# Recursion Theory

**Theodore A. Slaman**

**University of California, Berkeley**

**March 2001**



## $\boxed{\text{Recursion Theory}}$

**Mathematical Logic:** That part of mathematics which involves understanding language, semantics, syntax, proof, etc.

**Recursion Theory** That part of mathematical logic which is focused on definability, especially for subsets of the natural numbers ($\omega$) and of the real numbers ($2^\omega$).

We will take a short and mostly nontechnical tour of the subject.

## Desiderata

We want *complete understanding* of language and meaning.

Including but not limited to:

- A common perspective on definability across mathematical disciplines.
- Quantitative tools to calibrate the types of definitions and constructions of various sorts.
- A convincing meta-theory to say that our perspective is correct and adequate.



## Definability

As logicians, we know how to formalize definability.

> A relation $R$ is definable within a structure $\mathfrak{M}$ if and only if there is a formula $\varphi$ in the language of $\mathfrak{M}$ such that for all $\vec{m}$ in $\mathfrak{M}$,
>
> $$\vec{m} \in R \iff \mathfrak{M} \models \varphi[\vec{m}]$$

Tarski's definition of $\models$ seems unassailable. Consequently, we have a notion of definability once we specify a language and a structure $\mathfrak{M}$ which interprets it.

## Arithmetic

We will limit ourselves to definability within first and second order arithmetic, though even this is too much to cover in one talk.

**Language:** Usual operations of arithmetic $0, 1, +, \times, exp, \ldots$ on $\mathbb{N}$. Depending on context: extra unary predicate symbols $A, B, \ldots$ and/or second order variables $X, Y, \ldots, \epsilon$ to denote membership $n \in A$ or $n \in X$.

**Variations:**
- Pure definability in the standard model of arithmetic.
- Fix one or several predicates on $\mathbb{N}$ and obtain arithmetic relative definability.
- Vary the scope of the second order variables to either all countable sets or to a more limited domain.



## Features and Questions

The syntax of arithmetic divides these definable sets into classes according to the form of their definitions. We have a variety of questions to consider.

**Natural classes:**
- Do the syntactic classes correspond with naturally occurring classes?
- Is every naturally occurring class accounted for in this way?

**Stratification of particular classes:** Fixing a particular syntactic class:
- Can the sets within that class be generated by simple operations?
- Is there a natural hierarchy associated with the generating process?

**Diagonalizing against a definable class:** How does one build a set which is not in the class?

**Basis Questions:** Given a set in the class, what sorts of elements does it have?

- Simple elements.
- Complicated elements.



Subsets of $\omega$

| | |
|---|---|
| $\Sigma^0_2$ | Recursive limit infimum |
| $\Delta^0_2$ | Recursive limit, recursive in $0'$ |
| $\Sigma^0_1$ | Recursively enumerable |
| $\Delta^0_1$ | Recursive |

$\boxed{\Delta^0_1}$

**Church's thesis:** Isolation of the class of $\Delta^0_1$ sets and identification of that class with the class of computable sets.. Turing, Church, Kleene, Post, Gödel, etc. (1930's).

**Relativized computability:** Define $A \geq_T B$ if and only if $B$ is $\Delta^0_1(A)$; obtain the Turing degrees $\mathcal{D}$.

**Basis question:** "How complicated are the numbers in $A$?" This question leads in a step or two to Chaitin-Kolmogorov complexity.



$\boxed{\Sigma^0_1}$

**Kleene Enumeration Theorem:** There is a universal computing machine. It does not always return a value.

**Kleene Fixed Point Theorem:** Recursion theoretic manifestation of Gödel's diagonal argument.

**The halting problem:** The natural definable set which is not recursive. $0'$ is the Turing degree of the halting problem.

**The priority method:** A combinatorial tool to build simple (eg. recursively enumerable) sets while ensuring that they have complicated properties.

## Non-Recursive

**It is not so easy:** There is a recursive subtree $T$ of $2^{<\omega}$ as follows.

- Every node in $T$ extends to an infinite path in $T$.
- $T$ has a perfect subtree.
- If $A$ is an infinite path in $T$ and $A$ is $\Delta^1_1$, then $A$ is recursive.

**Hidden aspect of non-recursive:** Posner and Robinson (1981) showed that $A$ is not recursive if and only if there is a $G$ such that $A \oplus G \equiv_T G'$. The analogous property holds throughout the arithmetic hierarchy.



## Recursively Enumerable Turing Degrees

**Post's problem:** Friedberg and Mučnik (1956-7) introduced the priority method when they showed that there are recursively enumerable sets of incomparable Turing degree. In fact, Harrington and Slaman showed that the structure is as complicated as possible; it interprets true arithmetic.

