Maps preserving measures

R. M. Solovay

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1 Introduction

The goal of this note is to present a proof of the following theorem:

Theorem 1 *If μ is an ultrafilter on a set I and $f : I \mapsto I$ has the property that $X \in \mu$ iff $f^{-1}[X] \in \mu$, then f is the identity function on a set in μ.*

I'm not sure whether or not the result is due to me. I proved this many years ago and I can't recall whether or not I was just reconstructing a proof of someone elses result.

1.1

Lemma 2 *Let $A \subseteq I$ such that $\mu(A) = 1$. Then*

$$\mu(A \cap f[A]) = 1$$

Proof: Note that if $\mu(f[A]) = 0$, our assumptions on f would imply $\mu(A) = 0$. Hence $\mu(f[A]) = 1$. The lemma is now clear.

1.2

We say that $x \in I$ is *periodic* if for some positive integer n, $f^n(x) = x$. The least such n is called the period of x.

Let $A = \{x \in I \mid x \text{ is periodic with a period} > 1\}$. We shall show that $\mu(A) = 0$. Towards a contradiction, assume that $\mu(A) = 1$.

Define an equivalence relation $\sim$ on A by putting $x \sim y$ if for some integer $i \in \omega$, $f^i(x) = y$.

Let $D \subseteq A$ contain precisely one element from each equivalence class of A.

Define a function, $\alpha : A \mapsto \omega$ as follows: Let $x \in A$. Let y be the unique element of D such that $x \sim y$. Then $\alpha(x)$ is the least $n \in \omega$ such that $f^n(y) = x$.

Let B_0 (resp. B_1) be the set of x in A such that $\alpha(x)$ is even (resp. odd). Since $\mu(A) = 1$, one of B_0, B_1 must have measure 1.

Let $B_2 = \{x \in A \mid \alpha(x) = 0\}$. Then one easily computes:

$$B_0 \cap f[B_0] \subseteq B_2$$

$$B_1 \cap f[B_1] \subseteq \emptyset$$

Using Lemma 2, we see that $\mu(B_2) = 1$. But $B_2 \cap f[B_2] = \emptyset$. Again, applying Lemma 2, we get our desired contradiction. We have shown that $\mu(A) = 0$.

1.3

The remaining cases of our proof are quite easy. First, let C be the set of those x such that x is not periodic, but for some positive n, $f^n(x)$ is periodic. We shall show that $\mu(C) = 0$.

We reinitialize our notation and let α be a function mapping C to ω defined as follows: let x in C. Then $\alpha(x)$ is the least $n \in \omega$ such that $f^n(x)$ is periodic.

Define a partition of C into two subsets, C_0, C_1 by letting C_0 (resp. C_1) be the set of x in C such that $\alpha(x)$ is even (resp. odd). Then clearly if j is either 0 or 1, we have:

$$C_j \cap f[C_j] = \emptyset$$

Hence, by Lemma 2, we have $\mu(C_0) = \mu(C_1) = 0$. It follows that $\mu(C) = 0$.

1.4

Now let E be the set of those x in I such that for no $i \in \omega$ do we have $f^i(x)$ periodic. We shall show that $\mu(E) = 0$.

Reinitializing our notation, we define an equivalence relation $\sim$ on E by $x \sim y$ if there exist i and j in ω such that $f^i(x) = f^j(y)$.

Lemma 3 *Let x and y in E such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$. Then*

$$i - j = i' - j'$$

The proof will be left as an exercise for the reader. The definition of E plays a crucial role in the proof.

Now let $D' \subseteq E$ be a subset of E that meets each equivalence class of $\sim$ in precisely one element.

Let $\mathbb{Z}$ be the ring of integers. Define a map $\alpha : E \mapsto \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose i and j in ω such that $f^i(y) = f^j(x)$. Set $\alpha(x) = i - j$. By Lemma 3, this is well defined.

It should be clear that if $x \in E$, then $f(x) \in E$. Moreover, $\alpha(f(x)) = \alpha(x) + 1$.

Let E_0 (resp. E_1) consist of those $x \in E$ such that $\alpha(x)$ is even (resp. odd). Then clearly $E_0 \cap E_1 = \emptyset$. Also $f[E_0] \subseteq E_1$ and $f[E_1] \subseteq E_0$. Hence by Lemma 2, $\mu(E_0) = \mu(E_1) = 0$. It follows that $\mu(E) = 0$.

1.5

We have now shown that A, C, and E have measure 0. But the points which lie in none of these sets are precisely those x such that $f(x) = x$. The theorem is proved.