Turing Degrees and Definability of the Jump

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Outline

- Lecture 3
  - Effective bounds on the values of π
  - Invariance of the double-jump
  - Arithmetic representations of automorphisms
- Lecture 4
  - Interpreting $Aut(\mathfrak{D})$ within $\mathfrak{D}$
  - Join theorem for the double-jump
  - Definability of the jump

Interpreting $Aut(\mathfrak{D})$ within $\mathfrak{D}$

Definition

An assignment of reals consists of

- A countable ω-model $\mathfrak{M}$ of T ($T = \Sigma_1$-ZFC).
- A function f and a countable ideal $\mathcal{I}$ in $\mathfrak{D}$ such that $f : \mathfrak{D}^{\mathfrak{M}} \to \mathcal{I}$ surjectively and for all x and y in $\mathfrak{D}^{\mathfrak{M}}$, $\mathfrak{M} \models x \geq_T y$ if and only if $f(x) \geq_T f(y)$ in $\mathcal{I}$.

Definition

For assignments $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ and $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$, $(\mathfrak{M}_1, f_1, \mathcal{I}_1)$ extends $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ if and only if

- $\mathfrak{D}^{\mathfrak{M}_0} \subseteq \mathfrak{D}^{\mathfrak{M}_1}$,
- $\mathcal{I}_0 \subseteq \mathcal{I}_1$,
- and $f_1 \restriction \mathfrak{D}^{\mathfrak{M}_0} = f_0$.

Interpreting $Aut(\mathfrak{D})$ within $\mathfrak{D}$

Definition

An assignment $(\mathfrak{M}_0, f_0, \mathcal{I}_0)$ is extendable if

$$
\begin{array}{l}
\forall z_1 \exists (\mathfrak{M}_1, f_1, \mathcal{I}_1) \\
\quad \left[\begin{array}{l}
(\mathfrak{M}_1, f_1, \mathcal{I}_1) \text{ extends } (\mathfrak{M}_0, f_0, \mathcal{I}_0)),\ z_1 \in \mathcal{I}_1, \text{ and} \\
\quad \forall z_2 \exists (\mathfrak{M}_2, f_2, \mathcal{I}_2) \\
\qquad \left(\begin{array}{l}
(\mathfrak{M}_2, f_2, \mathcal{I}_2) \text{ extends } (\mathfrak{M}_1, f_1, \mathcal{I}_1),\ z_2 \in \mathcal{I}_2, \text{ and} \\
\quad \forall z_3 \exists (\mathfrak{M}_3, f_3, \mathcal{I}_3) \left[\begin{array}{c}
(\mathfrak{M}_3, f_3, \mathcal{I}_3) \text{ extends} \\
(\mathfrak{M}_2, f_2, \mathcal{I}_2) \text{ and } z_3 \in \mathcal{I}_3
\end{array} \right]
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\end{array}
$$

Interpreting $Aut(\mathfrak{D})$ within $\mathfrak{D}$

extendable assignments

Theorem

If $(\mathfrak{M}, f, \mathcal{I})$ *is an extendable assignment, then there is a* $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$ *such that for all* $x \in \mathfrak{D}^{\mathfrak{M}}$, $\pi(x) = f(x)$.

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Proof

We chase the inclusions between the Turing degrees of the domain models and the range ideals. Sets coded in the range $\mathcal{I}$ belong to the domain $\mathfrak{M}$. Sets in the range which together with $0'$ can only code elements of $\mathfrak{M}$ must belong to $\mathfrak{M}$.

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We conclude that if $(\mathfrak{M}, f, \mathcal{I})$ is an extendable assignment, then $f : \mathfrak{D}^{\mathfrak{M}} \to \mathcal{I}$ extends to a persistent automorphism of a larger ideal. Hence, it extends to an automorphism of $\mathfrak{D}$. □

Defining relative to parameters

Theorem

If g is the Turing degree of an arithmetically definable 5-generic set, then the relation $R(\overrightarrow{c}, d)$ given by

$$R(\overrightarrow{c}, d) \iff \overrightarrow{c} \text{ codes a real } D \text{ and } D \text{ has degree } d$$

is definable in $\mathfrak{D}$ from g.