**Uniformity?** Is there a definable recursively enumerable degree other than 0 and $0'$?

## $\boxed{\Delta^0_2}$

**Shoenfield limit lemma:** The $\Delta^0_2$ sets are those which can be recursively approximated. They are also characterized as the collection of sets recursive in $0'$.

**Generic behavior:** The $\Delta^0_2$ sets include examples of various almost everywhere behavior: 1-generic, Martin-Löf random, paths through infinite recursive binary trees (Kleene Basis Theorem).

**Difference hierarchy:** Fix a system of notations for the recursive ordinals (such as from Kleene's $\mathcal{O}$). Ershov (1968) showed that the class of $\Delta^0_2$ sets can be generated relative to this representation of $\omega_1^{CK}$ by transfinitely iterating set difference.



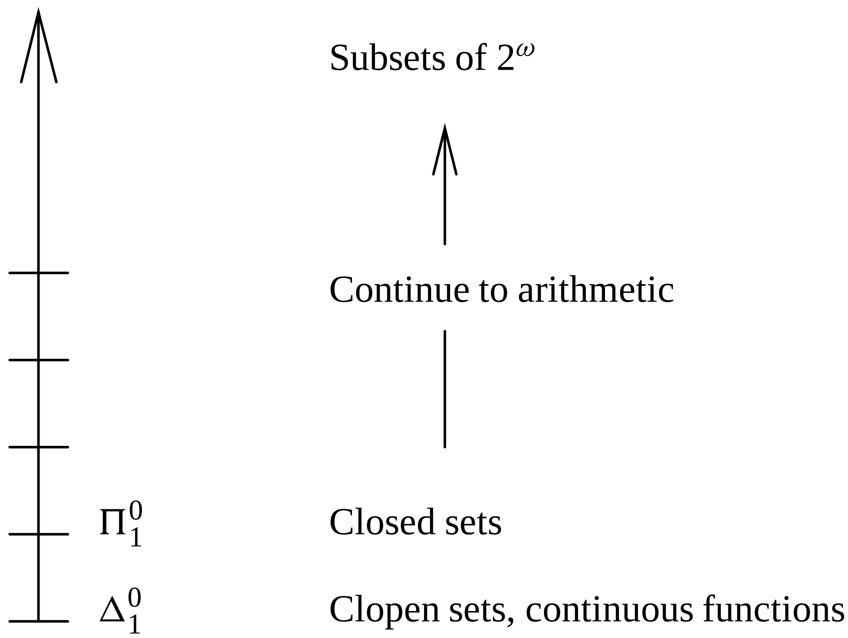
## **From $\omega$ to $2^\omega$**

We have just discussed various arithmetic classes of subsets of $\omega$.

If we consider the language of first order arithmetic with a unary predicate symbol $X$, we obtain the classes of arithmetically definable subsets of $2^\omega$:

$$\{X : (\mathbb{N}, X) \models \varphi\}$$

So we have a direct route from recursion theory on $\omega$ to effective descriptive set theory on $2^\omega$.






$$\boxed{\Delta^0_1}$$

**Sets:** By compactness, these subsets of $2^\omega$ are just the finite unions of intervals.

**Functions:** If $f : 2^\omega \to 2^\omega$ is $\Delta^0_1$ or even $\Delta^0_1$ in a real parameter, then $f$ is continuous. Every continuous function is obtained in this way.

$\boxed{\Pi^0_1}$

**Closed sets:** The closed sets are exactly those which are $\Pi^0_1$ in a real parameter.

**Trees:** A $\Pi^0_1$ set is represented as the collection of infinite paths through a recursive subtree of $2^{<\omega}$. It's leftmost infinite path has recursively enumerable degree.

**Low Basis Theorem:** Jockusch and Soare (1972) showed that for every infinite recursive subtree $T$ of $2^{<\omega}$ there is a path $G$ in $T$ such that $G' \leq_T 0'$.

**Question:** Suppose that $T$ is a nonrecursive perfect subtree of $2^{<\omega}$. Is there a branch $A$ through $T$ such that $T \geq_T A >_T 0$?

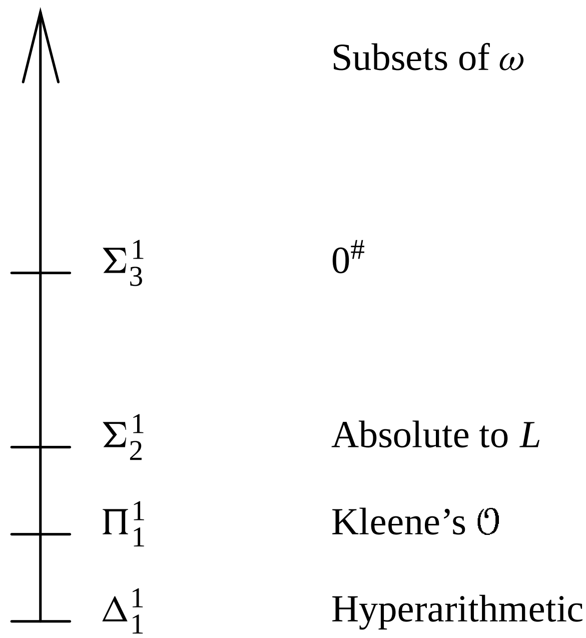


## Connections with models of second order arithmetic

There is a close connection between

- what can be constructed using a particular resource (recursion theory)
- and what can be proven assuming that resource is available (reverse math).

