On Martin's Conjecture

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The Hierarchy of Definability

We are all familiar with the hierarchies of definability which appear in recursion theory, fine structure, and descriptive set theory in the context of determinacy.

From these notions of definability, we obtain the canonical functions from 2^ω to 2^ω such as $x \mapsto x'$, $x \mapsto \mathcal{O}^x$, and $x \mapsto x^\sharp$. Similarly, we obtain the canonical functions from 2^ω to 2^{2^ω} such that $x \mapsto \Delta^0_2(x)$, $x \mapsto \Delta^1_1(x)$, and $x \mapsto 2^\omega \cap L(x)$.

We will discuss some attractive conjectures which assert that these functions are examples from naturally characterized and ordinally parametrized classes.

Notation

Definition 1 1. A *cone of reals* is a set $\{y : y \geq_T x\}$, for some x.

2. A property P on D *contains a cone* iff there is a cone of reals all of whose degrees satisfy P.

3. A function $f : 2^\omega \to 2^\omega$ is *degree invariant* iff for x and y, if $x \equiv_T y$ then $f(x) \equiv_T f(y)$.

For functions from D to D, we define order preserving on a cone, constant on a cone, and other notions, similarly.

For degree invariant f and g, $f \geq_M g$ iff $\{x : f(x) \geq_T g(x)\}$ contains a cone.

Martin's Conjecture

Martin has made the following conjecture: Assume *ZF+AD+DC*.

I. If f is degree invariant and not increasing on a cone, then f is constant on a cone.

II. $\leq_M$ prewellorders the set of degree invariant functions which are increasing on a cone. If f has $\leq_M$-rank α, then f' has $\leq_M$-rank $\alpha + 1$, where $f' : x \mapsto f(x)'$ for all x.

Martin's conjecture was prompted by Sacks's question, "Is there a degree invariant solution to Post's problem?"

Though Martin's conjecture is open, there is a small but interesting collection of partial results around it.

Regressive Functions

Theorem 2 (Slaman and Steel, 1988) *Assume AD. For any degree invariant function f, if $f(x) <_T x$ on a cone then f is constant on a cone.*

Comments on the proof:

- Use determinacy to obtain a pointed perfect set U and a recursive function $\{e\}$ such that for all $x \in U$, $\{e\}(x) \equiv_T f(x)$.
- Give a recursion theoretic argument, analyzing the rate at which the computations to evaluate $\{e\}$ converge, to show that $\{e\}$ is constant on a cone.

Uniformly Degree Invariant Functions

Definition 3

- We say that $x \equiv_T y$ *via* (i, j) iff $\{i\}(x) = y$ and $\{j\}(y) = x$.
- $f : 2^\omega \to 2^\omega$ is *uniformly degree invariant* iff there is a point perfect set U and a function $t : \omega \times \omega \to \omega \times \omega$ such that

$$\forall x, y \in U[x \equiv_T y \text{ via } (i, j) \to f(x) \equiv_T f(y) \text{ via } t(i, j)]$$

For example, $x \mapsto x'$ is uniformly degree invariant.

Steel's Theorem

Theorem 4 (Steel, 1982–88) *Martin's conjecture is true when restricted to uniformly degree invariant functions.*

Theorem 4 extends an earlier result of Lachlan, that there is no uniformly degree invariant solution to Post's problem. Steel's proofs are direct (not straightforward) applications of *AD*.

Steel has made the following conjecture: Assume *ZF+AD+DC*.

III. Every degree invariant function is uniformly degree invariant.

Becker's Analysis

Becker (1988) extended the understanding of those uniformly degree invariant functions which are increasing on a cone. Given a uniformly degree invariant function f, Becker constructs a pointclass Γ such that for each x, $f(x)$ has the same degree as the universal $\Gamma(x)$ subset of ω.

Theorem 5 (Becker) *If f is a uniformly degree invariant function, then f is uniformly order preserving.*

Order Preserving Functions

Theorem 6 (Slaman and Steel, 1988) *Suppose that* $f : 2^\omega \to 2^\omega$ *is order preserving and increasing on a cone. Then either,*

- $\exists \alpha < \omega_1 [f(x) \equiv_T x^\alpha]$ *on a cone or*
- $\forall \alpha < \omega_1^x [x^\alpha <_T f(x)]$ *on a cone.*

Theorem 6 follows from a modest extension the following.

Theorem 7 (Posner and Robinson, 1981) *For any* $x \in 2^\omega$*, if* x *is not recursive then there is a* g *such that* $x + g \equiv_T g'$.

For every nonrecursive set x, x is equivalent to the jump relative to some set g.

Functions from 2^ω to 2^{2^ω}

Martin's conjecture is an assertion that the set of definable and degree invariant functions from 2^ω to 2^ω are naturally parametrized by the ordinals.

We will now make a similar conjecture for the functions from 2^ω to 2^{2^ω} that share some of the characteristics of maps from x to set of $y \in 2^\omega$ such that y is definable from x.

Closure Operators

Definition 8 A *closure operator* is a map $M : 2^\omega \to 2^{2^\omega}$ with the following properties.

1. For all $x \in 2^\omega$, $x \in M(x)$.
2. For all x and for all z, if z is recursive in finitely many elements of $M(x)$ then $Z \in M(x)$. *$M(x)$ is closed under relative computation.*
3. For all x and y in 2^ω, if x is recursive in y then $M(x) \subseteq M(y)$. *M is monotone.*

As in the Martin Conjecture, closure operators can be compared by eventual pointwise inclusion.

The Borel Case

Theorem 9 (Slaman) *If M is a closure operator such that the relation $y \in M(x)$ is Borel, then one of the following conditions holds.*

1. *There is a countable ordinal α such that M is equivalent to the map $x \mapsto \{y : y$ is recursive in $x^{(\alpha)}\}$.*
2. *There is a countable ordinal α such that M is equivalent to the map $x \mapsto \{y : (\exists\beta < \alpha)[y$ is recursive in $x^{(\beta)}]\}$.*
3. *M is equivalent to the map $x \mapsto 2^\omega$.*

Here $x^{(\alpha)}$ is evaluated relative to a fixed counting of α.

Question 10 Is something similar true under AD for those functions f such that for all x, $f(x) \subset L(x)$?

REA-operators

Definition 11 (Jockusch and Shore, 1984) • An *REA-operator* is a function j from 2^ω to 2^ω such that there is an e such that for all x, $j(x)$ is the join of x with the eth set which is recursively enumerable relative to x.

- An α-*REA*-operator is an α-length iteration of *REA*-operators, where the iteration is organized using a recursive presentation of α.

Theorem 12 • ***(Shore and Slaman, 1999)*** *Suppose that j is an α-REA operator and for all $\beta < \alpha$, $\emptyset^{(\beta)} \not\geq_T x$. There is a g such that $x + g \equiv_T j(g)$.*

- ***(Woodin)*** *Suppose that x is not Δ^1_1. Then there is a g such that $x + g \equiv_T \mathcal{O}^g$.*

What Next?

The proof of Theorem 9 uses the join theorem for α-*REA*-operators to work through the hyperarithmetic hierarchy and then applies Woodin's join theorem for the hyperjump eliminate any remaining possibilities.

This approach loses momentum as one attempts to use it to work through through the ordinals.

A Counter Point

Kechris has conjectured that Turing equivalence is a universal countable Borel equivalence relation. If this conjecture is true, then Martin’s conjecture is false.