Turing Degrees and Definability of the Jump

Theodore A. Slaman

University of California, Berkeley

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Outline

- Lecture 2
  - Absoluteness
  - Generic Persistence
  - Definable representations of automorphisms
- Lecture 3
  - Effective bounds on the values of π
  - Invariance of the double-jump
  - Arithmetic representations of automorphisms

Evaluating relative to a generic degree

Theorem

There is a family of dense open sets $\overrightarrow{D}$ and an injective continuous function $F(G, X)$ such that for all $\overrightarrow{D}$-generic G, if $\Pi(G)$ is a representative of $\pi(degree(G))$, then

$$(\forall X)[degree(F(G, X) \oplus \Pi(G)) = \pi(degree(X \oplus G))]$$

Definition

For F as defined above, let F_G be the function $X \mapsto F(G, X)$.

Evaluating relative to a generic degree

transferring perfect sets

Theorem

There is a countable family of dense open sets $\vec{D}$ such that for all $\vec{D}$-generic reals G the following conditions hold.

- *If P is a perfect set with tree T_P, then the range of F_G on P contains a perfect set Q with tree T_Q such that*

$$degree(T_Q) \leq_T \pi(degree(G) \vee degree(T_P)).$$

- *If Q is a perfect subset of the range of F_G with tree T_Q, then there is a perfect set P contained in the range of F_G^{-1} applied to Q with tree T_P such that*

$$degree(T_P) \leq_T (degree(G) \vee \pi^{-1}(degree(T_Q))).$$

Proof

- Let $\overrightarrow{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\overrightarrow{D}$-generic G and perfect tree T_P.

$\square$

Proof

- Let $\overrightarrow{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\overrightarrow{D}$-generic G and perfect tree T_P.
- For all H, $\Pi^{-1}(H) \oplus T_P$ is represented by the join of T_P with a path $X_{\Pi^{-1}(H)}$ in $[T_P]$.

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Proof

- Let $\overrightarrow{D}$ be the countable family of dense open sets of the previous theorem. Fix a $\overrightarrow{D}$-generic G and perfect tree T_P.
- For all H, $\Pi^{-1}(H) \oplus T_P$ is represented by the join of T_P with a path $X_{\Pi^{-1}(H)}$ in $[T_P]$.
- Now, use the representation of Π relative to G. There is an e such that for generic H,

$$\{e\}(H \oplus \Pi(T_P) \oplus \Pi(G)) = F_G(X_{\Pi^{-1}(H)})) \oplus \Pi(T_P) \oplus \Pi(G)$$

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Proof

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- For all H, $\Pi^{-1}(H) \oplus T_P$ is represented by the join of T_P with a path $X_{\Pi^{-1}(H)}$ in $[T_P]$.
- Now, use the representation of Π relative to G. There is an e such that for generic H,

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- Let T_Q be the tree $\{e\}(\sigma \oplus \Pi(T_P) \oplus \Pi(G))$, for $\sigma \in 2^{<\omega}$. T_Q is the desired perfect tree.

□

Coding Z into $\Pi(Z) \oplus \Pi(G)$

Theorem

For every $Z \subseteq \omega$, there is a countable family of dense open sets $\overrightarrow{D}$ such that for all $\overrightarrow{D}$-generic reals G,

$$\pi(\mathit{degree}(Z \oplus G))'' \geq_T \mathit{degree}(Z)$$

We use a countable forcing to efficiently code Z into the Turing degrees. We argue that there is are generics for this forcing which are recursive in $Z \oplus G$. (Recall the universal property of Cohen forcing.)

We view G as a quadruple of mutually generic reals G_1, G_2, G_3, and G_4.

Coding Z into $\Pi(Z) \oplus \Pi(G)$

1. coding the integers

- Fix a perfect binary tree T_1 such that T_1 is recursive in G_1 and any finite set of infinite paths in T_1 consists of reals which are mutually generic relative to $G_2 \oplus G_3 \oplus G_4$. Let P_1 be the perfect set of infinite paths in T_1.

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- Let F_{G_2} be the injective continuous function associated with G_2. Fix perfect sets Q and P_2 as in the previous theorem.

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- Let F_{G_2} be the injective continuous function associated with G_2. Fix perfect sets Q and P_2 as in the previous theorem.
- Let $(H_j : j \in \omega)$ be the sequence of elements of P_2 given by the leftmost branches in P_2 off of the rightmost branch of P_2. We note that any finite subset of $(H_j : j \in \omega)$ consists of reals which are mutually Cohen generic relative to $G_2 \oplus G_3 \oplus G_4$. Note, $(H_j : j \in \omega) \leq_T G_1 \oplus G_2$.

