-2-\epsilon})$ and using a very different method (initiated by V. Minerbe), Chen-Chen (2015) proved that only ALE, ALF, ALG, ALH cases can occur. The technical assumption essentially guarantees a regular torus fibration at infinity.
Combining with previous results, we have

**Corollary**

Given a gravitational instanton \((X, g)\), there is a choice of a compatible complex structure \(J\) such that \((X, J)\) is bi-holomorphic to \(\overline{X} \setminus D\), where \(\overline{X}\) is an algebraic surface and \(D\) is an anti-canonical divisor.

This confirms Yau’s compactification conjecture in the case of gravitational instantons.
Our proof of is based on a combination of geometry and analysis, namely, the Cheeger-Fukaya-Gromov theory, the study of asymptotic cones, and the analysis of the hyperkähler equation.

In particular, this is related to a question of Cheeger-Fukaya-Gromov. They showed that when Riemannian metrics collapse with uniformly bounded curvature, it is possible to perturb to nearby metrics with nilpotent Killing symmetry. The question is whether this perturbation can be made to preserve curvature equations (for example, Einstein equations).

A key step in our work is to give a positive answer to this question in a suitable setting.
Conclusion

We now have a relatively satisfactory understanding of the collapsing geometry of hyperkähler 4 manifolds, including the singularity structure of the limit spaces and classification of rescaled limits.

It is reasonable to expect that with more work one can obtain a full bubble tree structure, and show that the known constructions essentially yield all the possible collapsing geometry on the K3 manifold.

Some ideas and techniques we developed here extend to more general setting, but many more challenges are yet to be overcome.