

# The consistency strength of NFUB

## Preliminary draft

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### Abstract

We show that the consistency strength of the system NFUB, recently introduced by Randall Holmes, is precisely that of  $ZFC$  + “There is a weakly compact cardinal”.

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# 1 Introduction

## 1.1 NF and some variants.

The unorthodox system of set-theory, New Foundations, [or NF for short] was introduced by Quine. His approach for blocking the paradoxes of naive set theory was to introduce a stratification condition in the comprehension axiom. It is still an open problem whether NF can be proved consistent relative to one of the usual flavors of set theory such as ZFC. Indeed there is no known proof of the consistency of NF even relative to one of the standard large cardinal axioms.

That the system NF has some rather counterintuitive properties was shown by Specker. He proved among other things that the axiom of choice was refutable in NF and [as a corollary] that the axiom of infinity was a theorem of NF.

Subsequently, Jensen introduced NFU, the [slight?] variant of NF in which the axiom of extensionality was weakened to allow elements which are not sets. Jensen was able to prove that this variant was consistent, and that it was compatible with the axioms of choice and infinity. I shall differ from Jensen’s terminology by taking the axioms of choice and infinity as two of the axioms of NFU. I shall also suppose that an ordered pair operation is introduced as one of the primitives of the system, and that [for purposes of stratification] the type of the ordered pair is the same as that of its members. [The usual Kuratowski ordered pair has type two more than the types of its constituents and is therefore less suitable in the context of NFU.]

The theory NFU has a universal class,  $V$ , which contains all of its subsets. One would suspect that a contradiction is near. But disaster is escaped because the obvious map of  $V$  onto the set of all one-element sets can not

be proved to exist because of failure of stratification. We say that a set  $S$  is *Cantorian* if there is a bijection of  $S$  with the set  $USC(S)$  consisting of all the singletons whose members lie in  $S$ . A set is *strongly Cantorian* if the obvious map [which sends  $x$  to  $\{x\}$ ] provides a bijection of  $S$  with  $USC(S)$ . Thus, in the NFU context, the effect of the Russell paradox is to show that  $V$  is not Cantorian.

Holmes considered the system NFUA which is obtained from NFU by adjoining the axiom that “Every Cantorian set is strongly Cantorian”. Correspondence with Holmes prompted me to work out the precise consistency strength of NFUA [in work which is as yet unpublished]. The theory NFUA is equiconsistent with the theory obtained from ZFC by adjoining [for each positive integer  $n$ ] an axiom which asserts the existence of an  $n$ -Mahlo cardinal.

We caution the reader about a subtle point. Naively, this looks the same as the theory which is obtained from ZFC by adjoining a single axiom that asserts “For every integer  $n$ , there is an  $n$ -Mahlo cardinal.” But in fact, this latter theory has strictly greater consistency strength than the theory involving an infinite list of axioms which we mentioned in the preceding paragraph.

## 1.2 NFUB.

Holmes also introduced a stronger theory NFUB, which we now describe. In NFU we can develop a theory of ordinals in the spirit of Whitehead and Russell so that an ordinal consists of the class of all well-orderings which are order-isomorphic to a given well-ordering. An ordinal is Cantorian if the underlying sets of the well-orderings which are its members are all Cantorian. It is easy to see that the Cantorian ordinals [in the theory NFUA] form an initial segment of the ordinals which is not represented by a set. [That the collection of Cantorian ordinals does not form a set is a variant of the Burali-Forti paradox.]

Let us say that a subcollection  $S$  of the Cantorian ordinals is *coded* if there is some set  $s$  whose members among the Cantorian ordinals are precisely the members of  $S$ . The system NFUB is obtained from the system NFUA by adding an axiom schema which asserts that any subcollection of the Cantorian ordinals which is definable by a formula of the language of NFUB [possibly unstratified and possibly with parameters] is coded by some set.

Holmes showed that the existence of a measurable cardinal implies the

consistency of NFUB. Prior to my work on the problem, it was open if a Ramsey cardinal implied the consistency of NFUB and whether or not NFUB implied the existence of  $0^\#$ .

It follows from our main theorem that [in *ZFC*] a weakly compact cardinal implies the consistency of NFUB and that NFUB does not prove “ $0^\#$  exists”.

Our main result is the following:

**Theorem 1.1** *The following theories are equiconsistent:*

1. *NFUB*.
2. *ZFC*– + “*There is a weakly compact cardinal*”,

Remarks:

1. *ZFC*– is the theory consisting of all the axioms of *ZFC* except the power set axiom. All the usual formulations of the notion of “weakly compact” remain equivalent under *ZFC*–. The formulation of “weak compactness” that we shall actually use is:  $\kappa$  is weakly compact iff  $\kappa$  is strongly inaccessible and every  $\kappa$ -tree has a branch.
2. Officially, our metatheory is *ZFC*– [or equivalently,  $2^{nd}$ -order number theory]. The reader who is familiar with turning model-theoretic consistency proofs into syntactic ones will have no trouble in formalizing our arguments in Peano arithmetic [or, indeed, in primitive recursive arithmetic].

The remainder of this paper is organized as follows. Section 2 will derive the consistency of *ZFC*– plus a weakly compact from that of NFUB. This proof is a very slight extension of earlier work of Randall Holmes. The proof of the other direction is much more difficult. Section 3 sets up the problem by showing that it suffices to construct a model of ZFC equipped with an automorphism and having various desirable properties. Section 4 outlines the transfinite construction of such a model, with the key difficulty of how to execute the successor step postponed to Section 5. In section 5, we show how to handle the successor step. Here we must finally exploit the weakly compact cardinal.

## 2 Getting weakly compact cardinals

### 2.1 Outline of the proof

Our construction of a model of  $ZFC- +$  “There is a weakly compact cardinal” depends heavily on earlier work of Holmes which we shall have to review. First, every model of NFU has associated to it a certain model of  $ZFC-$ ,  $Z$ , whose construction we will review. An important ingredient of the structure of  $Z$  is a certain endomorphism  $T$  which we shall also recall.

There is an appropriate notion of Cantorian for elements of  $Z$ : those elements fixed by  $T$ . The key axiom schema of NFUB implies that the Cantorian elements of  $Z$  are the sets of a certain model of KM, which we dub the *canonical model*. [KM is the variant of class-set theory exposed in the appendix to Kelley’s *General Topology*.] As we shall explain below, it makes sense to ask if the class of ordinals,  $OR$ , of a model of KM is weakly compact. We show that for the canonical model,  $OR$  is indeed weakly compact.

Given any model of KM, there is associated another model of KM in which, in an appropriate sense,  $V = L$  holds. If  $OR$  was weakly compact in the original KM model, it will remain weakly compact in the new  $L$ -like KM model.

Finally, it will be easy to turn this  $L$ -like KM model into a model of  $ZFC- +$  “there is a weakly compact cardinal”.

This ends our outline. We turn to the details.

### 2.2 Cardinals

Until further notice, we are working in NFU.

As we indicated in the introduction, the usual treatment of cardinals and ordinals in NFU is in the spirit of Russell and Whitehead.

**Definition 2.1** *Let  $X, Y$  be sets. Then  $X$  is equipotent with  $Y$  iff there is a bijection mapping  $X$  onto  $Y$ .*

*Let  $X$  be a set. Then*

$$card(X) = \{Y \mid Y \text{ is equipotent with } X\}$$

$\lambda$  is a cardinal iff there is a set  $X$  such that  $\lambda = card(X)$ .

$CARD = \{\lambda \mid \lambda \text{ is a cardinal}\}$ .

### 2.3 The $T$ operation on cardinals.

As we recalled in the introduction, if  $X$  is a set,

$$USC(X) = \{\{x\} \mid x \in X\}$$

This operation makes sense in NFU. [I.e., the definition of  $USC(X)$  is *stratified*.]

**Definition 2.2** *Let  $\kappa$  be a cardinal. Then  $T(\kappa) = \text{card}(USC(X))$  for some [any]  $X \in \kappa$ .*

Caution: The map on  $CARD$  which sends  $\kappa$  to  $T(\kappa)$  is *not* given by a set.

As usual,  $V = \{x \mid x = x\}$ . We set  $\kappa_0 = \text{card}(V)$  and define  $\kappa_n$  [by induction on  $n$  in the metatheory] by setting  $\kappa_{n+1} = T(\kappa_n)$ .

We have:

$$\kappa_0 > \kappa_1 > \kappa_2 > \dots$$

### 2.4 Ordinals

The treatment of ordinals is quite analogous to that of cardinals.

**Definition 2.3** *Let  $X$  be a set and  $R$  a binary relation on  $X$ .  $R$  is a linear ordering of  $X$  if:*

1.  $R$  is transitive.
2. For any  $x, y \in X$ , exactly one of  $xRy$ ,  $yRx$ ,  $x = y$  holds.

Comment: Thus our linear orderings are *strict*.

Let  $X$  be a set and  $R$  a linear ordering on  $X$ . Then  $\text{otp}(\langle X, R \rangle) = \{\langle X_1, R_1 \rangle \mid R_1 \text{ is a linear ordering of } X_1 \text{ and } \langle X_1, R_1 \rangle \text{ is order isomorphic to } \langle X, R \rangle\}$ .

**Definition 2.4** *Let  $X$  be a set and  $R$  a linear ordering of  $X$ . Then  $R$  is a well-ordering of  $X$  iff for every non-empty  $U \subseteq X$  there is a  $u \in U$  such that  $(\forall v \in U)(uRv \text{ or } u = v)$ .*

**Definition 2.5**  $\lambda$  is an ordinal if  $\lambda = \text{otp}(\langle X, R \rangle)$  for some well-ordering  $\langle X, R \rangle$ .

$$OR = \{\lambda \mid \lambda \text{ is an ordinal}\}$$

Since we have the axiom of choice we can identify each cardinal with the corresponding initial ordinal. That is, if  $\gamma = \text{card}(X)$ , then letting  $R$  be a well-ordering of  $X$  which is “as short as possible”, we identify  $\gamma$  with the ordinal  $\text{otp}(\langle X, R \rangle)$ . It is easy to check this doesn’t depend on the choices of  $X$  and  $R$ .

## 2.5 The $T$ operation on ordinals

Let  $\lambda$  be an ordinal. We define  $T(\lambda)$  as follows: Let  $\lambda = \text{otp}(\langle X, R \rangle)$ . Let  $X_1 = \text{USC}(X)$ . Define  $R_1 = \{\{\{x\}, \{y\}\} \mid \langle x, y \rangle \in R\}$ .

Then  $R_1$  is a well-ordering and we set  $T(\lambda) = \text{otp}(\langle X_1, R_1 \rangle)$ .

It is easy to check that the identification of cardinals with initial ordinals is compatible with the  $T$  operations in  $CARD$  and  $OR$ .

Suppose that  $\langle X, R \rangle$  is a well-ordering and that  $\lambda = \text{otp}(\langle X, R \rangle)$ . We define a new well-ordering as follows:

The underlying set  $X^*$  will consist the set of ordinals less than  $\lambda$ . The ordering  $R^*$  will be the restriction to  $X^*$  of the usual ordering on  $OR$ .