Coding Z into $\Pi(Z) \oplus \Pi(G)$

2. coding the successor

- We find A_0 and A_1 recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following properties for all U and for all $j \in \omega$.

$$(U \in P_1 \text{ and } G_1 \oplus G_2 \geq_T U) \rightarrow$$
$$[A_0 \oplus G_2 \oplus H_{2j} \geq U \iff (U = H_{2j} \text{ or } U = H_{2j+1})]$$

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- The successor operation is obtained by recursion using positive instances of $\leq_T$ and various simpler properties.

Coding Z into $\Pi(Z) \oplus \Pi(G)$

3. coding Z

We find B_0 and B_1 recursively in $G_1 \oplus G_2 \oplus G_3$ to satisfy the following properties for all $j \in \omega$.

$$B_0 \oplus G_2 \geq_T H_j \iff j \notin Z$$
$$B_1 \oplus G_2 \geq_T H_j \iff j \in Z$$

Coding Z into $\Pi(Z) \oplus \Pi(G)$

4. decoding Z

- The sequence $(F_{G_2}(H_j) : j \in \omega)$ can be generated in order recursively in $\Pi(G)''$. For example, $F_{G_2}(H_1)$ is the unique set such that the following conditions hold.
  - $F_{G_2}(H_1) \in Q$
  - $F_{G_2}(H_1) \leq_T \Pi(G_1) \oplus \Pi(G_2)$
  - $F_{G_2}(H_1) \leq_T A_0 \oplus \Pi(G_2)$

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  - $F_{G_2}(H_1) \leq_T \Pi(G_1) \oplus \Pi(G_2)$
  - $F_{G_2}(H_1) \leq_T A_0 \oplus \Pi(G_2)$
- Z is recursive in $(\Pi(Z) \oplus \Pi(G))''$ by

$$\Pi(B_0) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \notin Z$$
$$\Pi(B_1) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \in Z$$

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For every $z \in \mathfrak{D}$, $z'' \geq_T \pi(z)$.

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Proof

- Fix Z and fix $\overrightarrow{D}$ as in the previous theorem relative to $\Pi(Z)$ and the automorphism π^{-1}.

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Proof

- Fix Z and fix $\overrightarrow{D}$ as in the previous theorem relative to $\Pi(Z)$ and the automorphism π^{-1}.
- Fix $\overrightarrow{D}^*$ so that if G^* is $\overrightarrow{D}^*$-generic, then $\Pi(G^*)$ computes a $\overrightarrow{D}$-generic.

$\square$

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- Fix $\overrightarrow{D}^*$ so that if G^* is $\overrightarrow{D}^*$-generic, then $\Pi(G^*)$ computes a $\overrightarrow{D}$-generic.
- Let G^* be $\overrightarrow{D}^*$-generic and let G be a $\overrightarrow{D}$-generic recursive in $\Pi(G^*)$.

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- Fix $\vec{D}^*$ so that if G^* is $\vec{D}^*$-generic, then $\Pi(G^*)$ computes a $\vec{D}$-generic.
- Let G^* be $\vec{D}^*$-generic and let G be a $\vec{D}$-generic recursive in $\Pi(G^*)$.
- Conclude that $\Pi(Z) \leq (Z \oplus G)''$ and so $\Pi(Z) \leq Z'' \oplus G$.

□

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- Fix $\vec{D}^*$ so that if G^* is $\vec{D}^*$-generic, then $\Pi(G^*)$ computes a $\vec{D}$-generic.
- Let G^* be $\vec{D}^*$-generic and let G be a $\vec{D}$-generic recursive in $\Pi(G^*)$.
- Conclude that $\Pi(Z) \leq (Z \oplus G)''$ and so $\Pi(Z) \leq Z'' \oplus G$.
- G was any generic, so $\Pi(Z) \leq Z''$.

□

The cone above $0''$

Corollary

For any 2-generic set G,

$$\mathit{degree}(G) \vee 0'' \geq_T \pi(\mathit{degree}(G)).$$

Theorem

Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$.

- *For all $x \in \mathfrak{D}$, $x \vee 0'' \geq_T \pi(x)$.*
- *For all $x \in \mathfrak{D}$, if $x \geq 0''$ then $x = \pi(x)$.*

Proof

Degrees above $0''$ can be written as a join of 2-generic degrees. □

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Theorem

For every $Z \subseteq \omega$, there is a countable family of dense open sets $\overrightarrow{D}$ such that such that for all $\overrightarrow{D}$-generic G, $\pi(\mathit{degree}(Z \oplus G))'' \geq_T \mathit{degree}(Z'')$

Proof

- Use $(H_j : j \in \omega)$ for $\mathbb{N}$ and A_0 and A_1 for the successor.

