Turing Degrees and Definability of the Jump

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Outline

▶ Lecture 2
  ▶ Absoluteness
  ▶ Generic Persistence
  ▶ Definable representations of automorphisms

▶ Lecture 3
  ▶ Effective bounds on the values of $\pi$
  ▶ Invariance of the double-jump
  ▶ Arithmetic representations of automorphisms
Evaluating relative to a generic degree

**Theorem**

There is a family of dense open sets $\mathcal{D}$ and an injective continuous function $F(G, X)$ such that for all $\mathcal{D}$-generic $G$, if $\Pi(G)$ is a representative of $\pi(\text{degree}(G))$, then

$$(\forall X)[\text{degree}(F(G, X) \oplus \Pi(G)) = \pi(\text{degree}(X \oplus G))]$$

**Definition**

For $F$ as defined above, let $F_G$ be the function $X \mapsto F(G, X)$.
Theorem

There is a countable family of dense open sets $\mathcal{D}$ such that for all $\mathcal{D}$-generic reals $G$ the following conditions hold.

1. If $P$ is a perfect set with tree $T_P$, then the range of $F_G$ on $P$ contains a perfect set $Q$ with tree $T_Q$ such that

$$\deg(T_Q) \leq_T \pi(\deg(G) \lor \deg(T_P)).$$

2. If $Q$ is a perfect subset of the range of $F_G$ with tree $T_Q$, then there is a perfect set $P$ contained in the range of $F_G^{-1}$ applied to $Q$ with tree $T_P$ such that

$$\deg(T_P) \leq_T (\deg(G) \lor \pi^{-1}(\deg(T_Q))).$$
Proof

Let $\overrightarrow{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\overrightarrow{D}$-generic $G$ and perfect tree $T_P$. 

$=$

Let $\overrightarrow{T}_Q$ be the tree $f_{\overrightarrow{e}}$ $g_{\overrightarrow{D}}(\overrightarrow{T}_P)(G)$, for $2_{_{2}}<!$. $\overrightarrow{T}_Q$ is the desired perfect tree.
Proof

Let $\overrightarrow{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\overrightarrow{D}$-generic $G$ and perfect tree $T_P$.

For all $H$, $\Pi^{-1}(H) \oplus T_P$ is represented by the join of $T_P$ with a path $X_{\Pi^{-1}(H)}$ in $[T_P]$.

Let $T_Q$ be the tree $f_{\overrightarrow{e}^g(H)}(T_P(G))$. $T_Q$ is the desired perfect tree.
Proof

- Let \( \overrightarrow{D} \) be the countable family of dense open sets of the previous theorem. Fix a \( \overrightarrow{D} \)-generic \( G \) and perfect tree \( T_P \).
- For all \( H \), \( \Pi^{-1}(H) \oplus T_P \) is represented by the join of \( T_P \) with a path \( X_{\Pi^{-1}(H)} \) in \([T_P]\).
- Now, use the representation of \( \Pi \) relative to \( G \). There is an \( e \) such that for generic \( H \),

\[
\{e\}(H \oplus \Pi(T_P) \oplus \Pi(G)) = F_G(X_{\Pi^{-1}(H)}) \oplus \Pi(T_P) \oplus \Pi(G)
\]
Let $\mathcal{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\mathcal{D}$-generic $G$ and perfect tree $T_P$.

For all $H$, $\Pi^{-1}(H) \oplus T_P$ is represented by the join of $T_P$ with a path $X_{\Pi^{-1}(H)}$ in $[T_P]$.

Now, use the representation of $\Pi$ relative to $G$. There is an $e$ such that for generic $H$,

$$\{e\}(H \oplus \Pi(T_P) \oplus \Pi(G)) = F_G(X_{\Pi^{-1}(H)}) \oplus \Pi(T_P) \oplus \Pi(G)$$

Let $T_Q$ be the tree $\{e\}(\sigma \oplus \Pi(T_P) \oplus \Pi(G))$, for $\sigma \in 2^{<\omega}$. $T_Q$ is the desired perfect tree.
Theorem

For every $Z \subseteq \omega$, there is a countable family of dense open sets $\mathcal{D}$ such that for all $\mathcal{D}$-generic reals $G$,

$$
\pi(\text{degree}(Z \oplus G))'' \geq_T \text{degree}(Z)
$$

We use a countable forcing to efficiently code $Z$ into the Turing degrees. We argue that there is are generics for this forcing which are recursive in $Z \oplus G$. (Recall the universal property of Cohen forcing.)