In orthodox set-theory,  $\langle X^*, R^* \rangle$  would be order isomorphic to  $\langle X, R \rangle$ . However, in NFU we have:

**Proposition 2.1** *The order type of  $\langle X^*, R^* \rangle$  is  $T^2(\lambda)$ . [I. e., it’s  $T(T(\lambda))$ .]*

In this draft, we won’t give the proof. A proof can certainly be found in Holmes’ forthcoming book on NFU.

A closely related fact is the following:

**Proposition 2.2** *The order type of  $OR$  [equipped with the usual ordering] is  $\kappa_2^+$ .*

Here, we are identifying cardinals with initial ordinals and writing  $\lambda^+$  for the least cardinal greater than the cardinal  $\lambda$ . [Of course, if  $\lambda = \kappa_0$ , then  $\lambda^+$  is undefined.]

## 2.6 The set $Z$ .

We continue to review Holmes’ work. It turns out that there is a natural model  $Z$  of  $ZFC-$  associated to any model of NFU. The construction of  $Z$  is just the adaptation to the NFU context of a familiar construction used, for example, to get a model of  $ZFC-$  from a model of  $2^{nd}$ -order number theory.



We shall differ slightly in the details of our development of  $Z$  from the treatment in Holmes' text. But the two treatments are completely equivalent and yield isomorphic versions of  $Z$ .

Let  $X$  be a set and  $R$  a binary relation on  $X$ . We list various properties that the relation  $R$  can have.

1.  $R$  is *extensional* if whenever  $x$  and  $y$  are two distinct members of  $X$ , then there is a  $z \in X$  such that  $zRx \neq zRy$ .
2.  $R$  is *well-founded* if whenever  $U$  is a non-empty subset of  $X$ , there is an element  $u \in U$  which is  $R$ -least in  $U$  in the sense that for no  $v \in U$  do we have  $vRu$ .
3.  $R$  is *topped* if there is an element  $t \in X$  [the *top*] such that for any  $x \in X$  there is a finite sequence  $x_0, \dots, x_n$  [ $n$  can be 0] with  $x_0 = x$ ,  $x_n = t$  and  $x_i R x_{i+1}$  for  $0 \leq i < n$ .

If  $X$  is a set and  $R$  is a binary relation on  $X$ , then we let  $iso(X, R)$  be the set of all pairs  $\langle X', R' \rangle$  such that  $R'$  is a binary relation on  $X'$  and  $\langle X', R' \rangle$  is isomorphic to  $\langle X, R \rangle$ .

We are now in a position to define  $Z$ . It consists of all such  $iso(X, R)$  where  $R$  is an extensional, well-founded, topped relation on  $X$ .

We remark that the top of a well-founded topped relation is unique.

To explain the intuition behind the definition of  $Z$  it helps to leave NFDU for the moment and work in  $ZFC$ . If  $\langle X, R \rangle$  is an extensional well-founded relation, then the Mostowski collapse theorem provides a transitive set  $T$  and an isomorphism  $\psi$  between  $\langle X, R \rangle$  and  $\langle T, \epsilon_T \rangle$ . [Here  $\epsilon_T$  is the restriction of the usual epsilon relation to  $T$ .]

Moreover, the set  $T$  and the map  $\psi$  are uniquely determined by  $X$  and  $R$ .

Now if  $\langle X, R \rangle$  is extensional, well-founded and topped [with top  $t$ ], then  $\langle X, R \rangle$  can be viewed as a code for  $\psi(t)$ .

## 2.7 The binary relation $E$ on $Z$

Still following Holmes, we now define an “ $\epsilon$ -relation” on  $Z$ , which we dub  $E$ .

We define it in terms of representatives. [It should, of course, be checked that the definition does not depend on the choice of representatives.]

So let  $z_1, z_2$  be elements of  $Z$  and let  $\langle X_i, R_i \rangle$  be a member of  $z_i$ . Let  $t_i$  be the top of  $\langle X_i, R_i \rangle$ .

Then  $z_1 E z_2$  iff there is a map  $\psi$  mapping  $X_1$  injectively into  $X_2$  such that:

1. For  $x, y \in X_1$ ,  $x R_1 y$  iff  $\psi(x) R_2 \psi(y)$ .
2.  $\text{Range}(\psi)$  is transitive in  $X_2$ . That is, if  $x \in X_1$ ,  $y \in X_2$  and  $y R_2 \psi(x)$ , then there is a  $z \in X_1$  with  $y = \psi(z)$ .
3.  $\psi(t_1) R_2 t_2$ .

## 2.8 The model $\langle Z, E \rangle$ .

To understand the model  $\langle Z, E \rangle$ , it helps to return once again to *ZFC*.

Recall that, for  $\kappa$  an infinite cardinal,  $H_\kappa$  is the collection of sets whose transitive closures have cardinality less than  $\kappa$ .

The model  $H_{\kappa^+}$  is a model of *ZFC*-. It is never a model of *ZFC* since it has a largest cardinal [namely  $\kappa$ ].

Let  $\kappa$  be an infinite cardinal. Here is a second-order characterization of when a relational structure  $\langle X, R \rangle$  is isomorphic to  $\langle H_{\kappa^+}, \in \rangle$ .

1.  $R$  is extensional and well-founded.
2. For every  $x \in X$ , the set  $\{y \mid y R x\}$  has cardinality  $\leq \kappa$ .
3. Let  $U \subseteq X$  with  $\text{card}(U) \leq \kappa$ . Then there is a  $u \in X$  such that  $U = \{y \in X \mid y R u\}$ .

Let us refer to this second order characterization [involving the parameter  $\kappa$ ] as  $\Phi(\kappa)$ . Returning to the NFU context, we have the following characterization of  $\langle Z, E \rangle$  up to canonical isomorphism:

**Proposition 2.3**  $\langle Z, E \rangle$  satisfies the second-order sentence  $\Phi(\kappa_2)$ .

We do not give the proof. The  $\kappa_2$  appears for the same reason that the  $\kappa_2^+$  appears in Proposition 2.2.

As an immediate corollary,  $\langle Z, E \rangle$  is a model of *ZFC*-.

Hence it has its own notion of ordinal: the usual von Neumann definition in which an ordinal  $\gamma$  is equal to the set of ordinals less than  $\gamma$ .

The ordinals we have previously used can be identified with elements of  $Z$  as follows. Let  $\lambda$  be an ordinal and let  $\langle X, R \rangle$  be some representative of  $\lambda$ . We may assume that the underlying set  $X$  is not all of  $V$ ; let  $x$  be an element not in  $X$ . We form a new binary structure as follows:  $X' = X \cup \{x\}$ .  $R'$  is the well-ordering of  $X'$  that agrees with  $R$  on  $X$  and puts  $x$  “at the end”. Then the structure  $\langle X', R' \rangle$  is extensional, well-founded and topped, and so determines an element of  $Z$ . It is this element which we identify with  $\lambda$ . It is not hard to see that this map of  $OR$  into  $Z$  is given by a set of NFU and that it maps  $OR$  onto the von Neumann ordinals of  $Z$  in an order-preserving fashion.

It is this writer’s opinion that the best way to define set-theoretic notions in NFU [at least in the presence of the axiom of choice] is to reduce matters to working in  $Z$  where stratification issues can for the most part be avoided. When they can’t be avoided, they can be reduced to a consideration of the  $T$  operator to which we now turn.

## 2.9 The $T$ operator on $Z$

The definition of  $T$  on elements of  $Z$  is totally analogous to the earlier definitions given for  $OR$  and  $CARD$ .

Let  $z \in Z$ . Let  $\langle X, R \rangle$  be a member of  $z$ . Let  $X_1 = USC(X)$  and let  $R_1 = \{\langle \{x\}, \{y\} \rangle \mid xRy\}$ . Then  $\langle X_1, R_1 \rangle$  is well-founded, extensional and topped. We set  $T(z) = iso(X_1, R_1)$ . It is clear that  $T(z)$  does not depend on the choice of representative  $\langle X, R \rangle$ .

We will need the following lemma in a moment:

**Lemma 2.4**  $2^{\kappa_1}$  is defined [and  $< \kappa_0$ ].

Proof sketch:

Work in NFU.

Since  $Z \subseteq V$  we have  $2^{\kappa_2} \leq \kappa_0$ .

Define the ordinal  $\eta_i$  by the requirement that  $\beth_{\eta_i} \leq \kappa_i$  and  $\eta_i$  is largest so that this is true. (This is really a scheme of definitions. We will only need the definitions for  $0 \leq i \leq 2$ .)

We have

$$\beth_{\eta_2+1} \leq 2^{\kappa_2} \leq \kappa_0$$

Hence  $\eta_2 + 1 \leq \eta_0$ .

I say that  $\eta_1 < \eta_0$ . Indeed we clearly have  $\eta_1 \leq \eta_0$ . But if  $\eta_1 = \eta_0$  then also  $\eta_2 = \eta_1$ . It follows that  $\eta_2 = \eta_0$  contradicting the conclusion of the preceding paragraph.

Now  $\eta_1$  and  $\eta_0$  have the same residue class mod 3. It follows that  $\eta_1 + 3 \leq \eta_0$ .

Hence

$$2^{\kappa_1} \leq \beth_{\eta_1+2} < \beth_{\eta_1+3} \leq \kappa_0 \quad \square$$

In order to understand the  $T$  map on  $Z$  we make the following construction. Let  $Z_1 = USC(Z)$ . Let  $E_1 = \{\langle \{x\}, \{y\} \rangle \mid xEy\}$ . Then the pair  $\langle Z_1, E_1 \rangle$  satisfies the second order sentence  $\Phi(\kappa_3)$  and so has cardinality  $2^{\kappa_3} \leq \kappa_2$ . It is also clearly well-founded and extensional. It follows that there is an isomorphism  $k$  mapping  $\langle Z_1, E_1 \rangle$  onto a transitive set of the model  $\langle Z, E \rangle$ .

**Proposition 2.5** *Let  $z \in Z$ . Then  $k(\{z\}) = T(z)$ .*

It follows that the range of the  $T$  map on  $Z$  is a set of the NFU model. The  $T$  map itself is definitely not given by a set of the NFU model.

The identification of the Russell-Whitehead ordinals with the von Neumann ordinals of  $Z$  is compatible with the  $T$  operations on the domain and range of the identification map.

## 2.10 Cantorian elements of $Z$

We say that an element of  $Z$  is *Cantorian* if  $T(z) = z$ . The main axiom of NFUA entails that the collection of Cantorian elements of  $Z$  is transitive. That is, if  $z \in Z$  is Cantorian and  $z_1 E z$  then  $z_1$  is also Cantorian.

Using the axiom of choice, we can translate the main axiom scheme of NFUB as follows. Let  $W$  be a subcollection of the elements of our NFUB model that is definable by a formula of  $\varphi$  of the language of NFUB, possibly containing names for particular elements of our NFUB model. [The formula  $\varphi$  need not be stratified.] Then there is an element  $w$  of  $Z$  which *codes* the intersection of  $W$  with the Cantorian elements of  $Z$ . That is, for any Cantorian element of  $Z$ , say  $c$ , we have  $c$  is in the collection  $W$  iff  $c E w$ .

