What's hat got to do with it?

A hat trick of prisoner hat puzzles

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Adapted from slides by Eric Neyman

Outline

The Finite Puzzle: Statement

- Ten prisoners in a line, called P_1 through P_{10}
- Each prisoner is wearing a hat, which is either white or black
- P_i can see P_j's hat whenever $i < j$
  - P_1 can see all hats but their own
  - P_{10} can't see any hats
- The executioner asks each prisoner in order for their own hat color
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If the prisoners strategize together beforehand, how many correct guesses can they guarantee?

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4. P_i knows the hat color of $P_2, P_3, \ldots, P_{i-1}$ from their guesses, and all the hats ahead of them by looking, so they can figure out their own hat color by the same parity comparison

Thus, every prisoner but P_1 is guaranteed to guess their own hat color correctly!

Infinite Puzzle I: Statement

- Infinitely many prisoners in a line, called $P_1, P_2, P_3, \ldots$
- Each prisoner is wearing a hat, which is either white or black
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  - P_1 can see all hats but their own
  - P_{10} can see the hats of $P_{11}, P_{12}, \ldots$
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If the prisoners strategize together beforehand, what is the minimal number of incorrect guesses they can guarantee?

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Example

Take our set to be $\mathbb{Z}$, and our equivalence relation to be congruence mod 3. Then $7 \sim 1$, and $-1 \not\sim 0$.

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Remark

Equivalence classes partition the set on which the relation is defined into disjoint subsets, whose union is the original set.

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Remark

The Axiom of Choice lets us pick representatives from our equivalence classes, no matter how complicated these equivalence classes are.

Infinite Puzzle I: Solution

Now we can finally solve the first infinite puzzle! (Remember the puzzle?)
The following strategy will guarantee that at most one of the infinitely many prisoners will die!

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- For any pair of equivalent sequences, there's a last difference between them
- The equivalence relation gives equivalence classes, so using the axiom of choice the prisoners pick representatives

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The solution is exactly the same as the finite case! (Once we do a bunch of complicated set up)

Infinite Puzzle II: Statement

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- Each prisoner is wearing a hat, which is either white or black
- P_i can see P_j's hat whenever $i < j$
  - P_1 can see all hats but their own
  - P_{10} can see the hats of $P_{11}, P_{12}, \ldots$
- The executioner asks each prisoner *simultaneously* for their own hat color
- Prisoners who guess wrong are brutally executed

If the prisoners strategize together beforehand, what is the minimal number of incorrect guesses they can guarantee?

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- Each prisoner guesses that their hat matches the hat in their position in the representative sequence
- Since there are only finitely many differences between the sequence of prisoners and the representative, it's impossible for infinitely many prisoners to die

Nobody has any information about their own hat, but by coordinating they can guarantee that "nearly every" prisoner survives!

Boxes of Numbers: Statement

- There is an infinite line of closed boxes, called $B_1, B_2, B_3, \ldots$
- In each box is a real number
- You may open any boxes, as long as you leave at least one closed
- After you're done looking at boxes, you must choose a closed box and guess the number inside it

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- We can now pick representatives as before (again using the axiom of choice)
- Notice that for any pair of equivalent sequences, there's a last difference between them

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    2. Compare each sequence with the corresponding representative
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The chance that our chosen row had the latest last difference was 1%, so we guess correctly 99% of the time!

Questions?

1. The Finite Puzzle
   - Statement
   - Solution
2. Infinite Puzzle I
   - Statement
   - Detour: Equivalence Classes and the Axiom of Choice
   - Solution
3. Infinite Puzzle II
   - Statement
   - Solution
4. Boxes of Numbers
   - Statement
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