This is the internal realization of the previous result that every automorphism is determined by its action on g.

Proof

Use following property of $\vec{c}$ and d. There are $\vec{m}$, $\vec{f}$, and $\vec{i}$ such that all of the following conditions are satisfied.

- $\vec{c}$ codes $\mathbb{N}$ with a unary predicate for a set D;
- $\vec{m}$ codes an ω-model $\mathfrak{M}$ of T;
- $\vec{i}$ codes a countable ideal $\mathcal{I}$ in $\mathfrak{D}$;
- $\vec{f}$ codes a function f from $\mathfrak{D}^{\mathfrak{M}}$ onto $\mathcal{I}$;
- $(\mathfrak{M}, f, \mathcal{I})$ is an extendable assignment;
- $g \in \mathcal{I}$, $\mathit{degree}(G)^{\mathfrak{M}}$ is the Turing degree of G as identified in $\mathfrak{M}$ by G's arithmetic definition, and $f(\mathit{degree}(G)^{\mathfrak{M}}) = g$;
- the set D coded by $\vec{c}$ is an element of $\mathfrak{M}$, $\mathit{degree}(D)^{\mathfrak{M}}$ is the Turing degree of D as defined in $\mathfrak{M}$, and $f(\mathit{degree}(D)^{\mathfrak{M}}) = d$.

□

Invariance and Definability

Theorem

Suppose that R is a relation on $\mathfrak{D}$. The following conditions are equivalent.

- *R is induced by a projective, degree invariant relation R_{2^ω} on 2^ω.*
- *R is definable in $\mathfrak{D}$ using parameters.*

Proof

$\overrightarrow{x}$ satisfies R if and only if there is an extendible assignment such that $f(degree(\overrightarrow{Y})) = \overrightarrow{x}$ and $\overrightarrow{Y}$ satisfies R_{2^ω}. □

Definability of the double-jump

Theorem

The function $x \mapsto x''$ is definable in $\mathfrak{D}$.

Proof

We have already shown that the relation $y = x''$ is invariant under all automorphisms of $\mathfrak{D}$. It is clearly degree invariant and definable in second order arithmetic. Therefore, it is definable in $\mathfrak{D}$. □

Biinterpretability

Definition

$\mathfrak{D}$ is biinterpretable with second order arithmetic if and only if the relation on $\overrightarrow{c}$ and d given by

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Conjecture

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We begin by showing that $\mathcal{I}(\Delta^0_2)$, the ideal of Δ^0_2 degrees, is definable in $\mathfrak{D}$. Our definition is based on the following Join Theorem for the Double-Jump.

Theorem (Shore and Slaman, 1999)

For $A \in 2^\omega$, the following conditions are equivalent.

- *A is not recursive in $0'$.*
- *There is a $G \in 2^\omega$ such that $A \oplus G \geq_T G''$.*

So, $\mathcal{I}(\Delta^0_2)$ is definable in terms of order, join, and the double jump. Consequently, it is definable in $\mathfrak{D}$.

Join Theorem

The join theorem has precursors.

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Jockusch and Shore generalized the proof and extended the theorem to operators in the n-r.e. hierarchy.

Non-coding approach

Given A, we will build a functional Φ to satisfy

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Note, Φ is a collection of elements (x, y, σ); so it makes sense to take its jump.
Our problem will be to determine whether $n \in \Phi''$ without deciding the value of $\Phi(n, A)$.

Kumabe-Slaman Forcing

The following notion of forcing is due to Kumabe and Slaman. We use it to construct a generic functional Φ_G

Definition

Let P be the following partial order.

- The elements p of P are pairs $(\Phi_p, \overrightarrow{X}_p)$ in which Φ_p is a finite use-monotone Turing functional and $\overrightarrow{X}_p$ is a finite collection of subsets of ω.
- If p and q are elements of P, then $p \geq q$ if and only if
  - $\Phi_p \subseteq \Phi_q$ and for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
  - $\overrightarrow{X}_p \subseteq \overrightarrow{X}_q$,
  - for every x, y, and $X \in \overrightarrow{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

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Kumabe-Slaman Forcing

the single jump

Consider this forcing and the Posner-Robinson Theorem.

