Turing Degrees and Definability of the Jump

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Outline

- Lecture 1
  - Forcing in arithmetic
  - Coding and decoding theorems
  - Automorphisms of countable ideals
  - Persistence
- Lecture 2
  - Absoluteness
  - Generic Persistence
  - Definable representations of automorphisms

Persistent Autormorphisms

review

Definition

An automorphism ρ of a countable ideal $\mathcal{I}$ is persistent if for every degree x there is a countable ideal $\mathcal{I}_1$ such that

- $x \in \mathcal{I}_1$ and $\mathcal{I} \subseteq \mathcal{I}_1$;
- there is an automorphism ρ_1 of $\mathcal{I}_1$ such that the restriction of ρ_1 to $\mathcal{I}$ is equal to ρ.

We have shown that persistent automorphisms are locally presented and that persistent automorphisms of ideals which contain $0'$ have persistent extensions. Thus, given a persistent automorphism $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$, a countable jump ideal $\mathcal{J}$ extending $\mathcal{I}$, and a presentation J of $\mathcal{J}$, we can find an extension of ρ to $\mathcal{J}$ which is arithmetic in J.

Persistent Autormorphisms

absoluteness

Theorem

The property I is a representation of a countable ideal $\mathcal{I}$, $0' \in \mathcal{I}$, and R is a presentation of a persistent automorphism ρ of $\mathcal{I}$ is Π^1_1.

Proof

ρ is persistent if and only if for every presentation J of a jump ideal $\mathcal{J}$ extending $\mathcal{I}$, there is an arithmetic in J extension of ρ to $\mathcal{J}$. This property is Π^1_1. □

Corollary

The properties R is a presentation of a persistent automorphism and There is a countable map $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$ such that $0' \in \mathcal{I}$, ρ is persistent and not equal to the identity are absolute between well-founded models of ZFC.

Persistent Autormorphisms
model theoretically

Let T be the fragment of ZFC in which we include only the instances of replacement and comprehension in which the defining formula is Σ_1.

Definition

Suppose that $\mathcal{M} = (M, \in^{\mathcal{M}})$ is a model of T.

1. $\mathcal{M}$ is an ω-model if $\mathbb{N}^{\mathcal{M}}$ is isomorphic to the standard model of arithmetic.
2. $\mathcal{M}$ is well-founded if the binary relation $\in^{\mathcal{M}}$ is well-founded. That is to say that there is no infinite sequence $(m_i : i \in \mathbb{N})$ of elements of $\mathcal{M}$ such that for all i, $m_{i+1} \in^{\mathcal{M}} m_i$.

Persistent Autormorphisms
model theoretically

Theorem

Suppose that $\mathcal{M}$ is an ω-model of T. Let $\mathcal{I}$ be an element of $\mathcal{M}$ such that

$$\mathcal{M} \models \mathcal{I} \textit{ is a countable ideal in } \mathfrak{D} \textit{ such that } 0' \in \mathcal{I}.$$

Then, every persistent automorphism of $\mathcal{I}$ is also an element of $\mathcal{M}$.

Proof

$\mathfrak{M}$ is closed under arithmetic definability. $\square$

Persistent Autormorphisms
model theoretically

Corollary

Suppose that $\mathcal{M}$ is an ω-model of T and that ρ and $\mathcal{I}$ are elements of $\mathcal{M}$ such that $0' \in \mathcal{I}$, $\rho : \mathcal{I} \tilde{\to} \mathcal{I}$, and $\mathcal{I}$ is countable in $\mathcal{M}$. Then,

$$\rho \textit{ is persistent} \Longrightarrow \mathcal{M} \models \rho \textit{ is persistent.}$$

Proof

Persistent automorphisms extend persistently. Hence, their extensions belong to $\mathfrak{M}$. □

Generic Persistence

We now extend the notion of persistence to uncountable ideals. In what follows, V is the universe of sets and G is a V-generic filter for some partial order in V.

