Turing Degrees and Definability of the Jump

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Outline

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The Turing Degrees

basic definitions

Definition

- $\mathfrak{D}$ denotes the partial order of the Turing degrees. $a + b$ denotes the join of two degrees. ($A \oplus B$ denotes the recursive join of two sets.)
- A subset $\mathcal{I}$ of $\mathfrak{D}$ is an ideal if and only if $\mathcal{I}$ is closed under $\leq_T$ ($x \in \mathcal{I}$ and $y \leq_T x$ implies $y \in \mathcal{I}$) and closed under $+$ ($x \in \mathcal{I}$ and $y \in \mathcal{I}$ implies $x + y \in \mathcal{I}$). A jump ideal is closed under the Turing jump ($a \mapsto a'$) as well.

Definition

- A Turing functional Φ is a set of sequences (x, y, σ) such that x is a natural number, y is either 0 or 1, and σ is a finite binary sequence. Further, for all x, for all y_1 and y_2, and for all compatible σ_1 and σ_2, if $(x, y_1, \sigma_1) \in \Phi$ and $(x, y_2, \sigma_2) \in \Phi$, then $y_1 = y_2$ and $\sigma_1 = \sigma_2$.
- We write $\Phi(x, \sigma) = y$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(x, y, \tau) \in \Phi$. If $X \subseteq \omega$, we write $\Phi(x, X) = y$ to indicate that there is an ℓ such that $\Phi(x, X \upharpoonright \ell) = y$, and write $\Phi(X)$ for the function evaluated in this way.

Note, we did not require that Φ be definable. If $\Phi(A) = B$ then $A \oplus \Phi \geq_T B$.

Forcing in arithmetic

In recursion theory, forcing is typically used to construct arithmetically exotic subsets of ω.

Example

- (Kleene-Post, 1954) constructed a pair of sets with incomparable Turing degree by forcing with finite conditions.
- (Friedberg, 1957) inverted the Turing jump on the Turing degrees above $0'$ also using finite conditions.
- (Spector, 1956) constructed a set of minimal Turing degree by forcing with perfect recursive trees.

Friedberg built a set G which is sufficiently generic so that atomic facts about G' are decided by forcing, but not so generic that the Turing degree of G' is incomparable with the given set above $0'$.

Generic Filters

Definition

Suppose that D is a subset of the partially order set P.

- D is open if and only if for all p and q in P, if $p \geq_P q$ and $p \in D$, then $q \in D$.
- D is dense if and only if for all $p \in P$ there is a $d \in D$ such that $p \geq_P d$.

Generic Filters

Definition

Suppose that G is a subset of the partially order set P.

- G is a filter on P if and only if the following conditions hold.
  - $G \neq \emptyset$.
  - For all p and q, if $p \geq_P q$ and $q \in G$, then $p \in G$.
  - For all p and q, if $p \in G$ and $q \in G$, then there is an $r \in G$ such that $p \geq_P r$ and $q \geq_P r$.
- If $\vec{D}$ is a collection of dense open subsets of P, then G is a $\vec{D}$-generic filter on P if and only if G is a filter on P and for all $D \in \vec{D}$, $G \cap D \neq \emptyset$.

When the partial order P is understood, we will simply say that G is generic for $\vec{D}$.

Cohen Forcing

Consider the partial order $2^{<\omega}$, ordered by extension. This partial order adds a Cohen real. It has a universal property among countable partial orders.

Theorem

Suppose that P is a countable partial order which is presented recursively in Z. There is a function $\lambda p.f(p, Z)$ which is recursive in Z, maps $2^{<\omega}$ to P, preserves order, and has the property that if D is a dense subset of P, then $f^{-1}(D)$ the pointwise inverse image of D is a dense subset of $2^{<\omega}$.

Consequently, it is possible to build P-generic sets from Cohen generic sets.

The Coding Theorem

Definition

A countable n-place relation $\mathcal{R}$ on $\mathfrak{D}$ is a countable subset of the n-fold Cartesian product of $\mathfrak{D}$ with itself. In other words, R is a countable subset of the set of length n sequences of elements of $\mathfrak{D}$.

