Random Sequences and Recursive Measures

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Randomness

There are at least as many views of effective randomness as there are viewers, and not one of them is completely satisfactory.

We will discuss and compare features of the two most prominent of them, measure theoretic randomness and algorithmic randomness.

Randomness

Points of comparison:

- The class of random reals.
- Natural examples and recursively enumerable reals.
- Dependence on the underlying measure.

Randomness

Definition 1 Σ^* is the set of finite binary sequences and Σ^ω the set of infinite ones. If $A \subseteq \Sigma^*$ then $A\Sigma^\omega$ is the open set of extensions of A.

Randomness: Measure

Definition 2 1. A *Martin-Löf test* is a uniformly recursively enumerable sequence $(A_n : n \geq 1)$ of subsets of $2^{<\omega}$ such that for each n, $\mu(A_n \Sigma^\omega) \leq 1/2^n$.

2. For $\alpha \in \Sigma^*$, α is *Martin-Löf random* iff for every Martin-Löf test $(A_n : n \geq 1)$, $\alpha \notin \bigcap_{n \geq 1} A_n \Sigma^\omega$.

3. A Martin-Löf test is *universal* iff for every α, if $\alpha \notin \bigcap_{n \geq 1} A_n \Sigma^\omega$ then α is Martin-Löf random.

Randomness: Complexity

Definition 3 For f a partial function from Σ^* to Σ^*, f is *self-delimiting* iff the domain of f is an antichain.

We obtain natural examples of self-delimiting functions on Σ^* from recursive functions on Σ^ω by considering the uses of their convergent computations.

$f(a)\downarrow$ indicates that a is in the domain of f.

Randomness: Complexity

Definition 4 (Levin, Chaitin) Suppose that f is a self-delimiting recursive function.

1. If b is in the range of f then the f-complexity of b is the least $|a|$ where $f(a) = b$. Otherwise, the f-complexity of b is ∞. We write $H_f(b)$ for the f-complexity of b.
2. The *halting probability* of f is $\sum_{f(a)\downarrow} 1/2^{|a|}$.

The halting probability of f is a real number in $[0, 1]$. Identify such reals as elements of Σ^ω.

Randomness: Complexity

Definition 5 (Chaitin) A recursive function u is *Chaitin-universal* iff the following conditions hold:

1. u is self-delimiting.
2. For any self-delimiting recursive function f there is a constant C such that for all a, $H_u(a) \leq H_f(a) + C$.

Theorem 6 (Chaitin) *There is a Chaitin-universal function.*

Randomness: Complexity

Definition 7 (Chaitin) For $\alpha \in \Sigma^*$, α is *Chaitin-random* iff there is a Chaitin-universal function u and a constant C such that for all n, $H_u(\alpha \upharpoonright n) > n - C$. (A similar notion is due to Kolmogorov.)

One could say that Chaitin-random reals α are incompressible. It takes roughly n items of data to describe $\alpha \upharpoonright n$. In other words, up to a constant, the string complexity of $\alpha \upharpoonright n$ (namely, $H_u(\alpha \upharpoonright n)$) is n.

Randomness: Complexity

Chaitin provided a class of natural examples of Chaitin-random reals.

Definition 8 (Chaitin) An Ω-*number* is the halting probability of a Chaitin-universal function.

Theorem 9 (Chaitin) *Every* Ω*-number is Chaitin-random.*

Randomness: Equivalence

It is straightforward to check that Martin-Löf-random reals are Chaitin-random. Schnorr proved the converse.

Theorem 10 (Schnorr) *For every $\alpha \in \Sigma^\omega$, if α is Chaitin-random then α is Martin-Löf random.*

Henceforth, we will simply say that α is random.

Recursive Enumerability

Definition 11 For $\alpha \in \Sigma^\omega$, α is recursively enumerable iff there is a lexicographically nondecreasing recursive sequence $(a_n : n \in \omega)$ such that $\lim_{n\to\infty} a_n = \alpha$.

For example, the Ω-numbers are recursively enumerable.

Recursive Enumerability

Definition 12 (Solovay) Let $(a_n : n \geq 1)$ and $(b_n : n \geq 1)$ be recursive lexicographically nondecreasing sequences in Σ^* which converge to α and β, respectively.

1. $(a_n : n \geq 1)$ dominates $(b_n : n \geq 1)$ iff there is a positive constant C such that for all n, $C(\alpha - a_n) > (\beta - b_n)$, viewed in $[0, 1]$.
2. $(a_n : n \geq 1)$ is *universal* if it dominates every such recursive sequence.

RE: Complexity

Definition 13 α is Ω-like if it is the limit of a universal recursive sequence.

Theorem 14 (Solovay) *If α is Ω-like then α is random.*

Theorem 15 (Calude, Hertling, Khoussainov and Wang) *If α is Ω-like then α is an Ω-number*

RE: Complexity

Theorem 16 (Kučera and Slaman) *If* α *is recursively enumerable and random then* α *is* Ω*-like and, hence, an* Ω*-number.*

The more structural theorem is that every recursively random real is Ω-like. This is the dynamic property which characterizes randomness among recursively enmerable reals.

RE: Measure

The measure theoretic treatment of effective randomness provides a second class of natural examples, the measures of the sets appearing universal Martin-Löf tests.

Theorem 17 on the next slide combines results of Kučera and Slaman and results of Merkle, Mihailović, and Slaman.

RE: Measure

Theorem 17 *Suppose that* $(\alpha_n : n \geq 1)$ *is a uniformly recursively enumerable sequence of reals such that for all* n, $\alpha_n \leq 1/2^n$. *The following are equivalent.*

1. *For each* n, α_n *is random.*
2. *There is a universal Martin-Löf test* $(A_n : n \geq 1)$ *such that for all* n, $\mu(A_n) = \alpha_n$.

Measures

Lebesque measure μ is the natural uniform measure on $[0, 1]$. But it is not the only finitely additive measure worth considering. In the last few slides, we consider the ways in which these aspects of randomness change as the underlying measure does.

In the following, μ^* will denote a finitely additive probability measure on $[0, 1]$.

Measures: Continuity

Definition 18 μ^* is *continuous* iff for all $\alpha^* \in \Sigma^*$, $\mu^*(\{\alpha^*\}) = 0$.

Remark 19 If μ^* is continuous and α^* is μ^*-random (in the sense of Martin-Löf), then there is a Martin-Löf random real α such that $\alpha^* \geq_{tt} \alpha$.

Measures: Continuity

Corollary 20 *If μ^* is continuous and α^* is μ^*-random, then there is a nondecreasing, nonconstant, recursive function g such that for all n, the string complexity of $\alpha^* \restriction g(n)$ is greater than or equal to n. In other words, the compressibility of α^* is bounded by a recursive function.*

Measures: Discontinuity

Finally, we consider discontinuous probability measures. In a some sense, if μ^* concentrates on α^* then α^* is μ^*-random. However, this only occurs trivially.

Remark 21 If μ^* is recursive and $\mu^*(\{\alpha^*\}) \neq 0$ then α^* is recursive.

Of course, initial segments of recursive reals have no string complexity.

Measures: Discontinuity

Theorem 22 (Reimann and Slaman) *There is a recursive probability measure μ^* and a nonrecursive real α^* such that the following conditions hold.*

1. *α^* is μ^*-random.*
2. *For every recursive function g, there is an n such that the string complexity of $\alpha^* \upharpoonright g(n)$ has string complexity less than or equal to n.*

The connection between measure theoretic and algorithmic randomness is a feature of continuity.