Holmes has shown that we get a model of  $KM$  [Kelley-Morse set theory including the global axiom of choice] as follows. The sets of the model are the Cantorian elements of  $Z$ . The  $\epsilon$ -relation of the model is the restriction of  $E$ . Finally, the classes of the model are the subcollections of the Cantorian sets which are coded by some element of  $Z$  in the manner just described.

## 2.11 Digression on weak compactness

We first have to explicate what it means to assert that the class of ordinals of a model of  $KM$  is weakly compact. But this is easy. The various definitions of “weakly compact” are given by  $\Pi_2^1$  formulas. We interpret the type 2 variables as ranging over the classes of the  $KM$  model. We interpret the type 1 variables as ranging over the sets of the  $KM$  model.

There are many different characterizations of what it means for a strongly inaccessible cardinal to be weakly compact. The notions that will be important for us in this paper are:

1.  $\kappa$  is  $\Pi_1^1$ -indescribable.
2.  $\kappa$  has the tree property. [There are actually two versions of the tree property that we will need to consider. This will be spelled out in a moment.]

We remark that the usual proofs that these formulations are equivalent carry over to the  $KM$  context without difficulty.

We turn to the precise formulation of the two versions of the tree property we will need. In the following  $\kappa$  is a strongly inaccessible cardinal. At a first cut, the reader may take us as working in  $ZFC$  though we will have occasion to apply these definitions later in the context of  $ZFC$ — and in the context of  $KM$  [with  $\kappa$  replaced by the class of all ordinals].

A *tree* is a pair  $\langle X, R \rangle$  where  $X$  is a set,  $R$  is a transitive binary relation on  $X$  and whenever  $a \in X$  and  $seg(a) = \{y \in X : yRa\}$  then  $seg(a)$  is well-ordered by the restriction of  $R$ .

If  $\langle T, <_T \rangle$  is a tree and  $t \in T$ , then the rank of  $t$  [notation:  $\rho(t)$ ] is the order type of the restriction of  $<_T$  to  $seg(t)$ .

$T$  is a  $\kappa$ -tree if:

1. For every  $t \in T$ ,  $\rho(t) < \kappa$ .
2. If  $\alpha < \kappa$ , then let  $T_\alpha = \{t \in T \mid \rho(t) = \alpha\}$ .

Then we require that for every  $\alpha < \kappa$ ,  $T_\alpha$  is non-empty and has cardinality less than  $\kappa$ .

A branch through a  $\kappa$ -tree  $T$  is a subset  $b$  of  $T$  with the following properties:

1. For every  $\alpha < \kappa$ ,  $b \cap T_\alpha$  has exactly one member.

2. If  $x \in b$  and  $y <_T x$ , then  $y \in b$ .

We can now give our first formulation of weak compactness:  $\kappa$  is weakly compact if  $\kappa$  is strongly inaccessible and every  $\kappa$ -tree has a branch.

It turns out that it is not necessary to consider all  $\kappa$ -trees.

A binary  $\kappa$ -tree is a set  $S$  with the following properties:

1. The elements of  $S$  are functions whose domain is an ordinal less than  $\kappa$  and whose range is included in  $\{0, 1\}$ .
2. If  $f \in S$  has domain  $\alpha$  and  $\beta < \alpha$ , then the restriction of  $f$  to  $\beta$  lies in  $S$ .
3. If  $\alpha$  is an ordinal less than  $\kappa$ , then there is an  $f \in S$  with  $\text{domain}(f) = \alpha$ .

Such an  $S$  gives rise to a  $\kappa$ -tree in our previous sense if we take  $<_S$  to be the restriction of  $\subset$  to  $S$ .

What a branch amounts to for a binary  $\kappa$ -tree  $S$  is a function  $F : \kappa \mapsto 2$  such that the restriction of  $F$  to any ordinal less than  $\kappa$  lies in  $S$ .

Then an equivalent formulation of the notion of weak compactness is that  $\kappa$  is weakly compact iff  $\kappa$  is strongly inaccessible and every binary  $\kappa$ -tree has a branch.

## 2.12 $OR$ is weakly compact

It is now easy to show that in the canonical model of  $KM$  [associated to some given model of NFUB]  $OR$  is weakly compact. Let then  $S$  be a class of the canonical model which gives a binary  $OR$ -tree. Let  $s$  be an element of  $Z$  which codes  $S$ . Then for arbitrarily large Cantorian ordinals  $\alpha$ ,  $\alpha$  is a Beth fixed point and  $s \cap V_\alpha$  is a binary  $\alpha$ -tree [in the sense of  $Z$ ]. So there must be a non-Cantorian  $\alpha \in Z$  such that  $Z$  thinks that  $\alpha$  is a Beth fixed point and that  $s \cap V_\alpha$  is a binary  $\alpha$ -tree. [Otherwise, there would be a set of  $Z$  consisting precisely of the Cantorian ordinals, which is absurd.]

Let  $b$  be an element of  $s \cap V_\alpha$  of non-Cantorian rank. Then it is easy to see that  $b$  codes a branch through  $S$  [consisting of the restrictions of  $b$  to Cantorian ordinals of  $Z$ ]. Our proof that  $OR$  is weakly compact in the canonical model of  $KM$  is complete.

### 2.13 Getting a model of $ZFC-$

The Gödel  $L$  construction applies to models of  $KM$  as follows: Given a well-ordering of  $OR$ , we can build a model of  $V = L$  whose ordinals have the order type of  $R$ . Say that a class is constructible if it appears in some such model. Then if we take the sets of our new model of  $KM$  to be the constructible sets of the old model, and the classes of our new model to be the constructible classes of our old model, we get a model of  $KM + V = L$  in which every class is constructible.

The usual proof that if  $\kappa$  is weakly compact, then it remains weakly compact in  $L$  works also in the present context. It gives that  $OR$  is still weakly compact in our  $L$ -like model of  $KM$ .

What we have gained by this  $L$ -construction is a definable well-ordering of the classes of our  $KM$  model. In our original model, we only had, a priori, a well-ordering of the class of all sets.

It is now routine [much as we built  $Z$  in NFUB] to build a model of  $ZFC- + V = L +$  “there is a weakly compact cardinal  $\kappa$ ” from our  $L$ -like model of  $KM$ . This direction of the equiconsistency proof is complete.

## 3 Getting a model of NFUB

### 3.1 Initial preparations

We are given a model of  $ZFC- +$  “there is a weakly compact cardinal”. Say  $\kappa$  is a weakly compact cardinal of the model. We pass to the constructible sets of the original model. Then we still have a model of  $ZFC-$  and  $\kappa$  remains weakly compact. Moreover  $V = L$  holds.

We may assume as well that  $\kappa$  is the largest cardinal of the model. For if not, let  $\lambda = \kappa^+$  [in the sense of our current model]. Then  $L_\lambda$  is a model of  $ZFC- + V = L +$  “There is a largest cardinal  $\kappa$ ” + “ $\kappa$  is weakly compact”.

The upshot is that we may assume given a model  $M$  of  $ZFC- + V = L +$  “There is a largest cardinal  $\kappa$  which is weakly compact”. We must construct a model of NFUB. We shall reserve the symbols  $M$  and  $\kappa$  for this model and this cardinal for the remainder of the paper.

By a class of  $M$  we mean a subcollection of  $M$  definable by a formula of the language of set-theory [with the definition possibly involving names for particular elements of  $M$ ]. Our final model  $N$  of NFUB will be a proper class model in the sense of  $M$ .

## 3.2 Precise description of NFU

We have been a bit lax in describing precisely the formal system NFU. We now need to remedy this.

[Probably in the next draft of the paper, this material should be placed much earlier.]

The language of NFU is a first-order language with no function symbols and with the following five predicate symbols:

1. A binary predicate  $=$ . [This is the usual equality predicate. The basic properties of  $=$  are part of first order logic and need not be specified in our axioms for NFU.]
2. A binary predicate  $\in$ . [ $a \in b$  is read “ $a$  is a member of  $b$ ”.]
3. A unary predicate  $S$ . [ $Sa$  is read “ $a$  is a set.” Allowing “urelements” which are not sets is the key difference between NFU and NF.]
4. A ternary predicate  $P$ .

[The intuition is as follows. Every pair of elements in NFU determines a unique ordered pair. The relation  $Pabc$  means “ $c$  is the ordered pair determined by  $a$  and  $b$ ”.]

NFU will have three groups of axioms:

### 3.2.1 Extensionality axioms

1. If  $a \in b$  then  $Sb$ .
2. Suppose that  $Sa$ ,  $Sb$  and  $a \neq b$ . Then there is a  $c$  such that  $c \in a \neq c \in b$ .

### 3.2.2 Pairing axioms

1. For every  $a$  and  $b$ , there is exactly one  $c$  such that  $Pabc$ .
2. If  $Pabc$  and  $Pa'b'c$ , then  $a = a'$  and  $b = b'$ .



### 3.2.3 Comprehension axioms

The description of these is a bit more technical. Let  $\varphi$  be a formula of the language of NFUB.  $\varphi$  is *stratified* if there is a map  $\sigma$  from the set of variables occurring in  $\varphi$  [either free or bound] into  $\omega$  [the set of non-negative integers] such that:

1. If  $v = w$  is a subformula of  $\varphi$  then  $\sigma(v) = \sigma(w)$ .
2. If  $Puvw$  is a subformula of  $\varphi$  then  $\sigma(u) = \sigma(v) = \sigma(w)$ .
3. If  $v \in w$  is a subformula of  $\varphi$  then  $\sigma(w) = \sigma(v) + 1$ .

Let now  $\phi$  be a stratified formula whose free variables are included among  $v_0, \dots, v_n$ . Then the following is an axiom of NFU:

$$\forall v_1, \dots, v_n \exists v_{n+1} \forall v_0 [v_0 \in v_{n+1} \iff \phi].$$

### 3.3 A procedure for getting models of NFU

We describe a known procedure for getting models of NFU. The conditions we impose on the starting model are far too stringent. There is no need to require that  $V = L$  holds in the model, or that the model is a model of full *ZFC*. Moreover the requirement that  $j$  be an automorphism can be considerably relaxed. Nevertheless, we will have these conditions in our applications and they serve to simplify the discussion.

We consider the following situation:

1.  $\langle N, \in_N \rangle$  is a model of *ZFC* +  $V = L$ .
2.  $j$  is an automorphism of the model  $\langle N, \in_N \rangle$ .  $j$  is not the identity.

It follows, [say since  $V = L$  holds in  $N$ ], that  $j$  moves some ordinal  $\eta$ . In fact,  $j$  must move some Beth fixed point. [Indeed,  $j$  moves the  $\eta^{th}$  Beth fixed point.]

3.  $\gamma$  is a Beth fixed point of the model  $N$  moved by  $j$ .

Without loss of generality, we can assume that  $j(\gamma) > \gamma$ . [Otherwise, simply replace  $j$  by  $j^{-1}$ .]

