Aspects of the Turing Jump

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The Turing Jump

Definition 1 The *Turing Jump* is the function which maps a set $X \subseteq \omega$ to X', the halting problem relative to X.

$$X' = \left\{ e : \begin{array}{c} \text{The } e\text{th Turing machine} \\ \text{with oracle } X \text{ halts} \end{array} \right\}$$

The Turing degree of X' depends only on the Turing degree of X, so the jump induces a function on the Turing degrees $\mathcal{D}$.

Empirical Observation

The degree $0'$ is just as well represented by the set of validities in first order logic, the set of true existential statements in first order arithmetic, or any of a variety of other undecidable sets.

In fact, it seems that for every canonical recursively enumerable set W, if W is not recursive then W has degree $0'$.

Topics to be Discussed

- Properties of the jump and of the hierarchy of definability which it generates.
- The jump's intrinsic role within any hierarchy of definability.
- A purely degree theoretic characterization of the jump.
- Recursive enumerability—examined in similar terms.

Two Prototype Theorems

Theorem 2 (Friedberg Inversion, 1957) *Suppose that* $c \geq_T 0'$. *Then there is a* g *such that* $g' = c$.

Every sufficiently complicated degree is the jump of some degree.

Theorem 3 (Posner and Robinson Join, 1981) *Suppose that* $0 \not\geq_T x$. *Then there is a* g *such that* $x + g = g'$.

Every nontrivial degree is the jump *relative to* some other degree.

The Arithmetic Hierarchy

Definition 4 1. Σ^0_0 and Π^0_0 denote the collection of formulas in first order arithmetic in which every quantifier is bounded.

2. Σ^0_{n+1} denotes the collection of formulas in first order arithmetic of the form $(\exists n_1, \ldots, n_k)\psi$ in which $\psi \in \Pi^0_n$.

3. Π^0_{n+1} denotes the collection of formulas in first order arithmetic of the form $(\forall n_1, \ldots, n_k)\psi$ in which $\psi \in \Sigma^0_n$.

Definition 5 1. $X \subseteq \omega$ is Σ^0_n or Π^0_n if and only if it is definable in arithmetic by a formula in the corresponding class.

2. $X \subseteq \omega$ is Δ^0_n if and only if it is both Σ^0_n and Π^0_n.

Correlation with the Jump

Theorem 6 (Post) *1. (a) X is Δ^0_1 if and only if it is recursive.*

(b) X is Σ^0_1 if and only if it is recursively enumerable.

2. (a) X is Δ^0_{n+1} if and only if it is recursive in $\emptyset^{(n)}$.

(b) X is Σ^0_{n+1} if and only if it is recursively enumerable in $\emptyset^{(n)}$.

The Hyperarithmetic Hierarchy

Davis (1950) extended the arithmetic hierarchy into the transfinite by iterating the jump along recursive wellorderings of ω. At limits λ, he used the recursive presentation of λ to form the recursive join of the sets associated (by that presentation) to ordinals less than λ.

Spector (1955) showed that any two sets associated with the same ordinal have the same Turing degree. So we have robust notions of Σ^0_α, Π^0_α, and Δ^0_α.

Example 7 $\{(n, m) : m \in \emptyset^{(n)}\}$ (the first order theory of arithmetic) represents the ωth iteration of the jump.

Correlation with Second Order Arithmetic

Kleene (1955) showed that a subset of ω appears in the hyperarithmetic hierarchy if and only if it is Δ^1_1. That is, it and it's complement are definable in second order arithmetic by formulas of the form $(\exists X \subseteq \omega)\psi$, where ψ is Π^0_2 relative to X.

So the hyperarithmetic hierarchy, based on iteration of the jump, resolves the class of Δ^1_1 subsets of ω.

Jockusch–Shore REA-Operators

Definition 8 (Jockusch and Shore, 1984) An *REA-operator* is a function $J : 2^\omega \to 2^\omega$ such that there is an e such that for all X, $J(X)$ is the join of X with the eth set which is recursively enumerable relative to X. $J(X)$ is **R**ecursively **E**numerable in and **A**bove X.

An α-REA-operator is an α-length iteration of REA-operators.

Example 9 The canonical 2-REA operator is the map $A \mapsto A''$.

