Three Theorems Between Computability and Diophantine Approximation

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Statements of Results

Émile Borel (1909): Normal Numbers

Definition

Let ξ be a real number.

- ξ is *simply normal to base b* iff in its base-b expansion, $(\xi)_b$, each digit appears with limiting frequency equal to $1/b$.
- ξ is *normal to base b* iff for all k, ξ is simply normal to base b^k.
- ξ is *absolutely normal* iff it is normal to every base b.

Normality to Different Bases

There is one readily-identified connection between normality to one base and normality to another.

Definition

For natural numbers b_1 and b_2 greater than 0, we say that b_1 and b_2 are *multiplicatively dependent* if they have a common power.

Theorem (Maxfield 1953)

If b_1 and b_2 are multiplicatively dependent bases, then, for any real ξ, ξ is normal to base b_1 iff it is normal to base b_2.

Multiplicative independence

An early result due to Cassels (1959) is that almost every element of the Cantor Middle-Third Set is normal to every base which is multiplicatively independent of 3.

Theorem (Schmidt 1961/62)

Let R be a subset of the natural numbers greater than or equal to 2 which is closed under multiplicative dependence. There is a real number ξ such that ξ is normal to every base in R and not normal to any base in the complement of R.

Remark

Suppose that ξ is a computable real number. Then, $\{b : \xi \text{ is normal to base } b\}$ is a Π^0_3 subset of $\mathbb{N}$.

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Suppose that ξ is a computable real number. Then, $\{b : \xi$ is normal to base $b\}$ is a Π^0_3 subset of $\mathbb{N}$.

The proof is by inspection: ξ is normal to base b iff

for every k and every rational number $r > 0$,
  there exists ℓ_0,
    for all $\ell > \ell_0$,
      the discrepancy in the first ℓ digits of the base b^k representation of ξ is less than r.

Logical Independence Between Bases.

Let $\mathcal{S}$ be the set of minimal representatives of the multiplicative dependence classes.

Theorem (Becher and Slaman 2013)

Let R be a Π^0_3 subset of $\mathcal{S}$. There is a real ξ such that ξ is normal to every base in R and not simply normal to any of the other elements of $\mathcal{S}$. Furthermore, ξ is computable uniformly in the Π^0_3 formula which defines R.

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An index set calculation:

Theorem (Becher and Slaman 2013)

The set of real numbers that are normal to at least one base is Σ^0_4-complete.

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The set of real numbers that are normal to at least one base is Σ^0_4-complete.

A fixed point:

Theorem (Becher and Slaman 2013)

For any Π^0_3 formula φ there is a computable real ξ such that for all b in $\mathcal{S}$, ξ is normal to base b iff $\varphi(\xi, b)$ is true.

An Application

Definition

A real number ξ is *rich* in base b iff every finite sequence of base-b digits appears as a block in the base-b representation of ξ.

Corollary

There is a computable real number ξ such that for every $b \in \mathcal{S}$, ξ is rich in base b and not normal in base b. In other words, ξ is an absolutely-abnormal lexicon.

Proof.

By the previous theorem, take ξ such that for all $b \in \mathcal{S}$,

$$\xi \text{ is normal in base } b \Leftrightarrow \xi \text{ is not rich in base } b.$$

□

Simple Normality

What is the version of Schmidt's Theorem for simple normality?

Necessary Conditions:

Theorem

For any base b and real number ξ, the following hold.

- *For any positive integers k and n, if ξ is simply normal to base b^{kn} then ξ is simply normal to base b^n.*
- *(Long 1957) If there are infinitely many positive integers m such that ξ is simply normal to base b^m, then ξ is simply normal to all powers of b.*

Simple Normality

What is the version of Schmidt's Theorem for simple normality?

Theorem (Hertling 2002)

Suppose that $s \in \mathcal{S}$ is not a perfect power and that m and n are such that n is not a divisor of m. There is a real number ξ such that ξ is simply normal to base s^m and not simply normal to base s^n.

Simple Normality

What is the version of Schmidt's Theorem for simple normality?

Necessary and Sufficient Conditions:

Theorem (Becher, Bugeaud and Slaman 2013)

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integers k and m, if $b^{km} \in M$ then then $b^m \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Part 2

Ingredients in the Proofs

Heuristics

Directly construct a real number ξ with the desired property. Go by recursion on stages t, at stage t define a closed interval $I_t \subset I_{t-1}$, and $\{\xi\} = \bigcap_t I_t$.