$\square$

Coding Z'' into $\Pi(Z) \oplus \Pi(G)$

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For every $Z \subseteq \omega$, there is a countable family of dense open sets $\vec{D}$ such that such that for all $\vec{D}$-generic G,
$\pi(\mathit{degree}(Z \oplus G))'' \geq_T \mathit{degree}(Z'')$

Proof

- Use $(H_j : j \in \omega)$ for $\mathbb{N}$ and A_0 and A_1 for the successor.
- Find B_0 and B_1 to represent Z'':

$$\Pi(B_0) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \notin Z''$$
$$\Pi(B_1) \oplus \Pi(G_2) \geq_T F_{G_2}(H_j) \iff j \in Z''$$

Modulate the potential codings of H_j into B_0 and B_1 using the Skolem functions for $j \in Z''$.

□

Invariance of the double-jump

Theorem

Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. *For all* $z \in \mathfrak{D}$, $z'' = \pi(z)''$.

Proof

- Fix Z and a sufficiently generic G^* so that $\Pi^{-1}(G^*)$ bounds a G which is itself sufficiently generic for the previous theorem to apply.

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Proof

- Fix Z and a sufficiently generic G^* so that $\Pi^{-1}(G^*)$ bounds a G which is itself sufficiently generic for the previous theorem to apply.
- Then, $\Pi(Z \oplus G)'' \geq_T Z''$.

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- Fix Z and a sufficiently generic G^* so that $\Pi^{-1}(G^*)$ bounds a G which is itself sufficiently generic for the previous theorem to apply.
- Then, $\Pi(Z \oplus G)'' \geq_T Z''$.
- Observe, $\Pi(Z \oplus G)'' \leq_T (\Pi(Z) \oplus G^*)'' \leq_T \Pi(Z)'' \oplus G^*$.

□

Invariance of the double-jump

Theorem

Suppose that $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$. For all $z \in \mathfrak{D}$, $z'' = \pi(z)''$.

Proof

- Fix Z and a sufficiently generic G^* so that $\Pi^{-1}(G^*)$ bounds a G which is itself sufficiently generic for the previous theorem to apply.
- Then, $\Pi(Z \oplus G)'' \geq_T Z''$.
- Observe, $\Pi(Z \oplus G)'' \leq_T (\Pi(Z) \oplus G^*)'' \leq_T \Pi(Z)'' \oplus G^*$.
- Since G^* was any sufficiently generic, $\Pi(Z)'' \geq_T Z''$. Similarly, $Z'' \geq_T \Pi(Z)''$

□

Invariance of the double-jump

Theorem

The relation $y = x''$ *is invariant under* π.

Proof

Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$.

$\square$

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The relation $y = x''$ *is invariant under* π.

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Suppose that $y = x''$.

- Since $y \geq_T 0''$, $\pi(y) = y$.
- By the previous theorem, $x'' = \pi(x)''$.

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- By the previous theorem, $x'' = \pi(x)''$.
- Consequently, $\pi(y) = \pi(x)''$.

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- Since $y \geq_T 0''$, $\pi(y) = y$.
- By the previous theorem, $x'' = \pi(x)''$.
- Consequently, $\pi(y) = \pi(x)''$.

By the same argument, if $\pi(y) = \pi(x)''$ then $y = x''$. □

Representing $Aut(\mathfrak{D})$ by arithmetic functions

Theorem

Suppose that $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$.

- *There is a recursive function* $\{e\}(X, Y)$ *such that for all* G, *if* G *is 5-generic, then* $\pi(degree(G))$ *is represented by* $\{e\}(G \oplus \emptyset'')$.
- *There is an arithmetic function* $F : 2^\omega \to 2^\omega$ *such that for all* $X \in 2^\omega$, $\pi(degree(X))$ *is represented by* $F(X)$.

Proof

Replay the proof that π is continuously represented using the new informaton that $\pi(degree(G)) \leq G''$. There is a fixed reduction which works for all 5-generic G's.

Since the 5-generics generate $\mathfrak{D}$, the representation on 5-generic propagates to a representation everywhere. $\square$

Consequences

Theorem

$AUT(\mathfrak{D})$ is countable.

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Theorem

If g is 5-generic and $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$, then π is determined by its action on g.