Extensions of the Prototypes

Theorem 10 (Jockusch and Shore Inversion, 1984) *Suppose that $C \geq_T \emptyset^{(\alpha)}$ and J is an α-REA operator. There is a G such that $J(G) \equiv_T C$.*

Every sufficiently complicated degree is in the range of J.

Theorem 11 (Shore and Slaman Join, 1999) *Suppose that J is an α-REA operator and for all $\beta < \alpha, \emptyset^{(\beta)} \not\geq_T X$. There is a G such that $X \oplus G \equiv_T J(G)$.*

Every nontrivial set represents the value of J *relative to* some set.

On the Proof of the Posner–Robinson Join Theorem

Suppose that $\emptyset \not\geq_T X$. Posner and Robinson constructed a set G with the following properties.

- Every atomic statement about G' is decided by a Cohen condition on G.
- The recursion used to construct G is recursive in $X \oplus G$.

The recursive enumerability of $\emptyset'$ is used in the recovery of the construction of G from $X \oplus G$.

Kumabe–Slaman Forcing

Kumabe and Slaman introduced the following notion of forcing in their join theorem for the ω-jump.

- A *condition* is a pair $p = (\Phi_p, \boldsymbol{X}_p)$ consisting of a finite description of a partial Turing functional and a finite subset of 2^ω.
- q *extends* p, if Φ_q adds values to Φ_p, but not relative to any of the elements of $\boldsymbol{X}_p$, and $\boldsymbol{X}_p \subseteq \boldsymbol{X}_q$.

On the Proof of the Shore–Slaman Join Theorem

Suppose that for all β less than α, $\emptyset^{(\beta)} \not\geq_T X$. By applying the Jockusch-Shore Inversion Theorem, it is enough to build a Φ with the following properties.

- Every atomic statement about $\Phi^{(\alpha)}$ is decided by a Kumabe-Slaman condition on Φ
- $\Phi(X) = \Phi^{(\alpha)}$. Hence, $X \oplus \Phi \geq_T \Phi^{(\alpha)}$.

Compactness, in the form of the Jockusch and Soare (1972) Low Basis Theorem, provides the conditions to decide atomic statements about $\Phi^{(\alpha)}$ without deciding any additional values of $\Phi(X)$.

Summary

- Iteration of the jump generates a proper hierarchy which stratifies the class of Δ^1_1 subsets of ω.
- For every J in the REA hierarchy of functions,
  - every sufficiently complicated set has Turing degree in the range of J,
  - for every nontrivial set X, there is a G such that $X \oplus G$ is equivalent to $J(G)$.

Skepticism

- Is the jump intrinsic?
  - Is there a canonical way to produce nonrecursive sets and avoid the Turing jump?
  - Is there another hierarchy for the Δ^1_1 sets?
- Is the jump degree theoretic?
  - Is the function $x \mapsto x'$ first order definable in $\mathcal{D}$?

Is It Intrinsic?

How could one argue that the hyperarithmetic hierarchy is *not intrinsic*?

Excellent. Give an alternate development of the continuum, and natural models of the metamathematics on which that development is based. Prove that for each $X \subseteq \omega$ the sets of reals in the models associated with X give an alternate stratification of the $\Delta^1_1(X)$ subsets of ω.

Good. Give an alternate method which, for each $X \subseteq \omega$, stratifies the $\Delta^1_1(X)$ subsets of ω, but without an accompanying semantics.

Poor. Construct a function which avoids the hyperarithmetic hierarchy, maps each X to a proper subset $M(X)$ of $\Delta^1_1(X)$, and resembles mapping X to a collection of sets definable from X.

Abstract Closure Operators

Definition 12 A *closure operator* is a map $M : 2^\omega \to 2^{2^\omega}$ with the following properties.

1. For all $X \in 2^\omega$, $X \in M(X)$.
2. For all X and for all Z, if Z is recursive in finitely many elements of $M(X)$ then $Z \in M(X)$. $M(X)$ *is closed under relative computation.*
3. For all X and Y in 2^ω, if X is recursive in Y then $M(X) \subseteq M(Y)$. M *is monotone.*

The functions mapping X to the collection of subsets of ω recursive in X, recursive in X', or arithmetically definable from X are closure operators.