Suppose that $p = (\Phi_p, \overrightarrow{X}_p)$ is a condition such that $A \not\in \overrightarrow{X}_p$ and n is the least number not in the domain of $\Phi_p(A)$.

Produce a condition q stronger than p such that the following conditions hold.

- q forces that $\Phi'_G(n) = i$
- $\Phi_q(n, A) = i$ and n is the only number added to the domain of $\Phi_p(A)$ by Φ_q
- $A \not\in \overrightarrow{X}_p$

Kumabe-Slaman Forcing

The easy case occurs when there is a Φ_q extending Φ_p such that

- $n \in \Phi'_G$ based on a witness verified by Φ_q,
- for each $X \in \overrightarrow{X}_p$, $\Phi_q(X) = \Phi_p(X)$,
- and $\Phi_q(A) = \Phi_p(A)$.

Since $A \notin \overrightarrow{X}_p$, we can extend Φ_q to Φ_r so that $\Phi_r(n, A) = 1$ and for each $X \in \overrightarrow{X}_p$, $\Phi_r(X) = \Phi_p(X)$. Then, $r = (\Phi_r, \overrightarrow{X}_p)$ is the desired extension of p.

Otherwise, for every Φ_q extending Φ_p, if $n \in \Phi'_G$ based on a witness verified by Φ_q then Φ_q adds a computation relative to A or to an element of $\overrightarrow{X}_p$.

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Definition

Suppose $n \in \omega$ and Φ_p is a finite use-monotone Turing functional. For $\overrightarrow{\tau} = (\tau_1, \ldots, \tau_k)$ a sequence of elements of $2^{<\omega}$ all of the same length, we say that $\overrightarrow{\tau}$ is essential to $n \in \Phi'_G$ over Φ_p iff for all Φ_q extending Φ_p, if $n \in \Phi'_G$ based on Φ_q, then $\Phi_q \setminus \Phi_p$ includes a triple (x, y, σ) such that σ is compatible with at least one component of $\overrightarrow{\tau}$.

Kumabe-Slaman Forcing

the harder case

For each k, the set of essential sequences of length k forms a Π^0_1-tree T. (T can be replaced by a recursive tree.)

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But then, since A is not recursive, a theorem of Jockusch and Soare applies and we can conclude to there is another infinite path $\overrightarrow{Y}$ in that T_k such that $A \notin \overrightarrow{Y}$.

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Then, $q = (\Phi_p, \overrightarrow{X}_p \cup \overrightarrow{Y})$ forces $n \notin \Phi'_G$. We can then obtain the desired extension of p by extending q to r so as to define $\Phi_r(n, A) = 0$.

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- Relate the forcing relation for Π^0_1-sentences to the unboundedness of recursive trees of essential sequences.
- Show that being essential to forcing a Σ^0_2 sentence is Π^0_2.
- Apply the previous argument using the hypothesis that A is not Δ^0_2.

Defining the Jump

Theorem

The functions $\mathbf{a} \mapsto \mathcal{I}(\Delta^0_2(\mathbf{a}))$ *and* $\mathbf{a} \mapsto \mathbf{a}'$ *are definable in* $\mathfrak{D}$.

Proof

By relativizing the previous theorem. For each degree $\mathbf{a}$ and each $\mathbf{d}$ greater than or equal to $\mathbf{a}$, $\mathbf{d}$ is not Δ^0_2 relative to $\mathbf{a}$ if and only if there is an $\mathbf{x}$ greater than or equal to $\mathbf{a}$ such that $\mathbf{d} + \mathbf{x} \geq_T \mathbf{x}''$. Again, the double jump is definable in $\mathfrak{D}$, and this equivalence provides first order definitions as required. □

Defining Recursively Enumerable?

Question

Is the relation y is recursively enumerable relative to x definable in $\mathfrak{D}$?

A positive answer would follow from a proof of the Biinterpretability Conjecture.

Finis