Intuitively, I_t specifies more digits than I_{t-1}.

Convergence to normality:

- For a finite set of bases $r_1, \ldots, r_j$, ensure that the digits determined by I_t beyond those determined by I_{t-1} have discrepancy less than ϵ_t.
  - The number of new digits cannot be too large compared to the number of digits already determined, so that the convergence to normal for bases $r_1, \ldots, r_j$ is not disturbed.
    - *This is an invariance condition:* The number of digits needed to converge to within ϵ_t of normal should not depend on the number of digits already determined.
  - The number of iterations for $r_1, \ldots, r_j$ and ϵ can be large enough for the cumulative effect on the discrepancy of an initial segment of the digits in each base r_i to be on the order of ϵ_t.

Heuristics

Directly construct a real number ξ with the desired property. Go by recursion on stages t, at stage t define a closed interval $I_t \subset I_{t-1}$, and $\{\xi\} = \bigcap_t I_t$.

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Bias away from simple normality:

- For a base s, ensure that the digits determined by I_t beyond those determined by I_{t-1} have discrepancy greater than ϵ_s, where ϵ_s depends on the base s and not on the stage t.
  - The number of iterations for s and ϵ_s must be large enough for the cumulative effect on the discrepancy of an initial segment of the digits in each base s to be on the order of ϵ_s.

Heuristics

Directly construct a real number ξ with the desired property. Go by recursion on stages t, at stage t define a closed interval $I_t \subset I_{t-1}$, and $\{\xi\} = \bigcap_t I_t$.

Intuitively, I_t specifies more digits than I_{t-1}.

Existence of such extensions:

- At each stage, find an appropriate subinterval I_t of I_{t-1} by considering the initial segment of a the base s expansion of a random element in an appropriate Cantor-like subfractal of I_{t-1}.
  - Show that the properties of normality for random elements of the subfractal are invariant under choice of enclosing interval.

Heuristics

Directly construct a real number ξ with the desired property. Go by recursion on stages t, at stage t define a closed interval $I_t \subset I_{t-1}$, and $\{\xi\} = \bigcap_t I_t$.

Intuitively, I_t specifies more digits than I_{t-1}.

Overall construction:

- ϵ_t goes to 0 as t goes to infinity.
- Every base r for which ξ is supposed to be simply normal is eventually included in the list $r_1, \ldots, r_j$.
- Every base s for which ξ is supposed not to be simply normal is revisited infinitely often.

Specific Constructions

- To obtain an absolutely normal number, use Lebesgue measure to obtain convergence to normal.
- To obtain the logical theorems, combine these ingredients dynamically so as to reflect the satisfaction of an arbitrary Π^0_3 statement.
- To obtain Schmidt's Theorem, a number normal to all elements of R and not normal to any elements of S, use either the fractal that appeared in Schmidt's original proof or that used in a later adaptation by Pollington. Namely, for K appropriately large, use only the digits 0 and 1 in base s^K or omit the last digit or 2 in base s^K.
  - Schmidt's estimates apply in an invariant way.

Simple Normality

Recall Hertling's theorem.

Theorem (Hertling 2002)

Suppose that $s \in \mathcal{S}$ and that m and n are such that n is not a divisor of m. There is a real number ξ such that ξ is simply normal to base s^m and not simply normal to base s^n.

Simple Normality

Bugeaud gave an elegant proof of it.

- Consider two base s strings σ and τ of length nm: σ has an initial 1 and then is identically 0 and τ has a 1 at place $m+1$ and is 0 elsewhere.
  - Parsed as a set of size m blocks, they are equivalent.
  - Parsed as a set of size n blocks, they are not equivalent.
- Let γ be the sequence in base s obtained by concatenating all the base s sequences of length nm and then replacing the occurence of σ with an occurence of τ.
  - γ is perfectly normal in base s^m and biased in base s^n.
- Consider the fractal in base $s^{|\gamma|}$ obtained by omitting the digit corresponding to γ.
  - A random element will be simply normal in base s^m and not simply normal in base s^n.

Simple Normality

Now suppose that we have $s \in \mathcal{S}$, a finite set of numbers M and a number n such that n does not divide any element of M.

Definition

M and n are *fair* iff every residue class mod n is equally represented within M.

We may assume fairness.

Theorem (Haiman)

There are more sums of an even number of elements from M that are equivalent to 0 mod n than there are sums of odd numbers of elements from M that are equivalent to 0 mod n.