Definition

Suppose that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ and ρ is an automorphism of $\mathcal{I}$. We say that ρ is generically persistent if there is a generic extension $V[G]$ of V in which $\mathcal{I}$ is countable and ρ is persistent.

Generic Persistence

invariance under partial order

Theorem

Suppose that $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$ is generically persistent. If $V[G]$ is a generic extension of V in which $\mathcal{I}$ is countable then ρ is persistent in $V[G]$.

Proof

Generics for any two forcings can be realized simultaneously. Apply absoluteness. $\square$

Applications to Aut($\mathfrak{D}$)

Theorem

Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. Then, π is generically persistent.

Proof

If not, then the failure of π to be generically persistent would reflect to a countable well-founded model $\mathfrak{M}$. Then, we could add a generic counting of $\mathfrak{D}^{\mathfrak{M}}$ to $\mathfrak{M}$ to obtain $\mathfrak{M}[G]$ in which $\pi \restriction \mathfrak{D}^{\mathfrak{M}}$ is not persistent. Contradiction to the persistence of $\pi \restriction \mathfrak{D}^{\mathfrak{M}}$ and the absoluteness of persistence. $\square$

Theorem

Suppose that $V[G]$ is a generic extension of V. Suppose that π is an element of $V[G]$ which maps that Turing degrees in V automorphically to itself (that is, $\pi : \mathfrak{D}^V \overset{\sim}{\to} \mathfrak{D}^V$). If π is generically persistent in $V[G]$, then π is an element of $L(\mathbb{R}^V)$. That is, π is constructible from the set of reals in V.

Proof

π is generically persistent, so π is arithmetically definable in any generic counting of $\mathfrak{D}^V$. Consequently, π must belong to the ground model for such countings, namely $L(\mathbb{R}^V)$. □

Applications to Aut($\mathfrak{D}$)

global extension of persistent automorphisms

Theorem

Suppose that $0'$ *is an element of* $\mathcal{I}$ *and* $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$ *is persistent. Then* ρ *can be extended to a global automorphism* $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$.

Proof

ρ can be persistently extended to $\mathfrak{D}^V$ in a generic extension of V. This extension belongs to $L[\mathbb{R}^V]$. □

Corollary

The statement There is a non-trivial automorphism of the Turing degrees is equivalent to a Σ^1_2 *statement. It is therefore absolute between well-founded models of ZFC.*

Applications to Aut($\mathfrak{D}$)

lifting automorphisms to generic extensions

Theorem

Let π be an automorphism of $\mathfrak{D}$. Suppose that $V[G]$ is a generic extension of V. Then, there is an extension of π in $V[G]$ to an automorphism of $\mathfrak{D}^{V[G]}$, the Turing degrees in $V[G]$.

Proof

There is a persistent extension π_1 of π in any generic extension of $V[G]$ in which $\mathfrak{D}^{V[G]}$ is countable. This π_1 belongs to $V[G]$. □

Representing Automorphisms

Definition

Given two functions $\tau : \mathfrak{D} \to \mathfrak{D}$ and $t : 2^\omega \to 2^\omega$, we say that t **represents** τ if for every degree x and every set X in x, the Turing degree of $t(X)$ is equal $\tau(x)$.

We will analyze the behavior of an automorphism of $\mathfrak{D}$ in terms of the action of its extensions on the degrees of the generic reals.

Representing Automorphisms

continuity on sufficiently generic sets

Theorem

Suppose that $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$. There is a countable family $\overrightarrow{D}$ of dense open subsets of $2^{<\omega}$ such that π is represented by a continuous function f on the set $\overrightarrow{D}$-generic reals.

The proof has several steps, which we will sketch. Also, we introduce the notation $\Pi(Z)$ for a representative of $\pi(\mathit{degree}(Z))$.

1. Let $V[\mathcal{G}]$ be a generic extension of V obtained by adding ω_1-many Cohen reals and let π_1 be an extension of π to $\mathfrak{D}^{V[\mathcal{G}]}$.