From this data, we are going to define a model of NFU. We shall view this procedure as “well-known” and not carry out the verification that the model we describe is, indeed, a model of NFU.

We shall call the model we are constructing  $Q$ .

1. The underlying set of  $Q$ ,  $|Q|$  is just  $L_\gamma$ .
2. For  $i \in \omega$ , we define the ordinal  $\gamma_i$  of the model  $N$  as follows:

$$\begin{aligned}\gamma_0 &= \gamma; \\ \gamma_{n+1} &= j^{-1}(\gamma_n).\end{aligned}$$

$Sx$  holds iff  $x \subseteq L_{\gamma_1}$ .

3.  $=$  is the ordinary equality. Thus  $x = y$  holds in  $Q$  iff  $x = y$ .
4. However, the definition of  $\in_Q$  is non-standard.  $x \in_Q y$  iff  $Sy$  and  $x \in_N j(y)$ .
5.  $Pxyz$  holds in  $Q$  iff  $z$  is, in  $L_\gamma$ , the usual Kuratowski ordered pair of  $x$  and  $y$ . [That is  $z = \{\{x\}, \{x, y\}\}$ .]

This completes the specification of the model  $Q$ . As we have already mentioned,  $Q$  is a model of  $NFU$ . It turns out that the  $T$  operation of  $Q$  is essentially identical with  $j^{-1}$ . The relevant facts will be recalled in the next subsection.

Remark: Let us write  $\langle x, y \rangle$  for the unique  $z$  such that  $Pxyz$  holds in  $Q$ . We caution the reader that, despite its origin using the Kuratowski ordered pair in  $L_\gamma$ ,  $\langle x, y \rangle$  is *not* the Kuratowski ordered pair in the sense of the model  $Q$ .

### 3.4 $T$ vs. $j^{-1}$

Let  $Z$  be as defined in Section 2.6 [with respect to the model  $Q$ ].

Let  $Z^* = \{x \in |Q| \mid x \in_Q Z\}$ . Here the definition takes place in the model  $N$  [using the parameter  $j(Z)$ ].

Similarly, let  $E^* = \{\langle x, y \rangle \mid xEy \text{ holds in } Q\}$ . Again, the definition can be given in  $N$  using the parameter  $j(E)$ .

So, in  $N$ ,  $E^*$  is a binary relation on  $Z^*$ . Intuitively, this is  $N$ 's copy of the structure  $\langle Z, E \rangle$  of  $Q$ .

In fact,  $Z^*$  can be given a direct description quite analogous to the definition of  $Z$  in  $Q$ . In place of considering binary relations on subsets of  $V$ , we consider binary relations whose underlying set is a subset of  $L_{\gamma_2}$ . Thus in our present context, for  $X \subseteq L_{\gamma_2}$  and  $R$  a binary relation on  $X$ , we let  $iso(X, R)$  consist of all pairs  $\langle X', R' \rangle$  where:

1.  $X' \subseteq L_{\gamma_2}$ ;
2.  $R'$  is a binary relation on  $X'$ ;
3. The structure  $\langle X', R' \rangle$  is isomorphic to the structure  $\langle X, R \rangle$ .

[Of course, this definition takes place within the model  $N$ .]

Finally  $Z^*$  consists of all sets of the form  $iso(X, R)$  where:

1.  $X \subseteq L_{\gamma_2}$ ;
2.  $R$  is a binary relation on  $X$  which is well-founded, extensional, and topped.

[Again, this definition takes place in  $N$ .]

The definition of  $E^*$  is the obvious analogue, in the present context, of the prior definition of  $E$ . (Cf. section 2.7)

An immediate corollary of our description of  $\langle Z^*, E^* \rangle$  is that this structure is isomorphic to  $L_{\gamma_2+}$  [with its usual  $\epsilon$ -relation].

Let us use  $k$  to describe this isomorphism. Thus if  $z \in_Q Z$ ,  $k(z)$  is the element of  $L_{\gamma_2+}$  that corresponds under the isomorphism just described.

The following proposition now explains the sense in which  $T$  can be identified with  $j^{-1}$ .

**Proposition 3.1** *Let  $z \in_Q Z$ . Then  $k(T(z)) = j^{-1}(k(z))$ .*

For the moment, I am taking this proposition as “well-known”. I may include a proof in a later draft of this paper.

### 3.5 Criteria for $Q$ to be a model of NFUB

First, here is a sufficient criterion for  $Q$  to be a model of NFUA.

Criterion 1: Suppose that  $N, j, \gamma$  are as in section 3.3. Suppose further that whenever  $j(\alpha) = \alpha$  [for  $\alpha \in OR$ ] and  $\beta < \alpha$  then  $j(\beta) = \beta$ . Then  $Q$  is a model of NFUA.

This is immediate from the identification of  $T$  with  $j^{-1}$ .

We call the Cantorian elements of  $N$  those elements fixed by  $j$ . If  $N$  satisfies criterion 1, they will form an initial segment of  $N$ . We let  $C$  denote the collection of Cantorian elements of  $N$ .

Let  $S$  be a subcollection of the Cantorian elements of  $N$ . We say that  $S$  is coded by the element  $s$  of  $N$  if whenever  $x \in C$ , then  $(x \in S) \iff (x \in_N s)$ .

Let  $N^*$  be the structure  $\langle N; \in_N, j \rangle$ . We have the obvious notion of a class of  $N^*$ . This is a subset of  $N$  definable by a formula of the language appropriate to  $N^*$  [possibly with parameters from  $N$ .] It is evident that any class of  $Q$  is a class of  $N^*$ . Hence the following criterion is easy to verify.

Criterion 2: Suppose that  $N$  satisfies criterion 1, and that whenever  $W$  is a class of  $N^*$ , then the intersection of  $W$  with  $C$  is coded by some element  $w$  of  $N$ . Then  $Q$  is a model of *NFUB*.

### 3.6 Some promises and their consequences

We are going to describe properties of a model  $N_\infty$  and argue that if we can construct a model with these properties then the construction of section 3.3 will yield a model of *NFUB*.

Recall the model  $M$  and its cardinal  $\kappa$  that we introduced in section 3.1. The model  $N_\infty$  will have the following properties:

1. It will be a class-sized model of  $ZFC + V = L$ . That is, both the “underlying set” and the  $\epsilon$ -relation of  $N_\infty$  will be classes of  $M$ .
2. There will be an elementary embedding  $i_\infty : L_\kappa \mapsto N_\infty$  which maps  $L_\kappa$  onto an initial segment of  $N_\infty$ .  
[ $i_\infty$  will be a set of  $M$ .]
3. There will be an automorphism  $j_\infty$  of the model  $N_\infty$  [again given by a class of  $M$ ].

The only elements of  $N_\infty$  left fixed by  $j_\infty$  are those in the range of  $i_\infty$ .

4. Let  $A$  be a subset of  $L_\kappa$  lying in  $M$ . Then there is an element  $a \in N_\infty$  which codes  $A$  in the sense that for all  $x \in L_\kappa$  we have  $x \in A \iff i_\infty(x) \in_{N_\infty} a$ .

It follows first from item 4 that  $N_\infty$  is a proper class of  $M$  and hence, since range  $i_\infty$  is a set of  $M$  that  $j$  is not the identity. Picking some Beth fixed point  $\gamma$  of  $N_\infty$  which is moved by  $j_\infty$ , we are in position to apply the construction of section 3.3 to get a model  $Q$  of *NFU*. Applying criterion 1 of section 3.5 and item 3 of the list of properties of  $N_\infty$  we see that  $Q$  is a model of *NFUA*.

It remains to see that  $Q$  is a model of NFUB. We seek to apply criterion 2. First note that we can identify the Cantorian elements of  $N_\infty$  with  $L_\kappa$  via the map  $i_\infty$ .

Because all the elements of  $N_\infty^*$  are classes of  $M$  and  $M$  is a model of  $ZFC-$ , the intersection of any class of  $N_\infty^*$  with the cantorian elements of  $N_\infty$  will correspond to a subset  $S$  of  $L_\kappa$  which appears in  $M$ . But then item 4 of the list of properties of  $N_\infty$  will guarantee that  $S$  is coded by some element of  $N_\infty$ .

So to complete the construction of a model  $Q$  of NFUB, it remains to construct a model  $N_\infty$  with the stated properties.

## 4 The transfinite construction

### 4.1 Introduction to this section

The model  $N_\infty$  will be constructed by a transfinite construction, carried out within  $M$ , whose stages are indexed by the ordinals of  $M$ . The various stages will have various additional components [in addition to being models of  $V = L$ ]. We introduce a category of  $A$ -models, so each of the stages will be such an  $A$ -model.

We let  $\langle S_\alpha \mid \alpha \in OR^M \rangle$  be a listing of the subsets of  $L_\kappa$ , lying in  $M$ , in order of construction. [I.e., as ordered by  $<_L$ .] The construction of  $N_{\alpha+1}$  will be devoted to insuring that  $S_\alpha$  is coded in the final model.

For limit ordinals  $\lambda$ ,  $N_\lambda$  will be constructed by a direct limit process; the final model  $N_\infty$  will also be constructed by a direct limit process. This is straightforward, but we choose to carefully present the direct limit process below.

We remark that for  $\alpha \in OR^M$ , the model  $N_\alpha$  will be a set of  $M$  [and in fact have cardinality  $\kappa$ ].

### 4.2 The category of $A$ -models: Objects

The following definition takes place within the model  $M$ .

An  $A$ -model consists of the following data:

1. A model  $N$  of  $ZFC + V = L$  of cardinality  $\kappa$ ;
2. An elementary embedding  $i : L_\kappa \mapsto N$ ;

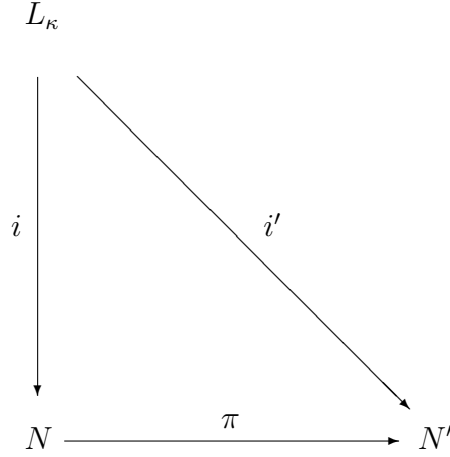


Figure 1:

3. An automorphism  $j$  of  $N$ .

These data are subject to the following requirements:

1.  $i$  maps  $L_\kappa$  onto an initial segment of  $N$  [with respect to the ordering that is  $N$ 's version of  $<_L$ ].
2. The points of  $N$  left fixed by  $j$  are precisely those in the range of  $i$ .

Note that there is the trivial  $A$ -model in which  $N = L_\kappa$  and  $i$  and  $j$  are identity maps.