Comparing Closure Operators

Definition 13 If M_1 and M_2 are closure operators, then $M_1 \leq M_2$ if there is a set of natural numbers B, such that for all X, if B is recursive in X, then $M_1(X) \subseteq M_2(X)$.

In other words, $M_1 \leq M_2$ if for all sufficiently complicated X, $M_1(X) \subseteq M_2(X)$. Say M_1 and M_2 are *equivalent* if for all sufficiently complicated X, $M_1(X) = M_2(X)$.

Remark 14 This notion is a $2^\omega \to 2^{2^\omega}$ variation on Martin's ordering of degree invariant functions on reals.

The Hyperarithmetic Hierarchy is Intrinsic

Theorem 15 *If M is a closure operator such that the relation $Y \in M(X)$ is Borel, then one of the following conditions holds.*

1. *There is a countable ordinal α such that M is equivalent to the map* $X \mapsto \{Y : Y \text{ is recursive in } X^{(\alpha)}\}$.
2. *There is a countable ordinal α such that M is equivalent to the map* $X \mapsto \{Y : (\exists \beta < \alpha)[Y \text{ is recursive in } X^{(\beta)}]\}$.
3. *M is equivalent to the map $X \mapsto 2^\omega$.*

Proof for a Special Case: The Jump

(Slaman and Steel, 1988) Suppose that M is a Borel closure operator such that for all X, $F(X)$ has an element which is not recursive in X.

1. Let A be a nonrecursive element of $F(\emptyset)$.
2. By the Posner–Robinson Join Theorem, fix G so that $A \oplus G \equiv_T G'$.
3. Then $A \in F(\emptyset) \subseteq F(G)$, $G \in F(G)$, and so $A \oplus G \equiv_T G' \in F(G)$.
4. Relativizing (1)–(3) yields, for arbitrarily complicated G, $G' \in F(G)$.
5. Apply Martin's (1975) Borel Determinacy to conclude that for all sufficiently complicated X, $X' \in F(X)$.

Is It Degree Theoretic?

- Cooper (1990) claimed that the jump $x \mapsto x'$ is first order definable in $\mathcal{D}$. Cooper's proof relied on the claim that there is a Lachlan-style priority construction to produce a d-r.e. degree with a nonsplitting property.
- Shore and Slaman (2000) proved that there is no d-r.e. degree as claimed. Shore and Slaman (1999) provided a different route to the definability of the jump, described in the next few slides.
- Cooper (2000) claimed that a variation on the 1990 approach can be used to define the jump.

We will say more about the d-r.e. approach when we turn from the jump to the relation of relative recursive enumerability.

Automorphisms of $\mathcal{D}$

Consider this mathematically specious but plausible argument: Suppose that π is an automorphism of $\mathcal{D}$.

Fact 1. (Jockusch and Shore, 1984) For every x, $\pi(x)$ is arithmetically definable relative to x.

Plausibility 2. There is one arithmetic definition Π, such that for all generic $G \in 2^\omega$, $\Pi(G)$ is an element of $\pi(degree(G))$.

Fact 3. (Jockusch and Posner, 1981) The degrees of any comeager subset of 2^ω generate $\mathcal{D}$ by meet and join.

Plausible conclusion. π is induced by an arithmetic function on 2^ω.

Slaman and Woodin Analysis of $\mathcal{D}$

Theorem 16 (Slaman and Woodin, 1990) *Suppose that π is an automorphism of $\mathcal{D}$.*

1. *π is represented by an arithmetic function on 2^ω. (Hence the automorphism group of $\mathcal{D}$ is countable.)*

2. *π is the identity on all degrees greater than or equal to $0''$.*

Definability in General and in Particular

Theorem 17 (Slaman and Woodin, 1990) *$\mathcal{D}$ is rigid if and only if every degree invariant relation on 2^ω which is definable in second order arithmetic is definable in $\mathcal{D}$.*

Question 18 Is there a nontrivial automorphism of $\mathcal{D}$?

Cooper has claimed that the answer is yes.