By adapting the proof of the low basis theorem, Harrington showed that any countable model of recursive comprehension ($RCA_0$) can be extended to a model of $WKL_0$ by adding reals. Consequently, every $\Pi^1_1$ consequence of $WKL_0$ follows from $RCA_0$.






## The Hyperarithmetic Hierarchy

Davis (1950) extended the arithmetic hierarchy into the transfinite by iterating the jump along recursive well-orderings of $\omega$. At limits $\lambda$, he used the recursive presentation of $\lambda$ to form the recursive join of the sets associated (by that presentation) to ordinals less than $\lambda$.

Spector (1955) showed that any two sets associated with the same ordinal have the same Turing degree. So we have robust notions of $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $\Delta^0_\alpha$.

For example, $\{(n, m) : m \in \emptyset^{(n)}\}$ (the first order theory of arithmetic) represents the $\omega$th iteration of the jump.

$$\boxed{\Delta^1_1}$$

Kleene (1955) showed that a subset of $\omega$ appears in the hyperarithmetic hierarchy if and only if it is $\Delta^1_1$. So the hyperarithmetic hierarchy, based on iteration of the jump, resolves the class of $\Delta^1_1$ subsets of $\omega$.

One of the most interesting aspects of this theory is the use of the fixed point theorem to define recursive functions as if by transfinite recursion.



$$\boxed{\Pi^1_1}$$

The canonical $\Pi^1_1$ subset of $\omega$ is $\mathcal{O}$, Kleene's system of notations for the recursive ordinals. It is complete among all $\Pi^1_1$ sets.

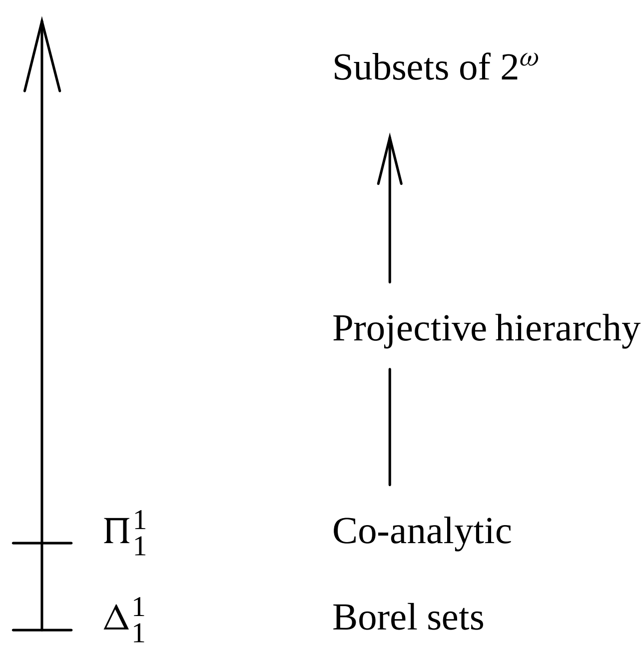
To really understand $\mathcal{O}$, one need only understand $L_{\omega_1^{CK}}$. $\mathcal{O}$ is equivalent to the existential theory of this structure.

## $\boxed{\Sigma^1_2 \text{ and } \Delta^1_3}$

**$\Sigma^1_2$:** In analyzing the $\Sigma^1_2$ sets, we are almost but not quite set theorists. Shoenfield (1961) showed that every $\Sigma^1_2$ relation is absolute to Gödel's $L$, in fact whether $A$ belongs to a $\Sigma^1_2$ set can be determine by examining $L_{\omega_1}(A)$.

**$\Delta^1_3$:** Silver's $0^\#$, the unique well-founded remarkable Ehrenfeucht-Mostowski set for $L$, is a $\Delta^1_3$ set. It's existence is tied to the existence of large cardinals and is the recursion theorist's passport to set theory.





## Classical Descriptive Set Theory

The sets of reals definable in second order arithmetic relative to real parameters were well known to the classical descriptive set theorists: $\Delta^1_1$ in a real is equivalent to Borel, $\Sigma^1_1$ in a real is equivalent to analytic.

The classical descriptive set theorists established a variety of topological regularity properties for the analytic sets: property of Baire, Lebesgue measurable, perfect set property.