### 4.3 The category of $A$ -models: Maps

Let  $\mathcal{N} = \langle N, i, j \rangle$  and  $\mathcal{N}' = \langle N', i', j' \rangle$  be  $A$ -models. A map of  $A$ -models,  $\pi : \mathcal{N} \mapsto \mathcal{N}'$  is an elementary embedding  $\pi : \langle N, \in_N \rangle \mapsto \langle N', \in_{N'} \rangle$  such that:

1. The map  $\pi$  should respect the embeddings  $i$  and  $i'$ . That is, the diagram indicated in figure 1 should commute.
2. The map  $\pi$  should appropriately intertwine the automorphisms  $j$  and  $j'$ . That is, the diagram indicated in figure 2 should commute.

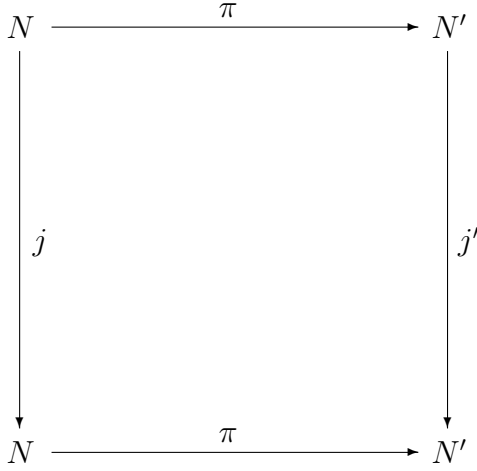


Figure 2:

It is easy to see that the identity map of an  $A$ -model is an  $A$ -model map and that  $A$ -model maps are closed under composition. I.e., the  $A$ -models form a category.

#### 4.4 Inductive requirements

We continue to work within the model  $M$ .

Our construction will proceed in stages indexed by the ordinals [of  $M$ ]. Let  $\lambda$  be such an ordinal. Here are the inductive requirements that we will maintain before stage  $\lambda$ :

1. For each  $\alpha < \lambda$  we will have defined an  $A$ -model,  $\mathcal{N}_\alpha$ .
2. If  $\alpha_1 \leq \alpha_2 < \lambda$ , we will have defined an  $A$ -model map  $\pi_{\alpha_1, \alpha_2} : \mathcal{N}_{\alpha_1} \mapsto \mathcal{N}_{\alpha_2}$ .
3. If  $\alpha < \lambda$ , then  $\pi_{\alpha, \alpha}$  is the identity map of  $\mathcal{N}_\alpha$ .
4. If  $\alpha_1 \leq \alpha_2 \leq \alpha_3 < \lambda$ , then the map  $\pi_{\alpha_1, \alpha_3}$  is equal to the composition  $\pi_{\alpha_2, \alpha_3} \pi_{\alpha_1, \alpha_2}$ .

5. Suppose that  $\alpha + 1 < \lambda$ . Let  $S_\alpha$  be the  $\alpha^{\text{th}}$  subset of  $L_\kappa$  in order of construction. Then there will be an element  $s \in N_{\alpha+1}$  that codes  $S$  in the sense described in section 3.6. That is, for any  $x \in L_\kappa$  we have  $x \in S_\alpha \iff i_{\alpha+1}(x) \in_{N_{\alpha+1}} s$ .

## 4.5 Continuing the construction

Suppose that we are at stage  $\lambda$  and that our inductive requirements hold at that stage. Here is what we do:

1.  $\lambda = 0$ . We take  $\mathcal{N}_0$  to be the trivial  $A$ -model. [So  $N_0 = L_\kappa$  and  $i_0$  and  $j_0$  are identity maps.]
2.  $\lambda = \alpha + 1$ . We use the following lemma which will be proved in section 5:

**Lemma 4.1** *Let  $S$  be a subset of  $L_\kappa$  and let  $\mathcal{N}$  be an  $A$ -model. Then there is an  $A$ -model  $\mathcal{N}'$  and an  $A$ -model map  $\pi : \mathcal{N} \mapsto \mathcal{N}'$  such that:*

- (a) *The map  $\pi$  does not map  $N$  onto  $N'$ .*
- (b) *There is an element  $s \in N'$  which codes the set  $S$ .*

We apply this lemma in the obvious way with  $\mathcal{N}_\alpha$  in the role of  $\mathcal{N}$  and  $S_\alpha$  in the role of  $S$ .

We set  $\mathcal{N}_{\alpha+1}$  equal to the  $A$ -model  $\mathcal{N}'$  provided by the lemma. and set  $\pi_{\alpha,\alpha+1}$  equal to the map  $\pi$  provided by the lemma. For other  $\xi \leq \alpha + 1$  we define  $\pi_{\xi,\alpha+1}$  in the unique way that maintains our inductive requirements.

3.  $\lambda$  is a limit ordinal. Then we take  $\mathcal{N}_\lambda$  to be the direct limit of the system of  $A$ -models and  $A$ -model maps already defined. This direct limit construction will be reviewed in the next subsection. It provides all the maps needed to maintain our inductive requirements.

## 4.6 The direct limit construction

Suppose we are at stage  $\lambda$  for some limit ordinal  $\lambda$ . We describe the construction of  $\mathcal{N}_\lambda$  and the associated maps.



Let  $\alpha < \lambda$  and let  $x \in N_\alpha$ . We say that  $x$  is *original* if for no  $\beta < \alpha$  and  $y \in N_\beta$  do we have:  $\pi_{\beta,\alpha}(y) = x$ . It is clear that for every  $x \in N_\alpha$  there is a  $\beta \leq \alpha$  and an original  $y \in N_\beta$  such that  $\pi_{\beta,\alpha}(y) = x$ .

The underlying set of  $\mathcal{N}_\lambda$  will consist of all pairs  $\langle \alpha, y \rangle$  where  $\alpha < \lambda$  and  $y \in N_\alpha$  is original.

Of course, we set  $\pi_{\lambda,\lambda}$  equal to the identity map.

Let now  $\alpha < \lambda$  and let  $x \in N_\alpha$ . We have to define  $\pi_{\alpha,\lambda}(x)$ . Let  $\beta \leq \alpha$  and  $y \in N_\beta$  with  $\pi_{\beta,\alpha}(y) = x$  and  $y$  an original element of  $N_\beta$ . [These requirements clearly uniquely determine  $y$  and  $\beta$ .] Then we set  $\pi_{\alpha,\lambda}(x) = \langle \beta, y \rangle$ .

Let  $\langle \alpha, y \rangle$  be an element of  $N_\lambda$ . We set  $j_\lambda(\langle \alpha, y \rangle) = \langle \alpha, j_\alpha(y) \rangle$ . [It must of course be checked that  $j_\alpha(y)$  is an original element of  $N_\alpha$ . This is not hard to do.]

Let  $R$  be one of the relations  $=$  and  $\in$ . Let  $x$  and  $y$  be elements of  $N_\lambda$ . We must determine whether or not  $xRy$  holds in  $\mathcal{N}_\lambda$ . Here is the procedure. Find  $\gamma < \lambda$  such that there are  $x'$  and  $y'$  in  $N_\gamma$  with  $\pi_{\gamma,\lambda}(x') = x$  and  $\pi_{\gamma,\lambda}(y') = y$ . Then  $xRy$  holds in  $\mathcal{N}_\lambda$  iff  $x'Ry'$  holds in  $\mathcal{N}_\gamma$ . [It must, of course, be checked that this procedure does not depend on the choice of  $\gamma$ .]

Finally, we take  $i_\lambda$  to be  $\pi_{0,\lambda}$ .

The check that, with these definitions,  $\mathcal{N}_\lambda$  is an  $A$ -model and our inductive conditions continue to hold left to the reader.

We remark only that the reason that  $N_\lambda$  has cardinality at most  $\kappa$  is that this is true of  $\lambda$ .

## 4.7 Defining $\mathcal{N}_\infty$

The definition of  $\mathcal{N}_\infty$  is totally analogous to the direct limit procedure of the preceding section. The only difference is that we take the direct limit of the full system  $\langle \mathcal{N}_\alpha \mid \alpha \in OR^M \rangle$  so  $N_\infty$  is a proper class.

We discuss only the coding property. Let then  $S \subseteq L_\kappa$  be a set in  $M$ . Then  $S = S_\alpha$  for some ordinal  $\alpha$  and there is an element  $s \in N_{\alpha+1}$  that codes  $S$ . But then  $\pi_{\alpha+1,\infty}(s)$  codes  $S$  in  $N_\infty$ .

## 5 Using the weakly compact cardinal

### 5.1 Introduction to this section

It remains to prove Lemma 4.1. Our proof of this is rather mysterious in that for most of the proof we are engaged in constructions having nothing to do with the main lemma, and at the last minute we return to it and prove it. Perhaps the following comments will dispel some of the mystery.

First, it turns out that the heart of the problem is the special case when the initial  $A$ -model is the trivial one. In this case, the new model  $\mathcal{N}'$  will be generated [in an appropriate sense] from  $L_\kappa$  and a sequence of “indiscernibles”  $\langle \xi_i \mid i \in \mathbf{Z} \rangle$ . [Here  $\mathbf{Z}$  is the set of *all* integers, positive, negative, or zero.]

Everything boils down to determining a suitable EM-blueprint for the indiscernibles. It’s natural to use the partition properties associated with a weakly compact, but since we have  $\kappa$  many partitions to tame, this doesn’t seem to work. What does work is to imitate the proof of the partition properties, which we briefly recall.

Let  $[\kappa]^n$  be the set of  $n$ -element subsets of  $\kappa$ . [Recall that  $\kappa$  is equal to the set of ordinals less than  $\kappa$ .] We are given a map  $F : [\kappa]^n \mapsto 2$  and we seek to find a homogeneous set for  $F$  of size  $\kappa$ . The approach is to build a tree  $T$  so that on any branch  $b$  through the tree  $F$  does not depend on its last coordinate. This allows us to reduce the problem from  $n$  to  $n - 1$ .

In our case, this approach of simplifying situations on the branches of trees will still apply. We will, in fact, start with a length  $\omega$  process of building trees and branches. Eventually, this will help us in building EM blueprints and thereby models with automorphisms.

### 5.2 The basic module

The following lemma is where we make use of the fact that  $\kappa$  is weakly compact. It is indeed not difficult to deduce that  $\kappa$  is weakly compact if the lemma holds.

**Lemma 5.1** *Suppose given the following data:*

1. *A subset of  $\kappa$ ,  $B$ , of cardinality  $\kappa$ .*
2. *A  $\kappa$ -sequence of ordinals  $\langle \gamma_i \mid i < \kappa \rangle$  such that  $0 < \gamma_i < \kappa$  for all  $i < \kappa$ .*
3. *A  $\kappa$ -sequence of functions  $\langle F_i \mid i < \kappa \rangle$  such that  $F_i : \kappa \mapsto \gamma_i$ .*

Then there is a subset  $B'$  of  $B$  of cardinality  $\kappa$  such that each  $F_i$  is constant on a tail of  $B'$ . That is, for every  $i < \kappa$  there is an  $\eta < \kappa$  and a  $\xi < \gamma_i$  such that  $F_i(\alpha) = \xi$  whenever  $\alpha \in B'$  and  $\alpha \geq \eta$ .