We view $G$ as a quadruple of mutually generic reals $G_1, G_2, G_3,$ and $G_4$. 

Coding $Z$ into $\Pi(Z) \oplus \Pi(G)$
1. coding the integers

Fix a perfect binary tree $T_1$ such that $T_1$ is recursive in $G_1$ and any finite set of infinite paths in $T_1$ consists of reals which are mutually generic relative to $G_2 \oplus G_3 \oplus G_4$. Let $P_1$ be the perfect set of infinite paths in $T_1$. 
1. coding the integers

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- Let $F_{G_2}$ be the injective continuous function associated with $G_2$. Fix perfect sets $Q$ and $P_2$ as in the previous theorem.
Coding $\mathbb{Z}$ into $\Pi(\mathbb{Z}) \oplus \Pi(G)$

1. coding the integers

- Fix a perfect binary tree $T_1$ such that $T_1$ is recursive in $G_1$ and any finite set of infinite paths in $T_1$ consists of reals which are mutually generic relative to $G_2 \oplus G_3 \oplus G_4$. Let $P_1$ be the perfect set of infinite paths in $T_1$.

- Let $F_{G_2}$ be the injective continuous function associated with $G_2$. Fix perfect sets $Q$ and $P_2$ as in the previous theorem.

- Let $(H_j : j \in \omega)$ be the sequence of elements of $P_2$ given by the leftmost branches in $P_2$ off of the rightmost branch of $P_2$. We note that any finite subset of $(H_j : j \in \omega)$ consists of reals which are mutually Cohen generic relative to $G_2 \oplus G_3 \oplus G_4$. Note, $(H_j : j \in \omega) \leq_T G_1 \oplus G_2$. 
We find $A_0$ and $A_1$ recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following properties for all $U$ and for all $j \in \omega$.

\[(U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \rightarrow [A_0 \oplus G_2 \oplus H_{2j} \geq U \iff (U = H_{2j} \text{ or } U = H_{2j+1})] \]

\[(U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \rightarrow [A_1 \oplus G_2 \oplus H_{2j+1} \geq U \iff (U = H_{2j+1} \text{ or } U = H_{2j+2})] \]
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(U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \rightarrow \\
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\]

The successor operation is obtained by recursion using positive instances of $\leq_T$ and various simpler properties.
Coding $Z$ into $\Pi(Z) \oplus \Pi(G)$

3. coding $Z$

We find $B_0$ and $B_1$ recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following properties for all $j \in \omega$.

\[
B_0 \oplus G_2 \geq_T H_j \iff j \not\in Z
\]

\[
B_1 \oplus G_2 \geq_T H_j \iff j \in Z
\]
The sequence \( (F_{G_2}(H_j)) : j \in \omega \) can be generated in order recursively in \( \Pi(G)'' \). For example, \( F_{G_2}(H_1) \) is the unique set such that the following conditions hold.

- \( F_{G_2}(H_1) \in Q \)
- \( F_{G_2}(H_1) \leq_T \Pi(G_1) \oplus \Pi(G_2) \)
- \( F_{G_2}(H_1) \leq_T A_0 \oplus \Pi(G_2) \)
The sequence \((F_{G_2}(H_j) : j \in \omega)\) can be generated in order recursively in \(\Pi(G)''\). For example, \(F_{G_2}(H_1)\) is the unique set such that the following conditions hold.

- \(F_{G_2}(H_1) \in Q\)
- \(F_{G_2}(H_1) \leq_T \Pi(G_1) \oplus \Pi(G_2)\)
- \(F_{G_2}(H_1) \leq_T A_0 \oplus \Pi(G_2)\)

\(Z\) is recursive in \((\Pi(Z) \oplus \Pi(G))''\) by

\[
\begin{align*}
\Pi(B_0) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) &\iff j \notin Z \\
\Pi(B_1) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) &\iff j \in Z
\end{align*}
\]
Bounding $\pi(z)$ by $z''$

**Theorem**

*For every $z \in \mathcal{D}$, $z'' \geq_T \pi(z)$.*
Bounding $\pi(z)$ by $z''$

**Theorem**

For every $z \in \mathcal{D}$, $z'' \geq_T \pi(z)$.

**Proof**

- Fix $Z$ and fix $\vec{D}$ as in the previous theorem relative to $\Pi(Z)$ and the automorphism $\pi^{-1}$. 
Bounding $\pi(z)$ by $z''$

**Theorem**

For every $z \in \mathcal{D}$, $z'' \geq_T \pi(z)$.