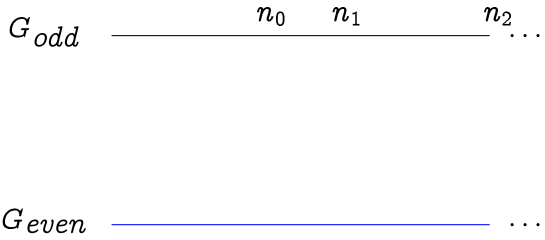
2. Since $\pi_1 \in L[\mathbb{R}^{V[\mathcal{G}]}]$, fix $X \in \mathbb{R}^{V[\mathcal{G}]}$ so that π_1 is ordinal definable from X in $L[\mathbb{R}^{V[\mathcal{G}]}]$. Work in $V[X]$ and note that $V[\mathcal{G}]$ is a generic extension of $V[X]$ obtained by adding ω_1-many Cohen reals. (The forcing factors.)

3. Consider a set G, of degree g, which is Cohen generic over $V[X]$. $\pi_1(g)$ is arithmetically definable relative to g and $\pi^{-1}(0')$. We can find an e and a k such that it is forced that $\pi_1(g)$ is represented by $\{e\}((G \oplus \Pi^{-1}(\emptyset'))^{(k)})$. Since G is Cohen generic, we can assume that e has the form $\{e\}(G \oplus \Pi^{-1}(\emptyset')^{(k)})$. Thus, π_1 is continuously represented on the set of $V[X]$-generic reals.

4. We make an aside to exploit a phenomenon first observed by Jockusch and Posner, 1981: For any $\vec{D}$, the $\vec{D}$-generic degrees generate $\mathfrak{D}$ under meet and join. We fix a mechanism by which this coding can be realized.

Join Coding
definition

Let G and Y be given. Define $\mathbb{C}(Y, G)$ by injecting the values of Y into G_{even} at those places where G_{odd} is not zero.



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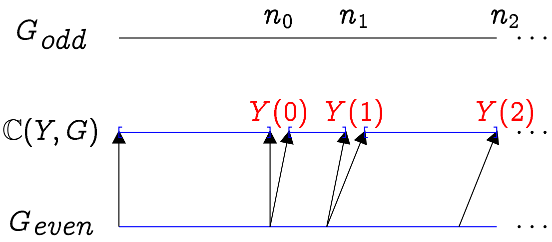
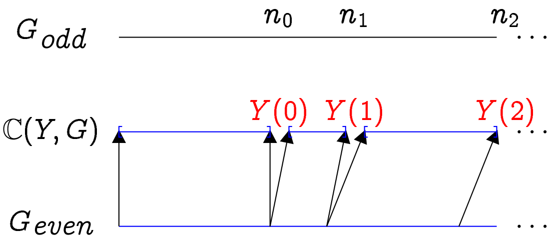


Join Coding

degree equivalence and genericity



Lemma

If G_{odd} is infinite, then $\mathbb{C}(Y,G) \oplus G \equiv_T Y \oplus G$.

Join Coding

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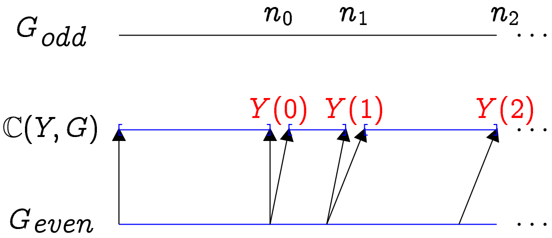

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Lemma

For any dense open subset of $2^{<\omega}$, D, there is a dense open set D^*, such that for all D^*-generic G and all Y, $\mathbb{C}(Y,G)$ is D-generic. In particular, for all G, Y, and Z, if G is generic over $V[Z]$, then so is $\mathbb{C}(Y,G)$.

Join Coding
generating $\mathfrak{D}$

Definition

For $Y \in 2^\omega$, let (Y) denote the set $\{Z : Z \leq_T Y\}$.