We have stated the lemma in a form most useful for applications. The version when  $B$  is taken to be  $\kappa$  is obviously equivalent.

Proof: We define a  $\kappa$ -tree  $T_1$  as follows. The nodes of  $T_1$  consist of functions  $h$  whose domain is an ordinal  $\alpha < \kappa$  and such that  $h(\beta) < \gamma_\beta$  for all  $\beta < \alpha$ .

Let  $\langle b_\xi \mid \xi < \kappa \rangle$  be an enumeration of  $B$  in increasing order. We will, by induction on  $\xi$ , assign a node of  $T_1$  to  $b_\xi$ . We will do this so that:

1. At most one ordinal is assigned to each node of  $T_1$ .
2. Suppose that  $x$  and  $y$  are nodes of  $T_1$  with  $x <_{T_1} y$ . Then no ordinal can be assigned to  $y$  while the node  $x$  is still unoccupied.

So suppose that for all  $\alpha < \xi$ ,  $b_\alpha$  has been assigned to some node of  $T_1$ . We show how to assign  $b_\xi$ . Define a function  $G_\xi : \kappa \mapsto \kappa$  as follows.  $G_\xi(\alpha) = F_\alpha(b_\xi)$ . It is clear that for all  $\alpha < \kappa$ , the restriction of  $G_\xi$  to  $\alpha$  is a node of  $T_1$ . We take  $\alpha$  as small as possible so that the restriction of  $G_\xi$  to  $\alpha$  is currently unoccupied, and assign  $b_\xi$  to this restriction.

Let  $T$  be the set of nodes of  $T_1$  that are assigned some element of  $B$  in the procedure just described. It is clear that  $B$  has cardinality  $\kappa$  and hence that it is a  $\kappa$ -tree. Let  $G : \kappa \mapsto \kappa$  give a branch through  $T$ . [It is here that we are using the weak compactness of  $\kappa$ .] Let  $B'$  be the set of elements of  $B$  which have been attached to nodes on this branch. Then if  $G(\alpha) = \xi$ , then it is clear that  $F_\alpha(\theta) = \xi$  for a tail of  $\theta$ 's in  $B'$ .

### 5.3 A technical lemma

Recall that throughout we are working in the model  $M$  which is a model of  $ZFC + V = L +$  "There is a largest cardinal  $\kappa$  which is weakly compact".

**Lemma 5.2** *Let  $A \subseteq \kappa$ . Let  $\kappa < \alpha$ . Then there is a  $\beta > \alpha$  such that  $A \in L_\beta$  and  $L_\beta$  is a model of  $ZFC - + V = L +$  " $\kappa$  is the largest cardinal".*

Proof: This is an easy consequence of the fact that  $\kappa$  is  $\Pi_1^1$ -indescibale. Note that we do not claim [and cannot claim] that  $\kappa$  is weakly compact in  $L_\beta$ .

## 5.4 The length $\omega$ construction

As we mentioned in the introduction to this section, the construction that follows is inspired by the constructions of finite length used to prove that weakly compact cardinals have partition properties.

Let  $A_0 \subseteq L_\kappa$ . We are going to define the following [for  $i \in \omega$ ].

1. A subset  $A_i \subseteq L_\kappa$ .
2. A subset  $B_i \subseteq \kappa$ .  
 $B_0$  will be  $\kappa$ .  $B_{i+1}$  will be a  $\kappa$ -sized subset of  $B_i$ .
3. An ordinal  $\xi_i > \kappa$

Here is a description of the inductive construction of these objects. First we handle the case  $n = 0$ :

1.  $A_0$  is given as the input to this construction.
2.  $B_0 = \kappa$ .
3.  $\xi_0$  is the least ordinal greater than  $\kappa$  such that  $A_0 \in L_{\xi_0}$  and  $L_{\xi_0}$  is a model of  $ZFC-$ . [It follows that  $\xi_0$  has cardinality  $\kappa$  and that  $L_{\xi_0}$  thinks that  $\kappa$  is the largest cardinal.]

Next we handle the case of  $i = n + 1$  assuming these objects have been defined for  $i = n$ :

1.  $A_{n+1}$  is some simple encoding of the pair of sets  $A_n$  and  $B_n$ . To be definite,  $A_{n+1} = [\{0\} \times A_n] \cup [\{1\} \times B_n]$ .
2.  $B_{n+1}$  will be obtained from  $B_n$  using Lemma 5.1.

We fix the  $L$ -least enumeration of length  $\kappa$  of the set of all pairs  $\langle F, \gamma \rangle$  such that  $F \in L_{\xi_n}$ ,  $F : \kappa \mapsto \gamma$  and  $0 < \gamma < \kappa$ . We apply the lemma to this enumeration and the set  $B_n$ , getting back a set which we take for  $B_{n+1}$ . [Actually, the lemma only asserts that there *exists* a set  $B'$  with certain properties. We take  $B_{n+1}$  to be the  $L$ -least set satisfying the conclusions of the lemma.]

3.  $\xi_{n+1}$  is the least ordinal  $> \xi_n$  such that  $A_{n+1} \in L_{\xi_{n+1}}$  and  $L_{\xi_{n+1}}$  is a model of  $ZFC-$ . It follows that  $\xi_{n+1}$  has cardinality  $\kappa$  and that  $L_{\xi_{n+1}}$  thinks that  $\kappa$  is the largest cardinal.

We set  $\xi_\infty = \sup\{\xi_n \mid n \in \omega\}$ .

The intuition is that the sets in  $L_{\xi_\infty}$  are those over which the length  $\omega$  construction has control. Cf. the partition result of the next subsection.

For the remainder of the proof, we keep the notation of this subsection. We will presently exploit our freedom to choose the starting set  $A_0$ .

## 5.5 Partition theorems

Let  $G : \kappa \mapsto \kappa$ .

An increasing  $n$ -tuple of ordinals less than  $\kappa$ :

$$\alpha_1 < \alpha_2 < \dots < \alpha_n$$

is  $G$  spread apart iff:

1.  $G(0) < \alpha_1$ ;
2. For  $1 < i \leq n$ ,  $G(\alpha_{i-1}) < \alpha_i$ .

Recall that  $[\kappa]^n$  is the set of size  $n$  subsets from  $\kappa$  [or what is much the same thing, the set of increasing  $n$ -tuples from  $\kappa$ ].

**Lemma 5.3** *Let  $\gamma$  be an ordinal with  $0 < \gamma < \kappa$ . Let  $F : [\kappa]^n \mapsto \gamma$  with  $F \in L_{\xi_\infty}$ . Then there is an  $m \in \omega$ , a  $G : \kappa \mapsto \kappa$  with  $G \in L_{\xi_\infty}$  and an  $\eta < \gamma$  such that whenever  $\alpha_1 < \dots < \alpha_n$  is an increasing  $n$ -tuple of ordinals from  $B_m$  which is  $G$  spread apart then*

$$F(\alpha_1, \dots, \alpha_n) = \eta$$

*Moreover, the value of  $\eta$  does not depend on the choices of  $m$  and  $G$ .*

Proof: We first prove the claim of the first paragraph of the lemma and proceed by induction on  $n$ .

Consider first the case when  $n = 1$ . Pick  $r$  large enough that  $F \in L_{\xi_r}$ . Then the pair  $\langle F, \gamma \rangle$  was considered when constructing  $B_{r+1}$ . It follows that there is an  $\eta < \gamma$  and an  $\alpha < \kappa$  such that  $F(\beta) = \eta$  whenever  $\beta > \alpha$  and  $\beta \in B_{r+1}$ . So it suffices to take  $G$  the function constantly equal to  $\alpha$  and  $m = r + 1$ .

We proceed to the inductive step where  $n = m + 1$  and we know the first part of the lemma when  $n = m$ . [Here  $n \geq 2$ .] Let again  $F \in L_{\xi_r}$ . Then

clearly for every choice of  $\alpha_1 < \dots < \alpha_m$ , there is an ordinal  $\beta$  and an ordinal  $\eta < \gamma$  such that  $F(\alpha_1, \dots, \alpha_m, \beta^*) = \eta$  whenever  $\beta^* \in B_{r+1}$  and  $\beta^* > \beta$ .

We can find a function  $G^* : \kappa \mapsto \kappa$  and a function  $H : [\kappa]^m \mapsto \gamma$  (both in  $L_{\xi_{r+1}}$ ) which express the dependence of  $\beta$  and  $\eta$  in the preceding paragraph on  $\alpha_1, \dots, \alpha_m$ . Namely, whenever

1.  $\alpha_1 < \dots < \alpha_m < \kappa$ ;
2.  $\kappa > \beta^* > G^*(\alpha_m)$ ;
3.  $\beta^* \in B_{r+1}$

then  $F(\alpha_1, \dots, \alpha_m, \beta^*) = H(\alpha_1, \dots, \alpha_m)$ .

We can now apply our inductive hypothesis to  $H$  getting an integer  $s \in \omega$ , a function  $G_1 : \kappa \mapsto \kappa$  in  $L_{\xi_\infty}$  and an  $\eta < \gamma$  such that whenever  $\alpha_1 < \dots < \alpha_m$  lie in  $B_s$  and are  $G_1$  spread apart, then  $H(\alpha_1, \dots, \alpha_m) = \eta$ .

Define  $G$  by  $G(\alpha) = \max(G^*(\alpha), G_1(\alpha))$ . Let  $s^* = \max(r + 1, s)$ . Then clearly whenever  $\alpha_1, \dots, \alpha_{m+1}$  lie in  $B_{s^*}$  and are  $G$  spread apart, then

$$F(\alpha_1, \dots, \alpha_{m+1}) = \eta.$$

We have successfully completed the inductive step.

We turn to the last paragraph of the lemma. Suppose that  $m, G$  and  $\eta$  are such that whenever  $\alpha_1, \dots, \alpha_n$  lie in  $B_m$  and are  $G$  spread apart, then  $F(\alpha_1, \dots, \alpha_n) = \eta$ .

Suppose further that  $m', G'$  and  $\eta'$  are such that whenever  $\alpha_1, \dots, \alpha_n$  lie in  $B_{m'}$  and are  $G'$  spread apart, then  $F(\alpha_1, \dots, \alpha_n) = \eta'$ . We must show that  $\eta = \eta'$ .

This is not difficult. Let  $m^* = \max(m, m')$ . Define  $G^* : \kappa \mapsto \kappa$  by  $G^*(\alpha) = \max(G(\alpha), G'(\alpha))$ .

We can clearly find  $\alpha_1, \dots, \alpha_n$  in  $B_{m^*}$  which are  $G$  spread apart. But then:

$$\eta = F(\alpha_1, \dots, \alpha_n) = \eta'.$$

## 5.6 The “ultrapower” construction

The word “ultrapower” is in quotes since what we do, though inspired by the ultrapower construction, is somewhat different.

Throughout this section  $\mathcal{A} = \langle A; R_1, \dots, R_k, f_1, \dots, f_p \rangle$  is a first-order structure which is a member of  $L_{\xi_\infty}$ . Thus  $A$  is a set [necessarily of cardinality

$\leq \kappa$ ],  $R_i$  is a relation on  $A$  of arity  $n_i$  and  $f_i$  is an operation of arity  $m_i$ . We let  $\mathcal{L}$  be the first order language appropriate to the similarity type of  $\mathcal{A}$ .

