Undecidability of the $\alpha$-degrees

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Overview

- Interpreting models within degree structures
- Techniques and their applications for the Turing degrees
- Joy and woe of trying the same for the $\alpha$-degrees
Interpretations of First Order Structures

Let $\mathcal{M}$ and $\mathcal{N}$ be first order structures with finite signature.

**Definition**

An *interpretation of $\mathcal{N}$ in $\mathcal{M}$* consists of the following.

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  - With the property that $\mathcal{N} \sim (N/ \equiv, c_j^\mathcal{M}, f_j^\mathcal{M}, R_j^\mathcal{M})$. 
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If there is an interpretation of $N$ in $M$, then there is a recursive translation from $Th(N)$, the first order theory of $N$, to $Th(M)$.

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- Keeping the formulas fixed and varying the parameters, we obtain a family of structures uniformly interpreted in $\mathcal{M}$.
- There there is a translation from the common theory of the models in this family to $Th(\mathcal{M})$. 
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*Every countable initial segment of the Turing degrees is the intersection of two principal ideals, i.e. has an exact pair.*
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To reverse the comparison and interpret second order arithmetic in $D$, we need to interpret second order quantifiers over a countable interpreted model.

**Theorem (Slaman and Woodin)**

*Every countable relation on $D$ is uniformly definable from finitely many parameters in $D$.***
Interpretations of First Order Structures

the degree of the theory of $\mathcal{D}$

Using the coding theorem for countable relations, the following are available to us.

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The first order theory of $\mathcal{D}$ is recursively isomorphic to the second order theory of arithmetic.

Earlier proofs by Simpson and Nerode-Shore used interpretations of first order arithmetic by initial segments and Spector's exact pair theorem to interpret set quantifiers.
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**Observation**

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**Proof**

There are parameters $\overrightarrow{p}$ which code $(\mathbb{N}, A)$ such that $\overrightarrow{p}$ is arithmetic in $A$. 
Observation

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Proof

There are parameters $\overrightarrow{p}$ which code $(\mathbb{N}, A)$ such that $\overrightarrow{p}$ is arithmetic in $A$.

If we can define a reasonable definability-neighbor of $a$ within $\mathcal{D}$, then we can interpret a countable set of reals which includes a representative of $a$. 
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Theorem (Jockusch and Shore)

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\( \mathcal{D} \) and \( \mathcal{D} (\geq_T \mathcal{O}) \) are not isomorphic.

Proof

The reals interpreted using parameters arithmetic in \( \mathcal{O} \) and above \( \mathcal{O} \) contains an element which is not arithmetic.
The Bi-interpretability Conjecture

**Conjecture (Slaman and Woodin)**

The relation

\[ R(\overrightarrow{p}, a) \iff \overrightarrow{p} \text{ interprets a representative of } a \]

is definable in \( D \).
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Theorem (Slaman and Woodin)

- The Bi-interpretability Conjecture is true relative to parameters.
- It is equivalent to \( D \)'s being rigid.
- It implies that a relation is definable in \( D \) iff it is induced by a degree-invariant relation definable in second order arithmetic.
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**Theorem (Shore and Slaman)**

*The Turing jump is definable in $D$.*

Recently, Shore has given a localized proof of the definability of the jump.
The Joy and Woe of the $\alpha$-degrees

**Definition**

An ordinal $\alpha$ is $\Sigma_1$-admissible iff $L_\alpha$ satisfies $\Sigma_1$-replacement and $\Delta_1$-comprehension.
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Examples of admissible ordinals:

- $\omega$
- $\omega_1, \omega_2, \ldots, \omega_\omega, \ldots$ any cardinal
- $\omega_1^{CK}$, the supremum of the recursive ordinals
Let $L_\alpha$ be fixed and consider definability within $L_\alpha$.

**Definition**

- An $\alpha$-reduction is a $\Sigma_1$ (in parameters) subset $\Phi$ of $L_\alpha$ consisting of quadruples $(P, N, P^*, N^*)$.
- For $A$ and $B$ contained in $\alpha$, $\Phi(A) = B$ iff for every disjoint $P$ and $N$ in $L_\alpha$,

$$
(P \subseteq B \land N \cap B = \emptyset) \iff \\
\exists (P^*, N^*)[P^* \subseteq A \land N^* \cap A = \emptyset \land (P, N, P^*, N^*) \in \Phi]
$$

- $A \geq_\alpha B$ iff there is an $\alpha$-reduction $\Phi$ such that $\Phi(A) = B$. 
α-Recursion Theory
deconstructing recursion theory

We have introduced the following analogy between concepts in recursion theory and concepts in α-recursion theory.

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Why?

For example, to separate and clarify the roles of these concepts – eliminate the role confusion and role strain that they endure in the standard setting.
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*Consider Spector’s construction of a minimal degree.* A single step is determined by the Boolean value of a $\Pi_2$ condition. $\Sigma_1$-admissibility is not sufficient to show that the construction produces a set of minimal degree.

In the absence of a solution to the minimal degree problem, interpreting structures using initial segments of the $\alpha$-degrees is impossible. Gone are the interpretations of Lachlan, Simpson, and Nerode-Shore. Gone, too, is the analysis of cones of minimal covers.
Theorem (S. Friedman)

In $L$, the $\omega_{\omega_1}$-degrees greater than $0'$ are well-ordered.
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Relative to sets $A$ such that $\alpha$ is not $\Sigma_1$-admissible relative to $A$, the analogies between recursive and $\alpha$-recursive are not helpful. Reveal an uncountable singularity in $\alpha$ and all the similarities between the $\alpha$-degrees and the Turing degrees are gone.
Theorem (Chong Chi Tat and Slaman)

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We modify the apparatus of the Slaman-Woodin coding theorem for the Turing degrees.
In both the Turing degrees and the $\alpha$-degrees, we define an anti-chain $A = \{A_1, \ldots, A_n\}$ from parameters as follows.

- Let $B$ be the join of $A$.
- Find $C_1$ and $C_2$ such that the following conditions hold.
  - For all $A_i \in A$, there is a $G$ such that $G \not\leq A_i$, $G \leq A_i + C_1$, and $G \leq A_i + C_2$, i.e. $C_1 + A_i \land C_2 + A_i \neq A_i$.
  - For all $W \leq B$, either there is an $A_i \in A$ such that $W \geq A_i$ or $C_1 + W \land C_2 + W = W$. 
We begin by assuming that each $A_i$ is recursive in any of its infinite subsets. Replace $A_i$ by the set of its initial segments.
Defining Anti-Chains from Parameters

$\alpha = \omega$ (the Turing degrees)

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We begin by assuming that each $A_i$ is the set of initial segments of a regular and hyper-regular set $W_i$. That is, for all $\beta < \alpha$, $W_i \cap \beta \in L_\alpha$, and $L_\alpha[W_i]$ is $\Sigma_1$-admissible.
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Explicit use of regularity replaces the original implicit use of finiteness.
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existence of generic sets

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Analysis of the forcing relation and effective approximation replace the $\omega$-length recursion to build generic sets.
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  - Is there an automorphism of the $\alpha$-degrees?
Finis