Simple Normality

- For appropriate ℓ, consider two base s strings σ and τ: σ is the concatenation of all length ℓ strings which are identically 0 except for a 1 at the location of the sum of an even number of elements of M and τ is the same for sums of odd numbers of elements.
  - For m in M, when parsed as a set of size m blocks, σ and τ are equivalent.
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  - For m in M, when parsed as a set of size m blocks, σ and τ are equivalent.
  - Parsed as a set of size n blocks, they are not equivalent.
- Continue as in Bugeaud's argument to produce a string γ that is perfectly normal in bases s^m, for $m \in M$, and biased in base s^n and then produce the fractal which omits digit γ.
  - A random element of this fractal will be
    - simply normal in all bases s^m for $m \in M$ and not simply normal in base s^n,
    - normal in all bases which are multiplicatively independent to s.
  - Further, the convergence to normal is invariant in the earlier sense.

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    - normal in all bases which are multiplicatively independent to s.
  - Further, the convergence to normal is invariant in the earlier sense.
- Follow the heuristic pattern.

Part 3

Speculation and Work in Progress

Abnormal Distributions of Digits

Definition

A real number ξ is *absolutely base stable* iff for every base s and every digit $d \in \{0, \ldots, s-1\}$, the base s representation of ξ has a limiting frequency for the digit d.

- Let ξ be absolutely base stable. When s is a base and d is a digit in base s, $Freq(d, s, \xi)$ is the asymptotic frequency of the digit d in the base s representation of ξ.

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- Let ξ be absolutely base stable. When s is a base and d is a digit in base s, $Freq(d, s, \xi)$ is the asymptotic frequency of the digit d in the base s representation of ξ.

Example

- Absolutely normal numbers are absolutely base stable.
- As mentioned earlier, almost every element of the Cantor Middle-Third Set is absolutely base stable.
- The examples that we gave earlier to exhibit independence of normality with respect to multiplicatively-independent bases were not absolutely base stable.

Abnormal Distributions of Digits

Question

What are the possibilities for the functions $Freq(d, b, \xi)$ for absolutely base stable numbers ξ?

Base Stability and Discrepancy

Definition

Suppose that ξ is absolutely base stable. $Disc(s, \xi)$, the discrepancy of ξ relative to base s, is the supremum over all digits d in base s of $|Freq(d, s, \xi) - 1/s|$.

Note, $Disc(s, \xi)$ is less than or equal to $1 - 1/s$.

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Note, $Disc(s, \xi)$ is less than or equal to $1 - 1/s$.

Theorem (Becher, Reimann and Slaman (work in progress))

Suppose that $D : \mathcal{S} \to \mathbb{R}$ so that the following conditions hold.

- $\forall s \in \mathcal{S}\big(D(s) \in [0, 1 - 1/s]\big)$
- $\displaystyle\sum_{s \in \mathcal{S}} D(s) \cdot \frac{1}{1 - 1/s} \leq 1$

Then, there is an absolutely base stable number ξ such that for all $s \in \mathcal{S}$, $Disc(s, \xi) = D(s)$.

The condition $\sum_{s \in \mathcal{S}} D(s) \cdot \frac{1}{1-1/s} \leq 1$

We construct ξ according to our heuristic.

- $D(s) \cdot \frac{1}{1-1/s} \leq 1$ is the fraction of time sspent biasing the base s digits toward all zero.
  - Use Schmidt's original fractal with only digits 0 and 1 in base s^K, for K large.
- $\sum_{s \in \mathcal{S}} D(s) \cdot \frac{1}{1-1/s} \leq 1$ is the condition that the sum of the fractions of time spent biasing expansions is less than or equal to 1.
  - Unallocated time is filled with normality.

We do not have a method to bias the digits for two multiplicatively independent bases simultaneously.

Furstenberg's $2\times, 3\times$ (mod 1) conjecture

Conjecture (Furstenberg 1967)

Suppose that μ is a continuous measure on $[0, 1]$ and that μ is invariant under multiplication by 2 (mod 1) and by multiplication by 3 (mod 1). Then μ is Lebesgue measure.

Furstenberg's conjecture and its supporting evidence suggest that there is no notion of randomness which implies biases for representations in bases 2 and 3 simultaneously and which is suitable for constructions of the type we have discussed.

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Question

Is there an absolutely base stable irrational real number ξ such that ξ has discrepancy $1/2$ in base 2 and discrepancy $2/3$ in base 3?