Our construction will give us a new model  $\mathcal{A}^*$  of the same similarity type as  $\mathcal{A}$  together with an elementary embedding  $d : \mathcal{A} \mapsto \mathcal{A}^*$ .

What is novel about our construction and differs from the usual ultrapower construction is that there will also be a canonical automorphism  $k$  of  $\mathcal{A}^*$ . The points of  $\mathcal{A}^*$  which are fixed by  $k$  are precisely those in the range of the diagonal map  $d$ .

### 5.6.1 The “measure space”

We let  $X$  be the set of all functions mapping  $\mathbf{Z}$  into  $\kappa$ . [Any such function lies in  $L_\kappa$ , so the set  $X$  is known to  $L_{\xi_\infty}$ .] Here  $\mathbf{Z}$  is, of course, the set of integers [positive, negative, or zero].

Let  $f \in X$  and let  $s$  be a finite subset of  $\mathbf{Z}$ . Then we write  $f[s]$  to indicate the restriction of  $f$  to the set  $s$ .

### 5.6.2 The class of functions

In the ordinary definition of an ultrapower, we would consider all functions from the measure space  $X$  to the underlying set  $A$  of our target model. Here, however, we must put several restrictions on our functions.

We define a class  $\mathcal{F}$  of functions mapping  $X$  to  $A$  as follows.  $f \in \mathcal{F}$  iff:

1.  $f$  is in the model  $L_{\xi_\infty}$ .
2. There is a finite subset  $s$  of  $\mathbf{Z}$  such that the value of  $f(x)$  depends only on  $x[s]$ . [We say that  $s$  is a support for the map  $f$ .]

Thus an  $f \in \mathcal{F}$  can be described as the composition of three maps:

1. The map which sends an element  $x \in X$  to  $x[s]$ ;
2. Let  $s$  have  $n$  elements:  $s = \{s_1, \dots, s_n\}$  where  $s_1 < s_2 < \dots < s_n$ . Then we have the map which sends  $x[s]$  to an element of  $\kappa^n$ :  $x[s] \mapsto \langle x(s_1), \dots, x(s_n) \rangle$ .
3. The final element of the composition is some function  $F : \kappa^n \mapsto A$  with  $F \in L_{\xi_\infty}$ .

N. B. The set  $\kappa^n$  is the set of [not necessarily increasing]  $n$ -tuples from  $\kappa$ . It should not be confused with the set  $[\kappa]^n$  of strictly increasing  $n$ -tuples from  $\kappa$ .

### 5.6.3 The “ultrafilter”

For each triple  $\langle s, m, G \rangle$  where:

1.  $s$  is a finite subset of  $\mathbf{Z}$ ;
2.  $m \in \omega$ ;
3.  $G : \kappa \mapsto \kappa$  lies in  $L_{\xi_\infty}$

we associate a subset  $A_{s,m,G} \subseteq X$  as follows:

An element  $x \in X$  lies in  $A_{s,m,G}$  iff:

1. The function  $x[s]$  is strictly increasing on its domain.
2. Let  $s$  have  $n$  elements. Let these elements, listed in increasing order, be  $s_1, \dots, s_n$ . Then  $x(s_i) \in B_m$  for  $1 \leq i \leq n$ .
3.  $x(s_1), \dots, x(s_n)$  are  $G$ -spread apart.

The reader should verify that the set  $A_{s,m,G}$  is non-empty.

These sets form the base of a filter in the following sense. Let  $\langle s_1, m_1, G_1 \rangle$  and  $\langle s_2, m_2, G_2 \rangle$  be two triples of the sort just discussed.

Set  $s = s_1 \cup s_2$ . Set  $m = \max(m_1, m_2)$ . Define  $G : \kappa \mapsto \kappa$  by setting  $G(\alpha) = \max(G_1(\alpha), G_2(\alpha))$ . Then:

$$A_{s,m,G} \subseteq A_{s_1,m_1,G_1} \cap A_{s_2,m_2,G_2}.$$

The following terminology will make the analogy with the usual ultra-power construction more transparent. Say that a subset of  $X$  is *measurable* if its characteristic function lies in the obvious analogue of  $\mathcal{F}$ . [Make the same definition but replace reference to  $A$  by reference to  $\{0, 1\}$ .] We say that a measurable set has measure 1 if it contains a set of the form  $A_{s,m,G}$ . We say that a measurable set has measure 0 if its complement has measure 1. It follows from the fact that sets of the form  $A_{s,m,G}$  form the base for a filter, that no set has simultaneously measure 1 and measure 0. It is an immediate consequence of the main result of Section 5.5 that given a measurable set  $A \subseteq X$  precisely one of  $A$  and  $X - A$  has measure 1. In this way we have defined an ultrafilter on the Boolean algebra of measurable sets. It is definitely not countably complete.



#### 5.6.4 Construction of the “ultrapower” $\mathcal{A}^*$

We omit many details since this now parallels the usual ultraproduct construction.

We first put an equivalence relation on the functions of  $\mathcal{F}$ . Two such functions, say  $f_1$  and  $f_2$  are equivalent if  $\{x \mid f_1(x) = f_2(x)\}$  has measure 1.  $\mathcal{A}^*$  will consist of all the equivalence classes of functions in  $\mathcal{F}$  [for the equivalence relation just described].

The basic relations of  $\mathcal{A}^*$  are defined in terms of representatives; it must be verified that the result is independent of the choices made. Let  $R_i$  be a  $n$ -ary relation of  $\mathcal{A}$ . The analogue for  $\mathcal{A}^*$  ( $R_i^*$ ) is defined as follows:

$R_i^*([f_1], \dots, [f_n])$  holds in  $\mathcal{A}^*$  iff  $\{x \mid R_i(f_1(x), \dots, f_n(x))\}$  has measure 1.

Similarly, let  $g$  be an  $n$ -ary operation of  $\mathcal{A}$ . The corresponding operation of  $\mathcal{A}^*$ , call it  $g^*$ , is defined thus:  $g^*([f_1], \dots, [f_n])$  is represented by the function:

$$x \mapsto g(f_1(x), \dots, f_n(x))$$

This completes our definition of the structure  $\mathcal{A}^*$ . Note that the resulting structure is clearly a *set* of  $M$  [since the functions we employ are taken from the *set*  $L_{\xi_\infty}$ ; in particular, the cardinality of the underlying set  $\mathcal{A}^*$  is clearly at most  $\kappa$ ].

#### 5.6.5 The Los theorem

**Proposition 5.4** *Let  $\varphi(v_1, \dots, v_n)$  be a formula of  $\mathcal{L}$  [the language appropriate to  $\mathcal{A}$ ] having at most the indicated free variables. Let  $[f_1], \dots, [f_n]$  be elements of  $\mathcal{A}^*$ .*

*Then  $\varphi([f_1], \dots, [f_n])$  holds in  $\mathcal{A}^*$  iff*

$$\{x \in X \mid \varphi(f_1(x), \dots, f_n(x)) \text{ holds in } \mathcal{A}\}$$

*has measure 1.*

Proof: The usual proof applies without essential change.

#### 5.6.6 The diagonal map

We define a map  $d : \mathcal{A} \mapsto \mathcal{A}^*$  as follows:

Let  $a \in A$ . Let  $c_a : X \mapsto \{a\}$  be the constant map with value  $a$ . Set  $d(a) = [c_a]$ .

As usual, it is an immediate consequence of the Los theorem that  $d$  is an elementary embedding.

### 5.6.7 The automorphism $k$

We define a bijection  $s : X \mapsto X$  by  $s(x)(n) = x(n + 1)$  [for  $n \in \mathbf{Z}$ ].

We define a map  $K : \mathcal{F} \mapsto \mathcal{F}$  by  $K(f)(x) = f(s(x))$ . [The routine check that  $K(f)$  is indeed in  $\mathcal{F}$  is left to the reader.]

The reader should also verify the more precise statement that if  $f$  is supported by  $\{s_1, \dots, s_n\}$  then  $K(f)$  is supported by  $\{s_1 + 1, \dots, s_n + 1\}$ .

**Lemma 5.5** *Let  $\varphi(v_1, \dots, v_n)$  be a formula of the language  $\mathcal{L}$  appropriate to  $\mathcal{A}$  having the indicated free variables. Let  $f_1, \dots, f_n \in \mathcal{F}$ .*

*Then the sentence*

$$\varphi([f_1], \dots, [f_n]) \iff \varphi([K(f_1)], \dots, [K(f_n)])$$

*holds in  $\mathcal{A}^*$ .*

*Proof:* It is clear that if an element of  $\mathcal{F}$  is supported by a finite set  $s$ , then it is supported also by any larger finite set  $s'$ . Hence, we may assume that  $f_1, \dots, f_n$  are all supported by  $[a, b]$ . [Here,  $a$  and  $b$  are integers with  $a \leq b$ .  $[a, b] = \{x \in \mathbf{Z} \mid a \leq x \leq b\}$ .]

Applying the main result of section 5.5, we see that there is an integer  $m$  and a function  $G : \kappa \mapsto \kappa$  [lying in  $L_{\xi_\infty}$ ] and a truth value  $\tau \in \{0, 1\}$  such that whenever:

1.  $x(a) < x(a + 1) \dots < x(b)$ ,
2.  $x(i) \in B_m$  for  $a \leq i \leq b$ , and
3.  $x(a), \dots, x(b)$  are  $G$  spread apart

then  $\varphi(f_1(x), \dots, f_n(x))$  receives the value  $\tau$  in  $\mathcal{A}$ .

Let  $s = [a, b + 1]$ . It follows that whenever  $x \in A_{s, m, G}$  then

$$\varphi(f_1(x), \dots, f_n(x)) \iff \varphi(K(f_1)(x), \dots, K(f_n)(x))$$

receives the truth value 1.

The lemma now follows from the Los theorem.

One immediate consequence of the lemma just proved is that we can define a map  $k : A^* \mapsto A^*$  by setting  $k([f]) = [K(f)]$ . [I.e., the value of  $k([f])$  does not depend on the choice of representative.]

The other immediate consequence is that the map  $k$  is an automorphism of  $A^*$ .

### 5.6.8 Supports

Let  $x \in A^*$ . We say that a finite subset  $s \subseteq \mathbf{Z}$  is a support for  $x$  if it is a support for some  $f \in \mathcal{F}$  with  $x = [f]$ .

We say that  $s$  is a block support for  $x$  if  $s$  is a support for  $x$  of the form  $[a, b]$  [where  $a$  and  $b$  are members of  $\mathbf{Z}$  with  $a \leq b$ ]. Of course every  $x \in A^*$  has a block support.

**Lemma 5.6** *Let  $x \in A^*$ , Then  $x$  has a minimum block support  $s_0$  which is contained in every other block support for  $x$ .*

Remark: The lemma remains true if the word “block” is deleted throughout. We shall not prove this stronger result.