Theorem 19 (Slaman and Woodin, 1990) *The function $x \mapsto x''$ is definable in $\mathcal{D}$.*

Defining the Turing Jump

Theorem 20 (Shore and Slaman, 1999) $0'$ *is defined in* $\mathcal{D}$ *as the greatest degree* z *such that there is no* g *such that* $z + g = g''$.

Prove Theorem 20 by combining Shore-Slaman Join Theorem with the definability of the double-jump.

Theorem 21 (Shore and Slaman, 1999) *For each* x, x' *is defined from* x *in* $\mathcal{D}$ *as the greatest degree* z *such that there is no* $g \geq_T x$ *such that* $z + g = g''$.

Recursive Enumerability

The relation W *is recursively enumerable in* X generates a finer hierarchy of degrees.

Definition 22 (Ershov, 1968) The *Difference Hierarchy*:

1. A set X is n-recursively enumerable if it has a recursive approximation $X(k) = \lim_{s\to\infty} \psi(k, s)$ such that there are at most n many numbers s such that $\psi(k, s+1) \neq \psi(k, s)$.

2. X is α-recursively enumerable relative to a recursive system S of notations for α if and only if there is a partial recursive function ψ such that for all k, $X(k)$ is equal to $\psi(k, z)$, where z is the S-least notation b such that $\psi(k, b)$ converges.

The 2-r.e. sets are also called the d-r.e. sets.

Correlation with Δ^0_2

Jockusch and Shore (1984) showed that every α-r.e. set is α-REA. The converse fails, as every α-re set is Δ^0_2.

Theorem 23 (Ershov, 1968) *For every path S through $\mathcal{O}$, Kleene's complete set of notations for recursive ordinals, and for every set $X \in \Delta^0_2$, X is α-recursively enumerable relative to S.*

So the difference hierarchy resolves the class of Δ^0_2 subsets of ω.

Inversion and Join

Inversion. The Jockusch-Shore Inversion Theorem for α-REA operators applies to α-r.e. operators.

Join. It is not possible to strengthen the join theorem to apply to the sets whose degrees are not recursively enumerable.

Theorem 24 *There is a 2-r.e. operator* $D : Z \mapsto D(Z) \geq_T Z$ *and a 2-r.e. set* X *with the following properties.*

1. X *is not of recursively enumerable degree.*
2. *For all* $G \subseteq \omega$, $X \oplus G \not\equiv_T D(G)$.

Skepticism

- Is it intrinsic?
  - Is there a canonical way to produce nonrecursive Δ^0_2 sets which avoids recursive enumerability?
  - Is there another hierarchy for the Δ^0_2 sets?
- Is it degree theoretic?
  - Is the relation w *is recursively enumerable in* x first order definable in $\mathcal{D}$?

Σ-Closure Operators

Definition 25 A Σ-*closure operator* is a map $M : 2^\omega \to 2^{2^\omega}$ with the following properties.

1. For all $X \in 2^\omega$, $X \in M(X)$.
2. For all $X \in 2^\omega$ and all $\{Z_1, \ldots, Z_k\} \subseteq M(X)$, every set which is Turing equivalent to the join $\bigoplus_{i \le k} Z_i$ is an element of $M(X)$.
3. For all X and Y in 2^ω, if X is recursive in Y then $M(X) \subseteq M(Y)$.

The maps sending X to the collection of sets with degree recursively enumerable in X or the collection of sets with degree 2-r.e. in X are Σ-closure operators which are not closure operators.

REA is Intrinsic to Not Recursive

Theorem 26 *For any Borel* Σ*-closure operator* M*, if there is a cone of* X*'s for which* $M(X) \not\subseteq \Delta^0_1(X)$*, then there is a cone of* X*'s such that* $M(X)$ *contains all of the sets which are REA in* X*.*

Shore has pointed out that the REA and difference hierarchies give different resolutions of Δ^0_2, so there is no classification as simply presented as the one for full closure operators.

Is it Degree Theoretic?

Cooper (2000) claimed that the relation *w is recursively enumerable relative to x* is definable in $\mathcal{D}$, but his proof relied on a join principle for 2-r.e. operators which is contradicted by Theorem 24. Cooper has since claimed to use additional properties of specific 2-r.e. operators to circumvent this problem.

Question 27 Is the relation *w is recursively enumerable relative to x* first order definable in $\mathcal{D}$?