Theorem (Slaman-Woodin, 1986)

For every n there is a first order formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for every countable n-place relation $\mathcal{R}$ on $\mathfrak{D}$ there is a sequence of degrees $\overrightarrow{p} = (p_1, \ldots, p_m)$ such that for all sequences of degrees $\overrightarrow{d} = (d_1, \ldots, d_n)$,

$$\overrightarrow{d} \in \mathcal{R} \iff \mathfrak{D} \models \varphi(\overrightarrow{d}, \overrightarrow{p}).$$

The Coding Theorem
coding antichains

Lemma

Let $\overrightarrow{A} = (A_i : i \in \omega)$ be a sequence of reals whose degrees form a countable antichain in $\mathfrak{D}$ and let B be an upper bound on the elements of $\overrightarrow{A}$. There are reals G_1 and G_2 with the following properties.

- *For every A_i in $\overrightarrow{A}$, there is a C such that C is recursive in $G_1 \oplus A_i$ and recursive in $G_2 \oplus A_i$, but C is not recursive in A_i.*
- *For every Y below B, either*
  - *for every Z, if Z is recursive in $G_1 \oplus Y$ and recursive in $G_2 \oplus Y$, then Z is recursive in Y*
  - *or there is an A_i in $\overrightarrow{A}$ such that $Y \geq_T A_i$.*

Lemma (Dekker and Myhill, 1958)

Suppose that X is a subset of ω. There is a $Y \subseteq \omega$ with the same Turing degree as X, such that Y is recursive in each of its infinite subsets.

So, we can assume that our sets $A \in \overrightarrow{A}$ have the Dekker-Myhill property.

The Coding Theorem

the partial order

Conditions. A condition p is a triple $(p_1, p_2, F(p))$. Here p_1 and p_2 are finite binary sequences of equal length and $F(p)$ is a finite initial segment of $\overrightarrow{A}$. The set of conditions is denoted by P.

Order. For p and q in P, we say that q is stronger than p if q_1 extends p_1, q_2 extends p_2 and $F(q)$ extends $F(p)$. In addition, if k is less than the length of $F(p)$ and A is the kth element of $F(p)$, then the following condition holds. If a is an element of A and (k, a) is less or equal to the common length of q_1 and q_2 but greater than the common length of p_1 and p_2, then q_1 and q_2 have the same value at (k, a).

Once $A \in F(p)$, G_1 and G_2 are constrained to have common generic information on $\{k\} \times A$.

The Coding Theorem

why it works

Consider the following problem. Suppose $p \in P$, $Y \leq B$, and recursive Turing functionals Φ_1 and Φ_2 are given. Show that one of the following holds.

- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \leq_T Y]$
- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \neq \Phi_2(Y \oplus G_2)]$
- $(\exists A \in F(p))[Y \geq_T A_i]$

The Coding Theorem

why it works

- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \leq_T Y]$ occurs when there is a $q < p$ such that q decides all the values of $\Phi_1(Y \oplus G_1)$.

The Coding Theorem

why it works

- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \leq_T Y]$ occurs when there is a $q < p$ such that q decides all the values of $\Phi_1(Y \oplus G_1)$.
- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \neq \Phi_2(Y \oplus G_2)]$ occurs when there are q_1 and q_2 which are consistent on $F(p)$ and decide incompatible values for $\Phi_1(Y \oplus G_1)$ and $\Phi_2(Y \oplus G_2)$.

The Coding Theorem

why it works

- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \leq_T Y]$ occurs when there is a $q < p$ such that q decides all the values of $\Phi_1(Y \oplus G_1)$.
- $(\exists q < p)[q \Vdash \Phi_1(Y \oplus G_1) \neq \Phi_2(Y \oplus G_2)]$ occurs when there are q_1 and q_2 which are consistent on $F(p)$ and decide incompatible values for $\Phi_1(Y \oplus G_1)$ and $\Phi_2(Y \oplus G_2)$.
- $(\exists A \in F(p))[Y \geq_T A_i]$ occurs when incompatible values in $\Phi_1(Y \oplus G_1)$ occur only with incompatible information on $F(p)$. One argues that it is dense that there are conditions q_1 and q_1^* such that q_1 and q_2 disagree only at one point and such that $\Phi_1(Y \oplus q_1)$ and $\Phi_1(Y \oplus q_1^*)$ are inconsistent. The point of disagreement must belong to an element of $F(p)$.

The Coding Theorem

why it works

Defining arbitrary sets (and arbitrary relations) is reduced to defining antichains by freely joining the elements of the set with mutually Cohen generic reals.