Proof: Let  $s_0$  be a block support for  $x$  of minimum cardinality. We have to show that it is contained in every other block support for  $x$ . Suppose not, then there is another block support for  $x$  which neither contains or is contained in  $s_0$ .

So there is some element of  $s_0$  which is not contained in  $s_1$ . If we shrink  $s_1$  this will continue to be true. Thus we may suppose that  $s_1$  is *minimal* in the sense that no proper subset of it is a block support for  $x$ . [Obviously  $s_0$  is minimal as well.]

Since  $s_0$  and  $s_1$  are blocks, neither of which is contained in the other, the least elements of the two blocks must be different. Interchanging the two blocks, if necessary, we may assume that the least element of  $s_0$  is less than the least element of  $s_1$ . It follows easily that also the largest element of  $s_0$  is less than the largest element of  $s_1$ .

If we performed this interchange, we no longer know that  $s_0$  is of minimum cardinality. But we still have that  $s_0$  is minimal [since before the interchange both  $s_0$  and  $s_1$  were minimal].

We let  $s_0 = [a_0, b_0]$  and  $s_1 = [a_1, b_1]$ . Let  $n_i = \text{card}(s_i)$ . Then there are functions  $F_i : \kappa^{n_i} \mapsto A$  in  $L_{\xi_\infty}$  such that defining  $f_i : X \mapsto A$  by

$$f_i(x) = F_i(x(a_i), \dots, x(b_i))$$

we have  $x = [f_0] = [f_1]$ .

It follows [by the Łos theorem] that there is an integer  $m$  and a function  $G : \kappa \mapsto \kappa$  such that whenever

1.  $x(a_0) < \dots < x(b_1)$ ;
2.  $x(i) \in B_m$  [for  $a_0 \leq i \leq b_1$ ];
3.  $x(a_0), \dots, x(b_1)$  are  $G$  spread apart

then  $F_0(x(a_0), \dots, x(b_0)) = F_1(x(a_1), \dots, x(b_1))$ .

**Proposition 5.7** *Let  $\theta_0, \dots, \theta_{n_0}$  be an increasing sequence of ordinals from  $B_m$  which are  $G$  spread apart. Then  $F_0(\theta_0, \theta_2, \dots, \theta_{n_0}) = F_0(\theta_1, \dots, \theta_{n_0})$ .*

Proof: Let  $n_2 = \text{card}([a_0, b_1])$ . We can enlarge the sequence of thetas to  $\theta_0, \dots, \theta_{n_2}$  so that it is still true that the sequence of thetas is increasing, consists of elements of  $B_m$ , and is  $G$  spread apart.

Let  $r = n_2 - n_1 + 1$ . Note that our hypotheses on  $s_0$  and  $s_1$  imply that  $r \geq 2$ . It follows that

$$F_0(\theta_0, \theta_2, \dots, \theta_{n_0}) = F_1(\theta_r, \dots, \theta_{n_2}) = F_0(\theta_1, \theta_2, \dots, \theta_{n_0})$$

The proposition is now clear.

Let now  $\theta^*$  be the least element of  $B_m$  which is greater than  $G(0)$ . Define  $G' : \kappa \mapsto \kappa$  so that

1.  $G'(0) = \max(G(0), \theta^* + 1, G(\theta^*))$ ;
2.  $G'(\alpha) = G(\alpha)$  for  $\alpha > 0$ .

Then if  $\theta_1, \dots, \theta_{n_0}$  is  $G'$  spread apart, then the sequence  $\theta^*, \theta_1, \dots, \theta_{n_0}$  is increasing and  $G$  spread apart.

Define a function  $H : \kappa^{n_0-1} \mapsto A$  by  $H(\theta_2, \dots, \theta_{n_0}) = F_0(\theta^*, \theta_2, \dots, \theta_{n_0})$ . Define  $h : X \mapsto A$  by  $h(x) = H(x(a_0 + 1), \dots, x(b_0))$ . Then it follows readily from the preceding that  $[f_1] = [h]$ . But this contradicts the fact that  $s_0$  is a minimal support for  $x$ . The lemma is proved.

## 5.7 Proof of the main lemma

We turn now to the proof of Lemma 4.1. We let  $\mathcal{N}$  and  $S$  be as in the statement of that lemma. We choose our set  $A_0$  [the input to the “length  $\omega$  construction”] to encode both of these objects. There is no difficulty doing this since  $N$ , the underlying set of  $\mathcal{N}$ , has cardinality  $\kappa$ .

The model to which we will apply our ultrapower construction is  $\mathcal{A} = \langle N \mid \in_N, j \rangle$ .

The result of the ultrapower construction is  $\mathcal{A}^* = \langle N^* \mid \in_{N^*}, j^* \rangle$ .

$N^*$  will be the underlying set of the model  $\mathcal{N}'$  that we are constructing. The elementary embedding  $\pi$  will just be the diagonal map  $d$  of the preceding subsection.

We have to verify that  $\pi$  is not onto. This is easy. Let  $i$  be the given embedding of  $L_\kappa$  into  $N$ . Define  $H : X \mapsto N$  by  $H(x) = i(x(0))$ . Then it is easily checked that  $[H]$  is not in the range of  $\pi$ .

The map  $i'$  will be the composition  $\pi i$ . The commutativity of the diagram of Figure 1 is immediate, and since both  $\pi$  and  $i$  are elementary embeddings of models of set theory, so is  $i'$ .

We have to verify that the range of  $i'$  is an initial segment of  $N'$ . Suppose not. Then there is an element  $i'(x)$  and an element  $[f] \in N^*$  such that  $[f] \in_{N^*} i'(x)$  but  $[f]$  is not of the form  $i'(y)$ .

Now  $f(x) = F(x(a), \dots, x(b))$  for some  $F \in L_{\xi_\infty}$  and  $a \leq b$  in  $\mathbf{Z}$ . By Łos, on some set of measure 1,  $f$  takes values in the set  $\{i(y) \mid y \in x\}$  of cardinality less than  $\kappa$ . It follows readily from lemma 5.3 that on some smaller set of measure 1,  $f$  is constant. This shows that  $[f]$  has the form  $i'(y)$ .

It follows readily from the fact that  $k$  is an automorphism of  $\mathcal{A}^*$  and the fact that  $j^*$  is one of the components of  $\mathcal{A}^*$  that  $j^*$  and  $k$  commute.

We set  $j' = kj^*$ . Since both  $k$  and  $j$  are automorphisms of  $\langle N^* \mid \in_{N^*} \rangle$ , so is  $j'$ .

We must check that the diagram of Figure 2 commutes. Let  $a \in N$ . We have to show that  $j'(\pi(a)) = \pi(j(a))$ . We write  $j' = j^*k$ . Now  $\pi$  is just the diagonal map, and we know that elements in the range of the diagonal map are fixed by  $k$ . So we are reduced to proving that  $j^*(\pi(a)) = \pi(j(a))$ . But this is clear from the following facts:

1.  $\pi$  is an elementary embedding from  $\mathcal{A}$  to  $\mathcal{A}^*$ .
2.  $j$  is one of the basic operations of  $\mathcal{A}$ .

3.  $j^*$  is the corresponding operation of  $\mathcal{A}^*$ .

Let  $x$  be an element of  $N^*$  fixed by  $j'$ . We must show that  $x$  is in the range of  $i'$ .

We first show that  $x$  is in the range of  $\pi$ . This amounts to showing that  $x$  has support  $\emptyset$ . Suppose not toward a contradiction.

Let  $[a, b]$  be the minimum block support for  $x$ . Let  $x = [f]$  with  $f(x) = F(x(a), \dots, x(b))$  and  $F \in L_{\xi_\infty}$ . Then  $j'(x)$  is represented by the map  $\{x \mapsto j(F(x(a+1), \dots, x(b+1)))\}$ . Since  $j'(x) = x$  and  $[a, b]$  is the minimal block support for  $x$ , we conclude that  $[a, b] \subseteq [a+1, b+1]$  which is absurd. The upshot is that the element  $x$  has empty support and so is in the range of  $\pi$ . Say  $x = \pi(y)$ .

Now

$$\pi(y) = x = j'(x) = j'(\pi(y)) = \pi(j(y))$$

Since  $\pi$  is injective, we conclude that  $y = j(y)$ . Since  $\mathcal{N}$  is an  $A$ -model, we conclude that  $y = i(z)$  for some  $z$ . But then  $x = \pi(i(z)) = i'(z)$  as desired.

We have now checked that  $\mathcal{N}'$  is an  $A$ -model, and that  $\pi$  is an  $A$ -model map which is not onto. The only remaining point to check is that  $S$  is coded in  $\mathcal{N}'$ .

Consider the map  $h : X \mapsto N$  given by  $h(x) = i(S \cap L_{x(0)})$ . It is easy to check that the element  $[h]$  codes  $S$  in  $\mathcal{N}'$ . The proof of Lemma 4.1 and hence the proof of Theorem 1.1 is complete.

## 6 Postscript

After completing this draft, I discovered a more conceptual way to think about the “ultrapower” construction of section 5.6.

In the next draft of this paper, I will incorporate this improvement. In the meantime, I indicate the main ideas in this postscript.

1. One can view the length  $\omega$  construction of section 5.4 as defining a measure,  $\nu$ , on the subsets of  $\kappa$  lying in  $L_{\xi_\infty}$ . Given  $B \subseteq \kappa$  with  $B \in L_{\xi_\infty}$ , the construction insures that for some  $n$ , either a tail of  $B_n$  is included in  $B$  or a tail of  $B_n$  is included in  $\kappa - B$ . In the former case, we set  $\nu(B) = 1$ ; in the latter case, we set  $\nu(B) = 0$ .

2. By slightly modifying the construction, we can arrange that for any  $G : \kappa \mapsto \kappa$  with  $G \in L_{\xi_\infty}$ , a tail of  $B_n$  is  $G$  spread apart for all sufficiently large  $n$ .

This is easy to achieve by a suitable thinning of the output of the “basic module” of section 5.2. We take  $B_{n+1}$  to be this thinned down set rather than the output of the basic module applied to  $B_n$ .

3. The result of this is that the measure  $\nu$  will enjoy the following partition property. [Cf. the proof in section 5.5.]

If  $F : [\kappa]^n \mapsto \gamma$  [with  $\gamma < \kappa$  and  $F \in L_{\xi_\infty}$ ] then there is a set  $B \in L_{\xi_\infty}$  with  $\nu(B) = 1$  such that  $F$  is constant on  $[B]^n$ .

4. One can now go through the construction of section 5.6 much as before. But it looks much more familiar, resembling the usual iterated ultrapowers with respect to a measurable cardinal [with, however, the set  $\mathbf{Z}$  indexing the iterations].
5. There is one point to be cautious about. Unlike the usual measures constructed from weakly compact cardinals, there is no reason to suppose that the ultrapower of  $\kappa$  with respect to  $\nu$  [using only functions  $F : \kappa \mapsto \kappa$  lying in  $L_{\xi_\infty}$ ] is well-founded or even that there is a least non-constant function.