The Mathematics of Relative Definability

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Channeling Alan Turing

Alan Turing had the remarkably prescient insight that understanding the means by which we work with things can be as important as, or even equivalent to, understanding those things.

Applied beautifully as mathematics:

- Simple and unapologetic formulation
- Direct analysis, with sophistication as needed

Examples of Turing at Work

Famous:

Example

- Definition of computable, universal machine, and non-computability of first order validity
- Criteria for thinking machine

Less Famous:

Example

- Computable normal sequence, example taken from paper by Becher, Figueira, and Picchi
- Possible usefulness of a random source to reduce computation time

Recursion Theory

the hierarchy of definability and canonical models

Classifying the means to produce mathematical objects.

- Hierarchies of definability:
  - first order arithmetic
  - second order arithmetic
  - set theory
- Canonical models:
  - the natural numbers, with addition and multiplication, or equivalently the finite sets
  - the natural numbers with a collection of its subsets, such as recursive, arithmetic, hyperarithmetic
  - Gödel's universe of constructible sets and its generalizations to inner models for large cardinals

The Turing Degrees

The partial ordering of the Turing degrees is the standard algebraic representation of relative definability.

- A Turing degree is the equivalence class of a subset of ω under equi-computability
- The Turing degrees are ordered by relative computability

One can vary the sets being considered, as when considering the Turing degrees of the recursively enumerable sets, or vary the notion of relative definability, as when considering the spectrum from many-one degrees, truth-table, Turing, enumeration, arithmetic, hyperarithmetic, constructible, and so forth.

A Fundamental Question

Sacks, Martin, Kechris

Characterize those operations on real numbers which are invariant under equi-definability, such as the Turing jump $X \mapsto X'$ or the function mapping X to the set of reals which are arithmetically definable from X.

Excluding applications of the Axiom of Choice, all the known non-trivial examples come from notions of relative definability.

- Degree invariant functions from reals to reals come from universal sets.
- Degree invariant functions from reals X to sets of reals containing X come from closures under relative definability.

Degree Invariant Operations

Question (Sacks)

Is there an e such that the function $X \mapsto W_e^X$ satisfies the following conditions?

- *For all X, $X <_T W_e^X <_T X'$.*
- *For all X and Y, if $X \equiv_T Y$ then $W_e^X \equiv_T W_e^Y$.*

Theorem (Slaman and Steel)

If we replace the second condition to require that $X \mapsto W_e^X$ preserve $\geq_T$, then there is no such e.

Proof of Theorem

- Suppose that $X \mapsto W_e^X$ preserves $\geq_T$ and for all X, $W_e^X >_T X$.
- Then $W_e^\emptyset$ is not recursive.
- By the Posner-Robinson Theorem, there is a G such that $W_e^\emptyset + G \geq G'$.
- Note, $W_e^G \geq_T W_e^\emptyset$ by preservation of order and $W_e^G \geq_T G$ by assumption.
- Conclude, $W_e^G \equiv G'$.

Martin Measure

Definition

1. A *cone of reals* is a set $\{X : X \geq_T B\}$, for some base B.
2. A property P on the Turing degrees, D, *contains a cone* iff there is a cone of reals all of whose degrees satisfy P.

Theorem (Assuming the Axiom of Determinacy)

Suppose a set $A \subseteq 2^\omega$ is closed under $\equiv_T$. Then one of A or $2^\omega \setminus A$ contains a cone.

Under AD, the cone filter is a $\{0, 1\}$-valued measure.

Martin's Conjecture

For degree-invariant functions, we define order preserving on a cone, constant on a cone, and other notions, similarly. We define $F \geq_M G$ iff $F(X) \geq_T G(X)$ on a cone.

Conjecture (Martin)

Assume ZF+AD+DC.

I. *If F is degree invariant and not increasing on a cone, then F is constant on a cone.*
II. *$\leq_M$ is a prewellordering of the set of degree invariant functions which are increasing on a cone. Further, if f has $\leq_M$-rank α, then f' has $\leq_M$-rank $\alpha + 1$, where $f' : x \mapsto f(x)'$ for all x.*

Martin's Conjecture

Martin's Conjecture is a particular realization of the view that all notions of relative definability extending relative computability appear in the logical hierarchy based on first order quantification over the finite sets.

It is a recursion theoretic account of the apparent well-ordering among natural consistency strengths and among inner models of set theory.

A Digression

enter Borel equivalence relations

Definition

Suppose E and F are equivalence relations on Polish spaces X and Y.

- *E Borel reduces to F* iff there is a Borel function $f : X \to Y$ such that for all $x, y \in X$,

$$x \mathrel{E} y \iff f(x) \mathrel{F} f(y).$$

- We write $E \leq_B F$ to denote E Borel reduces to F. $\sim_B$ and $<_B$ are defined similarly.

Definition

A Borel equivalence relation is *countable* iff each of its equivalence classes is countable.

Kechris's Question

Question (Kechris)

Is $\equiv_T$ a universal countable Borel equivalence relation? That is, is every countable Borel equivalence relation Borel reducible to $\equiv_T$?

If yes, then there are many Turing degree invariant functions, including a Borel pairing function, and Martin's Conjecture must fail.