Theorem (Simpson, 1977)

There is a recursive interpretation of the second order theory of arithmetic in the first order theory of $\mathfrak{D}$.

Proof

Specifying a standard model of arithmetic involves specifying a countable set N, a distinguished element "0", and a unary "successor" function s, such that $\mathbb{N} = (N, 0, s)$ satisfies finitely many first order properties (P^-) together with second order induction. By the coding theorem, all of these are available in $\mathfrak{D}$. $\square$

Effective Coding Theorem

Theorem

Suppose that there is a presentation of the countable relation $\mathcal{R}$ which is recursive in the set R. There are parameters $\overrightarrow{p}$ which code $\mathcal{R}$ in $\mathfrak{D}$ such that the elements of $\overrightarrow{p}$ are below R'.

Proof

We build G_1 and G_2 by a finite injury forcing construction. For the sake of a requirement on Φ_1 and Φ_2, at a given stage either we can find a condition that decides all the values of $\Phi_1(Y \oplus G_1)$, or we can find a condition forcing $\Phi_1(Y \oplus G_1) \neq \Phi_2(Y \oplus G_2)$, or we can compute one more element in some set from $F(p)$. □

Representatives
decoding

Theorem (Decoding Theorem)

Suppose that $\overrightarrow{p}$ is a sequence of degrees which lie below y and $\overrightarrow{p}$ codes the relation $\mathcal{R}$. Letting Y be a representative of y, $\mathcal{R}$ has a presentation which is $\Sigma^0_5(Y)$.

Proof

By the way in which sets are coded by degrees, $\mathcal{R}$ is arithmetic in Y. Directly counting the quantifiers shows that it is $\Sigma^0_5(Y)$. □

Representatives
coding

Theorem

For any degree x and representative X of x, there are parameters $\overrightarrow{p}$ such that

- *$\overrightarrow{p}$ codes an isomorphic copy of $\mathbb{N}$ with a unary predicate for X;*
- *the elements of $\overrightarrow{p}$ are recursive in $x + 0'$.*

Proof

Use X to produce a representation of $\mathbb{N}$ with a unary predicate for X. Use $0'$ to meet the dense sets to code it. □

Theorem

Suppose that $\overrightarrow{p}$ is a sequence of degrees below y, and $\overrightarrow{p}$ codes an isomorphic copy of $\mathbb{N}$ together with a unary predicate U. Then, for $Y \in y$, U is $\Sigma^0_5(Y)$.

Applications to Aut($\mathfrak{D}$)

Nerode-Shore

Theorem (Nerode and Shore, 1980)

Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. *For every degree* x, *if* x *is greater than* $\pi^{-1}(0')$ *then* $\pi(x)$ *is arithmetic in* x.

Theorem (Nerode and Shore, 1980)

Suppose that $\pi : \mathfrak{D} \stackrel{\sim}{\to} \mathfrak{D}$. For every degree x, if x is greater than $\pi^{-1}(0')$ then $\pi(x)$ is arithmetic in x.

Theorem (Nerode and Shore, 1980)

Suppose $\pi : \mathfrak{D} \stackrel{\sim}{\to} \mathfrak{D}$ is an automorphism of $\mathfrak{D}$ and $x \geq_T \pi^{-1}(0')^{(5)} + \pi^{-1}(\pi(0')^{(5)})$. Then, $\pi(x) = x$. Consequently, π is the identity on a cone.

Proof

Given x above $\pi^{-1}(0')^{(5)}$, fix y_1 and y_2 so that $y_1 \vee y_2 = x$; $\pi(y_1)$ and $\pi(y_2)$ are greater than $0'$; and $y_1^{(5)}$ and $y_2^{(5)}$ are recursive in x. Each $\pi(y_i)$ is $\Sigma^0_5(Y_i)$ and hence recursive in X. Thus, $x \geq_T \pi(y_1) \vee \pi(y_2) = \pi(x)$. By symmetry, $\pi(x) \geq_T x$. □

Theorem (Odifreddi and Shore, 1991)

Suppose that π is an automorphism of $\mathfrak{D}$ and that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ which includes $0'$ such that π restricts to an automorphism of $\mathcal{I}$. For any real I, if there is a presentation of $\mathcal{I}$ which is recursive in I then the restriction of π to $\mathcal{I}$ has a presentation which is arithmetic in I.