**Proof**

- Fix $Z$ and fix $\vec{D}$ as in the previous theorem relative to $\Pi(Z)$ and the automorphism $\pi^{-1}$.
- Fix $\vec{D}^*$ so that if $G^*$ is $\vec{D}^*$-generic, then $\Pi(G^*)$ computes a $\vec{D}$-generic.
Bounding $\pi(z)$ by $z''$

**Theorem**

*For every $z \in \mathcal{D}$, $z'' \geq_T \pi(z)$.*

**Proof**

- Fix $Z$ and fix $\overrightarrow{D}$ as in the previous theorem relative to $\Pi(Z)$ and the automorphism $\pi^{-1}$.
- Fix $\overrightarrow{D}^*$ so that if $G^*$ is $\overrightarrow{D}^*$-generic, then $\Pi(G^*)$ computes a $\overrightarrow{D}$-generic.
- Let $G^*$ be $\overrightarrow{D}^*$-generic and let $G$ be a $\overrightarrow{D}$-generic recursive in $\Pi(G^*)$. 

□
Bounding $\pi(z)$ by $z''$

**Theorem**

For every $z \in \mathcal{O}$, $z'' \geq_T \pi(z)$.

**Proof**

- Fix $\mathcal{Z}$ and fix $\mathcal{D}$ as in the previous theorem relative to $\Pi(\mathcal{Z})$ and the automorphism $\pi^{-1}$.
- Fix $\mathcal{D}^*$ so that if $G^*$ is $\mathcal{D}^*$-generic, then $\Pi(G^*)$ computes a $\mathcal{D}$-generic.
- Let $G^*$ be $\mathcal{D}^*$-generic and let $G$ be a $\mathcal{D}$-generic recursive in $\Pi(G^*)$.
- Conclude that $\Pi(\mathcal{Z}) \leq (\mathcal{Z} \oplus G)''$ and so $\Pi(\mathcal{Z}) \leq Z'' \oplus G$.  

$\square$
Bounding $\pi(z)$ by $z''$

**Theorem**

For every $z \in \mathcal{D}$, $z'' \geq_T \pi(z)$.

**Proof**

- Fix $Z$ and fix $\vec{D}$ as in the previous theorem relative to $
\Pi(Z)$ and the automorphism $\pi^{-1}$.
- Fix $\vec{D}^*$ so that if $G^*$ is $\vec{D}^*$-generic, then $\Pi(G^*)$ computes a $\vec{D}$-generic.
- Let $G^*$ be $\vec{D}^*$-generic and let $G$ be a $\vec{D}$-generic recursive in $\Pi(G^*)$.
- Conclude that $\Pi(Z) \leq (Z \oplus G)''$ and so $\Pi(Z) \leq Z'' \oplus G$.
- $G$ was any generic, so $\Pi(Z) \leq Z''$. 

The cone above $0''$

**Corollary**

For any 2-generic set $G$,

$$\text{degree}(G) \vee 0'' \geq_T \pi(\text{degree}(G)).$$

**Theorem**

Suppose that $\pi : \mathcal{D} \overset{\sim}{\rightarrow} \mathcal{D}$.

- For all $x \in \mathcal{D}$, $x \vee 0'' \geq_T \pi(x)$.
- For all $x \in \mathcal{D}$, if $x \geq 0''$ then $x = \pi(x)$.

**Proof**

Degrees above $0''$ can be written as a join of 2-generic degrees.
## Coding $Z''$ into $\Pi(Z) \oplus \Pi(G)$

### Theorem

For every $Z \subseteq \omega$, there is a countable family of dense open sets $\vec{D}$ such that for all $\vec{D}$-generic $G$,\
$\pi(\text{degree}(Z \oplus G))'' \geq_T \text{degree}(Z'')$

### Proof

- Use $(H_j : j \in \omega)$ for $\mathbb{N}$ and $A_0$ and $A_1$ for the successor.
Coding $Z''$ into $\Pi(Z) \oplus \Pi(G)$

**Theorem**

For every $Z \subseteq \omega$, there is a countable family of dense open sets $\overrightarrow{D}$ such that for all $\overrightarrow{D}$-generic $G$, $\pi(\text{degree}(Z \oplus G))'' \geq_T \text{degree}(Z'')$

**Proof**

- Use $(H_j : j \in \omega)$ for $\mathbb{N}$ and $A_0$ and $A_1$ for the successor.
- Find $B_0$ and $B_1$ to represent $Z''$:

  \[
  \Pi(B_0) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \notin Z'' \\
  \Pi(B_1) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \in Z''
  \]

  Modulate the potential codings of $H_j$ into $B_0$ and $B_1$ using the Skolem functions for $j \in Z''$. 
Invariance of the double-jump

Theorem

Suppose that $\pi : \mathcal{D} \to \mathcal{D}$. For all $z \in \mathcal{D}$, $z'' = \pi(z)''.