## Basis Theorems for $\Sigma^1_1$ sets

- Gandy showed that every nonempty $\Sigma^1_1$ subset of $2^\omega$ has an element $G$ such that $\mathcal{O}^G \equiv_T \mathcal{O}$.
- Dually, Groszek and Slaman (1998) showed that if there is a nonconstructible real then every perfect set has a nonconstructible element, hence every uncountable analytic set does also.

$\boxed{\Sigma^1_2}$

We are now well into descriptive set theory,

The basis question for $\Sigma^1_2$ has a uniform answer which implies that every $\Sigma^1_2$ subset of $2^\omega$ has a $\Delta^1_2$ element.

The Kondô-Addison theorem asserts that for every $\Sigma^1_2$ relation $R(X, Y)$ on $2^\omega \times 2^\omega$, there is a $\Sigma^1_2$ function $f$ such that for all $X$, if there is a $Y$ such that $R(X, Y)$, then $R(X, f(X))$.



## The Wadge Hierarchy

Suppose that $A$ and $B$ are subsets of $2^\omega$. $B$ is Wadge reducible to $A$ ($A \geq B$) if and only if there is a continuous function $f$ such that for all $X \in 2^\omega$, $X \in B \iff f(X) \in A$.

**Wadge's lemma:** ($AD$) For every pair of sets $A$ and $B$, either $A \geq B$ or $2^\omega \setminus B \geq A$.

**Martin's theorem:** ($AD$) Let the Wadge degree of $A$ be the equivalence class generated from $A$ and $2^\omega \setminus A$ under $\leq$. The Wadge degrees are well-ordered by Wadge reducibility.

## Meta-theories of Definability

We have concentrated on the syntactic hierarchy for the definable subsets of $\omega$ and $2^\omega$.

Now we will look at attempts to represent aspects of definability abstractly.

- Degree structures
- Degree invariant functions



## Degree Structures

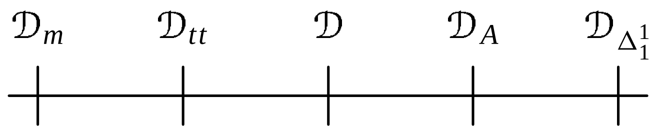
The Turing degrees $\mathcal{D}$ provide an algebraic representation of relative computability.

Surprisingly, the syntactic hierarchy is an invariant of $\mathcal{D}$.

**Defining the jump:** Slaman and Shore (1999) showed that the function $x \mapsto x'$ is first order definable in $\mathcal{D}$. The proof uses Slaman and Woodin's analysis of the automorphism group of $\mathcal{D}$.

**Defining relative recursive enumerability:** Is the relation *$y$ is recursively enumerable relative to $x$* definable in $\mathcal{D}$?

## The Spectrum of Degree Structures




homogeneous | rigid

many automorphisms

While the extreme points are well understood, the ones in the center and the transition from left to right are still mysterious.



## Variations on the Martin Conjecture

We have been cataloging syntactic forms and investigating the properties of the sets so defined. How can we know that this syntactic hierarchy is complete?

**Martin's conjecture** Consider the collection of functions from reals to reals which are invariant on Turing degree. Order these functions by pointwise domination on all sufficiently complicated reals. Conjecture: The Axiom of Determinacy implies that the nonconstant functions are pre-well-ordered with successor the Turing jump.

**Steel's theorem:** The Martin conjecture holds for uniformly degree invariant functions.

## Definable Closure Operators

A *closure operator* is a map $M : 2^\omega \to 2^{2^\omega}$ with the following properties.

1. For all $X \in 2^\omega$, $X \in M(X)$.
2. For all $X$ and for all $Z$, if $Z$ is recursive in finitely many elements of $M(X)$ then $Z \in M(X)$. $M(X)$ *is closed under relative computation.*
3. For all $X$ and $Y$ in $2^\omega$, if $X$ is recursive in $Y$ then $M(X) \subseteq M(Y)$. $M$ *is monotone.*

As in the Martin Conjecture, closure operators can be compared by eventual pointwise inclusion.



## The Borel Case

If $M$ is a closure operator such that the relation $Y \in M(X)$ is Borel, then one of the following conditions holds.

1. There is a countable ordinal $\alpha$ such that $M$ is equivalent to the map $X \mapsto \{Y : Y \text{ is recursive in } X^{(\alpha)}\}$.
2. There is a countable ordinal $\alpha$ such that $M$ is equivalent to the map $X \mapsto \{Y : (\exists \beta < \alpha)[Y \text{ is recursive in } X^{(\beta)}]\}$.
3. $M$ is equivalent to the map $X \mapsto 2^\omega$.

**Question:** Is something similar true under $AD$ for those functions $f$ such that for all $X$, $f(X) \subset L(X)$?