A Universal Example

Theorem (Slaman and Steel)

The equivalence relation given by equi-arithmetic definability is a universal countable Borel equivalence relation.

An Application

The recursion theoretic equivalence relations give perspective on the abstract structure of Borel equivalence relations.

Theorem (Marks)

If E is a universal countable equivalence relation on the Polish space X and $X = Y \sqcup Z$ is a partition of X into two Borel subsets, then either the restriction of E to Y is universal or the restriction of E to Z is universal.

A Question

Question (Following Sacks)

Is there an arithmetically invariant function F such that for all X, $X <_A F(X) <_A X^\omega$?

The Axiomatic Hierarchy

Parallel to the hierarchy of definability is the hierarchy of axiomatic theories, which formalize the basic properties of the canonical models.

- Axiomatic systems
  - Peano Arithmetic and its subsystems, such as $B\Sigma_n$, $I\Sigma_n$
  - Second Order Arithmetic and its subsystems, such as RCA_0, WKL_0, ACA_0, Π^1_1-CA_0
  - Zermelo-Fraenkel set theory, $0^\#$, measureables, supercompacts

These systems approximate the theories of their standard models.

Definability vs Provability

which tells us more about the nature of mathematical investigations?

The hierarchy of definability and the hierarchy of axiom systems within set theory are parallel attempts to quantitatively and systematically describe the ingredients of mathematical investigations.

They provide alternate means to answer questions like the following.

- Whether there is an object, such as a real number, which can be produced using methods, principles, techniques of Type A and which satisfies Property B
- Whether principles of Type A and be used to settle questions of Type B

A Recursion Theorist's Assumption

Both sorts of questions can be formulated and settled by directly considering the nature of "Type A," as Turing did with the nature of computation, with minimal reliance on the formalization of theories.

Reverse Mathematics

Reverse Mathematics is the program set forth to investigate the following question.

Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?

The reverse mathematical approach to the problem is two-fold.

- Frame the question in formal subsystems of second-order arithmetic.
- Locate the theorems of ordinary, non-set-theoretic mathematics within these subsystems. In a substantial number of cases, the theorems are formally equivalent to the subsystem in which they can be proven over a weaker subsystem (RCA_0).

Reverse Math vs. Recursion Theorist's Assumption

Caveat: there may not be universal agreement with this view

The typical reverse mathematics question asks whether RCA_0 can prove

$$\forall X_1 \exists Y_1 \psi_1 \to \forall X_2 \exists Y_2 \psi_2,$$

where ψ_1 and ψ_2 are arithmetic formulas. Typically, $\forall X_1 \exists Y_1 \psi_1$ is formulated in the language of ordinary, non-set-theoretic mathematics and $\forall X_2 \exists Y_2 \psi_2$ is formulated as a principle of logic.

Understanding $\forall X_1 \exists Y_1 \psi_1 \to \forall X_2 \exists Y_2 \psi_2$

Caveat: there may not be universal agreement with this view

The typical solution to a reverse mathematics question is one of two types.

- (Reversal) Show that for every X_2 there is an X_1, such that for any Y_1 satisfying $\psi_1(X_1, Y_1)$ there is a Y_2 recursive in Y_1 for which $\psi_2(X_2, Y_2)$. Typically, the proof is by translation and can be formalized in RCA_0, after the fact.
- (Non-reversal) Show that there is an ideal I in the Turing degrees as follows.
  - For every $X_1 \in I$ there is a $Y_1 \in I$ such that $\psi_1(X_1, Y_1)$
  - There is an $X_2 \in I$ such that for all $Y_2 \in I$, $\neg\psi_2(X_2, Y_2)$. Typically, X_2 is recursive.

If one only works with ω-models, then the formalism is not needed.

Challenging the Recursion Theorist's Assumption

Question

What are the finitary/number theoretic consequences of infinitary principles?

Here, one can ask about principles such as the existence of an infinite random source, infinite combinatorial principles such as Ramsey's Theorem, or set theoretic principles such as the existence of infinitely many cardinals or large cardinals.

Definability Theoretic Thinking in Non-ω-models

Our understanding of the fundamentals of definability applies perfectly well in non-standard models.

- Understanding the jump in the Jockusch-Soare Low Basis Theorem applies to conclude Harrington's Theorem that WKL_0 is conservative over RCA_0 for Π^1_1-statements.
- Understanding the double jump in Ramsey's Theorem for Pairs, applies to conclude Cholak-Jockusch-Slaman's theorem that RT^2_2 is conservative over $RCA_0 + I\Sigma_2$ for Π^1_1-statements.

Missing Ingredients

We have two powerful tools with which to analyze relative definability.

- The hierarchy of definability, based on the Turing jump.
- Forcing, interpreted broadly to include priority constructions and other effective implementations.

To have a widely applicable technology to answer questions about infinite/finite, we need a third ability.

We need tools to fine-tune the underlying structure of arithmetic in coordination with the tools that we already have.

A Final Thought

I was once asked (forced) to write about the long-term goals of recursion theory.

> *. . . we are all trying to understand the interaction between the mathematical objects and the means needed to speak about them. I am fascinated by this enterprise, which I find as fundamental as any other mathematical investigation.*

No one's contribution to this investigation is more fundamental than Alan Turing's.