Proof

Code a counting of $\mathcal{I}$ by parameters $\overrightarrow{p}$ which are arithmetic in I. The action of π on $\mathcal{I}$ is determined by the action of π on $\overrightarrow{p}$. Since $0' \in \mathcal{I}$, the Nerode and Shore Theorem implies that $\pi(\overrightarrow{p})$ is arithmetic in I. $\square$

Persistent Automorphisms

We now begin our account of the Slaman and Woodin analysis of $Aut(\mathfrak{D})$. Unless indicated otherwise, the results that follow belong to that body of work.

Definition

An automorphism ρ of a countable ideal $\mathcal{I}$ is persistent if for every degree x there is a countable ideal $\mathcal{I}_1$ such that

- $x \in \mathcal{I}_1$ and $\mathcal{I} \subseteq \mathcal{I}_1$;
- there is an automorphism ρ_1 of $\mathcal{I}_1$ such that the restriction of ρ_1 to $\mathcal{I}$ is equal to ρ.

We will show that ρ is persistent if and only if ρ extends to a automorphism of $\mathfrak{D}$.

Persistent Automorphisms

easy direction

Theorem

Suppose that $\pi : \mathfrak{D} \xrightarrow{\sim} \mathfrak{D}$. *For any ideal* $\mathcal{I}$, *if* π *restricts to an automorphism* $\pi \restriction \mathcal{I}$ *of* $\mathcal{I}$ *then* $\pi \restriction \mathcal{I}$ *is persistent.*

Thus, if $\mathfrak{D}$ is not rigid, then there is a nontrivial persistent automorphism of some countable ideal.

Theorem

Suppose that $\rho : \mathcal{I} \xrightarrow{\sim} \mathcal{I}$, that $\mathcal{J}$ is a jump ideal contained in $\mathcal{I}$ and that $\rho(0') \vee \rho^{-1}(0') \in \mathcal{J}$. Then $\rho \restriction \mathcal{J}$ is an automorphism of $\mathcal{J}$.

Proof

Follows from the effective coding and decoding theorems. $\square$

Corollary

Suppose that $\mathcal{I}$ is an ideal such that $0'$ is an element of $\mathcal{I}$ and suppose that ρ is a persistent automorphism of $\mathcal{I}$. For any countable jump ideal $\mathcal{J}$ extending $\mathcal{I}$, ρ extends to an automorphism of $\mathcal{J}$.

Theorem

Suppose that $\mathcal{I}$ is an ideal in $\mathfrak{D}$ such that $0'$ is an element of $\mathcal{I}$. Suppose that there is a presentation of $\mathcal{I}$ which is recursive in I. Finally, suppose that $\mathcal{J}$ is a jump ideal which includes I and ρ is an automorphism of $\mathcal{J}$ that restricts to an automorphism of $\mathcal{I}$. Then, the restriction $\rho \upharpoonright \mathcal{I}$ of ρ to $\mathcal{I}$ has a presentation which is arithmetic in I.

Proof

Apply the Odifreddi-Shore argument. There is a code $\overrightarrow{p}$ for a counting of $\mathcal{I}$ which is arithmetic in I. By the previous theorem, $\rho(\overrightarrow{p})$ is arithmetic in I. □

Corollary

Suppose that $\mathcal{I}$ is an ideal and $0'$ is an element of $\mathcal{I}$. If ρ is a persistent automorphism of $\mathcal{I}$, then ρ is arithmetically definable in any presentation of $\mathcal{I}$.

Consequently, persistent automorphisms of $\mathcal{I}$ are locally presented and there are at most countably many of them.

Theorem

Suppose that $\mathcal{I}$ is an ideal and $0'$ is an element of $\mathcal{I}$. Suppose that ρ is a persistent automorphism of $\mathcal{I}$. For any jump ideal $\mathcal{J}$ which extends $\mathcal{I}$, ρ extends to a persistent automorphism of $\mathcal{J}$.

Proof

Suppose that $\mathcal{J}$ were a jump ideal such that there is no persistent automorphism of $\mathcal{J}$ which extends ρ. Let J compute a presentation of $\mathcal{J}$. Choose x_e so that the eth arithmetic in J extension of ρ to $\mathcal{J}$ cannot be extended further to include x_e. Let x bound the x_e's. By its persistence, extend ρ to an automorphism ρ_1 of the jump ideal generated by x. Then, $\rho_1 \upharpoonright \mathcal{J}$ is arithmetic in J, contradiction. □