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4. (continued) Let Y be given with Turing degree y, and let G_1 and G_2 be mutually Cohen generic over $V[X \oplus Y]$. We can write the ideal generated by Y as the meet of joins of generic ideals.

$$\begin{aligned}(\mathbb{C}(Y, G_1) \oplus G_1) \cap (\mathbb{C}(Y, G_2) \oplus G_2) &= (Y \oplus G_1) \cap (Y \oplus G_2) \\ &= (Y)\end{aligned}$$

Representing Automorphisms

pulling back to arithmetic generics

5. The previous equality is preserved by π_1, as represented on generic reals.

$$\{Z : \text{the degree of } Z \text{ belongs to } (\pi^*(y))\} =$$

$$\Big(\{e\}(\mathbb{C}(Y,G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big)$$

$$\bigcap \Big(\{e\}(\mathbb{C}(Y,G_2) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_2 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big)$$

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When Y is also generic:

$$\Big(\{e\}(Y \oplus \Pi^{-1}(\emptyset')^{(k)})\Big) =$$
$$\Big(\{e\}(\mathbb{C}(Y,G_1) \oplus \Pi^{-1}(\emptyset')^{(k)}) \oplus \{e\}(G_1 \oplus \Pi^{-1}(\emptyset')^{(k)})\Big)$$
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Representing Automorphisms

pulling back to arithmetic generics

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expresses an arithmetic identity between generic reals. It holds when Y is arithmetically generic in V and G_1 and G_2 are $V[X]$-generic. In this case, the right-hand-side represents $(\pi(\mathit{degree}(Y)))$. Thus, we have an arithmetic (hence continuous) representation of π on all arithmetically generic reals.

Representing Automorphisms

pulling back to arithmetic generics

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Corollary

If $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$ then π has a Borel representation.

Arithmetic Representations

images of generic degrees

Theorem

Suppose that $\pi : \mathfrak{D} \overset{\sim}{\to} \mathfrak{D}$ *and that* $\overrightarrow{D}^*$ *is a countable collection of dense open subsets of* $2^{<\omega}$. *There is a countable collection of dense open subsets of* $2^{<\omega}$, $\overrightarrow{D}$, *such that for all* $\overrightarrow{D}$*-generic reals* G *there is a* G^* *such that* G^* *is* $\overrightarrow{D}^*$*-generic and the degree of* G^* *is less than or equal to* $\pi(\mathit{degree}(G))$

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- We chose $\vec{D}_0$ as above so that we have a continuous representation of π on the collection of $\vec{D}_0$-generic reals.
- We show that for any sufficiently generic real G, $\pi(degree(G))$ computes a function which has no a priori bound on its rate of growth. This reflects the fact that arbitrarily much of $\mathbb{C}(Y, G)$ can be specified between injecting values of Y.

Arithmetic Representations

images of generic degrees

- We define a real G^* by comparing the values of several of these functions. Hence, for any sufficiently generic G, G^* will be defined at every argument and will be recursive in any representative of $\pi(degree(G))$.
- We show that for any dense open set D^* and any finite condition p, there is an extension q of p such that for any G extending q, G^* meets D^*.
- At the end of the analysis, we calculate the genericity needed on G to ensure that G^* is $\overrightarrow{D}$-generic.

Evaluating relative to a generic degree

Theorem

There is a family of dense open sets $\overrightarrow{D}$ *and a continuous function* $F(G, X)$ *such that for all* $\overrightarrow{D}$*-generic* G,

$$(\forall X)[\pi(\mathit{degree}(X \oplus G)) = \mathit{degree}(F(G, X) \oplus \Pi(G))]$$

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Proof

Apply the continuous representation on generic reals to $\mathbb{C}(X,G)$. □

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Proof

Use G_{odd} to obtain a perfect tree of mutually generic reals. Let X pick a path $T(X)$. Use the previous F on G_{even} and $\mathbb{C}(T(X), G_{even}))$. □