Proof

- Fix $Z$ and a sufficiently generic $G^*$ so that $\Pi^{-1}(G^*)$ bounds a $G$ which is itself sufficiently generic for the previous theorem to apply.
Theorem

Suppose that \( \pi : \mathcal{D} \sim \to \mathcal{D} \). For all \( z \in \mathcal{D} \), \( z'' = \pi(z)'' \).

Proof

- Fix \( Z \) and a sufficiently generic \( G^* \) so that \( \Pi^{-1}(G^*) \) bounds a \( G \) which is itself sufficiently generic for the previous theorem to apply.
- Then, \( \Pi(Z \oplus G)'' \geq_T Z'' \).
Invariance of the double-jump

**Theorem**

Suppose that \( \pi : \mathcal{D} \sim \mathcal{D} \). For all \( z \in \mathcal{D} \), \( z'' = \pi(z)'' \).

**Proof**

- Fix \( Z \) and a sufficiently generic \( G^* \) so that \( \Pi^{-1}(G^*) \) bounds a \( G \) which is itself sufficiently generic for the previous theorem to apply.
- Then, \( \Pi(Z \oplus G)'' \preceq_T \Pi(Z)'' \).
- Observe, \( \Pi(Z \oplus G)'' \preceq_T \Pi(Z)'' \oplus G^* \).
Invariance of the double-jump

Theorem

Suppose that $\pi : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$. For all $z \in \mathcal{D}$, $z'' = \pi(z)''$.

Proof

- Fix $Z$ and a sufficiently generic $G^*$ so that $\Pi^{-1}(G^*)$ bounds a $G$ which is itself sufficiently generic for the previous theorem to apply.
- Then, $\Pi(Z \oplus G)'' \succeq_T Z''$.
- Observe, $\Pi(Z \oplus G)'' \preceq_T (\Pi(Z) \oplus G^*)'' \preceq_T \Pi(Z)'' \oplus G^*$.
- Since $G^*$ was any sufficiently generic, $\Pi(Z)'' \succeq_T Z''$. Similarly, $Z'' \succeq_T \Pi(Z)''$.
Theorem

The relation $y = x''$ is invariant under $\pi$.

Proof

Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$. 
Invariance of the double-jump

Theorem

The relation $y = x''$ is invariant under $\pi$.

Proof

Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$.
- By the previous theorem, $x'' = \pi(x)''$. 
Invariance of the double-jump

Theorem

The relation $y = x''$ is invariant under $\pi$.

Proof

Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$.
- By the previous theorem, $x'' = \pi(x)''$.
- Consequently, $\pi(y) = \pi(x)''$. 
Invariance of the double-jump

**Theorem**

The relation $y = x''$ is invariant under $\pi$.

**Proof**

Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$.
- By the previous theorem, $x'' = \pi(x)''$.
- Consequently, $\pi(y) = \pi(x)''$.

By the same argument, if $\pi(y) = \pi(x)''$ then $y = x''$. 

□
Representing $Aut(\mathfrak{D})$ by arithmetic functions

**Theorem**

Suppose that $\pi: \mathfrak{D} \sim \mathfrak{D}$.

- There is a recursive function $\{e\}(X,Y)$ such that for all $G$, if $G$ is 5-generic, then $\pi(\text{degree}(G))$ is represented by $\{e\}(G \oplus \emptyset'')$.

- There is an arithmetic function $F: 2^\omega \rightarrow 2^\omega$ such that for all $X \in 2^\omega$, $\pi(\text{degree}(X))$ is represented by $F(X)$.

**Proof**

Replay the proof that $\pi$ is continuously represented using the new information that $\pi(\text{degree}(G)) \leq G''$. There is a fixed reduction which works for all 5-generic $G$'s.

Since the 5-generics generate $\mathfrak{D}$, the representation on 5-generic propagates to a representation everywhere.
Consequences

Theorem

\( \text{AUT}(\mathcal{D}) \) is countable.
Consequences

**Theorem**

\ \textit{\textit{AUT}(D) is countable.}

**Theorem**

\textit{If g is 5-generic and }\pi : D \sim D, \textit{ then }\pi \textit{ is determined by its action on } g.