Higher Orders of μ-Randomness

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The Motivating Question

Question (Reimann)

For which sequences $X \in 2^\omega$ does there exist (a presentation of) a measure μ such that X is 1-random for μ?

It is natural to ask the same question with 1-random replaced by other degrees of randomness and to ask it for measures with particular properties.

We begin with a review of what is known about the 1-random case, which is now understood reasonably well. Then, we will discuss the fragmentary results for *n*-randomness for μ.

Basic Definitions

Definition

A *representation* m of a probability measure μ on 2^ω provides, for each $\sigma \in 2^{<\omega}$ and each k, $m_1(\sigma, k)$ and $m_2(\sigma, k)$ such that $m_2(\sigma, k) - m_1(\sigma, k) < 1/2^k$ and $\mu([\sigma]) \in [m_1, m_2](\sigma, k)$.

Definition

$X \in 2^\omega$ is n-random relative to a representation m of μ if and only if X passes every Martin-Löf test relative to $m^{(n-1)}$, in which the measures of the open sets of the test are evaluated using μ.

In what follows, we will speak of an X's being n-random for μ and leave it as understood that this means relative to a representation of μ. Similarly, we will speak of computing relative to μ, taking the Turing jump of μ, and so forth.

1-Randomness

remarks on constructing measures

Definition

For X, Y, and Z in 2^ω, we write $X \equiv_{T,Z} Y$ ($X \equiv_{tt,Z} Y$) to indicate that there are (total) functionals Φ and Ψ which are recursive in Z such that $\Phi(X) = Y$ and $\Psi(Y) = X$.

Lemma (in the style of of Demuth, Kautz, and Levin-Zvonkin)

For X and Z in 2^ω, the following conditions are equivalent.

- *There is an R such that R is n-random relative to Z and $X \equiv_{tt,Z} R$.*
- *There is a continuous measure μ which is recursive in Z such that X is n-random for μ.*

1-Randomness

Theorem (Reimann and Slaman)

For $X \in 2^\omega$, the following are equivalent.

- *X is not recursive.*
- *There is a measure μ such that $\mu(X) \neq 0$ and X is 1-random relative to μ.*

Theorem

- (Reimann and Slaman) *If X is not Δ^1_1, then there is a continuous measure μ such that X is 1-random relative to μ.*
- (Kjos-Hanssen and Montalban) *If X is an element of a countable Π^0_1 subset of 2^ω, then X is not 1-random relative to any continuous μ.*

1-Randomness

remarks on constructing measures

To prove X not recursive implies X is 1-random relative to some μ, we construct the measure as follows.

- Use a theorem of Posner and Robinson to find a G such that $X \oplus G \equiv_T G'$.
- Use Kučera's Theorem to find an R which is 1-random relative to G for which $G' \equiv_T R \oplus G$.
- Now that $X \equiv_{T,G} R$ for R a 1-random relative to G, use a compactness argument on the space of measures to obtain μ.

Note, μ can be obtained recursively in X''.

1-Randomness

remarks on constructing measures

To prove X not Δ^1_1 implies X is 1-random relative to some continuous μ, we construct the measure as follows.

- Use a theorem of Woodin to find a G such that $X \oplus G \equiv_{tt} G'$.
- Use Kučera's Theorem again to find an R which is 1-random relative to G for which $G' \equiv_T R \oplus G$.
- Now that $X \equiv_{T,G} R$ for R a 1-random relative to G with a tt-reduction in the direction from X to R, use the previous compactness argument to obtain a continuous μ.

In Woodin's construction, finding G from X requires the Turing jump of $\mathcal{O}^X$, the complete $\Pi^1_1(X)$ subset of ω. There are examples in which $X \notin \Delta^1_1$ yet there is no continuous μ in $\Delta^1_1(X)$ such that X is 1-random for μ.

Higher orders of randomness

basic observations

From this point on, we restrict ourselves to continuous measures.

Fact (Well-known)

Suppose that $n > 1$ and X is n-random for μ.

- μ' *is not recursive in* X.
- *Every function recursive in* X *is dominated by a function recursive in* μ'.

Higher orders of randomness

NR_n

Definition

Let NR_n be the set of X's for which there is no (continuous) measure μ such that X is n-random for μ.

Theorem

For every n, NR_n is countable.

As in the analysis of relative 1-randomness, we will show that every element of NR_n is definable. However, and this is a flaw in our method, as n increases the levels of definability involve unboundedly many iterations of the power set applied to ω.

Our proof is not sensitive to the value of n, so we take $n = 2$.

Higher orders of randomness

a cone of Turing degrees disjoint from NR_2

Lemma

There is a $B \in 2^\omega$, such that $X \geqslant_T B$ implies $X \notin NR_2$.

Proof

A Borel subset of $\neg NR_2$. Suppose $Z \in 2^\omega$, R is 3-random relative to Z, and $X \equiv_{T,Z} R$. Then, $X \equiv_{tt,Z'} R$, R is 2-random relative to Z', and so X is 2-random relative to some continuous measure.

$\neg NR_2$ contains a cone in $\mathcal{D}$. By the above, $\neg NR_2$ contains the cofinal and degree-invariant set

$$\{Y : \exists Z \exists R(R \text{ is 3-random in } Z \text{ and } Y \equiv_T Z \oplus R).\}$$

This set is clearly cofinal in $\mathcal{D}$. By Borel Determinacy, it contains a cone in $\mathcal{D}$.

Higher orders of randomness

an observation about Borel Determinacy

- Martin's proof of Borel Determinacy starts with a description of a Borel game and produces a winning strategy for one of the players.
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the strategy.
- The absoluteness of Π^1_1 sentences between well-founded models and the direct nature of Martin's proof imply that if G is a real parameter used to define a Borel game, then the winning strategy for that game belongs to the smallest $L_\beta[G]$ such that $L_\beta[G]$ is a model of a sufficiently large subset of ZFC.

To keep things simple, we will work with models of ZFC and assume that there is a well-founded model of ZFC. Let L_β be the smallest well-founded model of ZFC. Note, L_β is countable.

Higher orders of randomness

a join theorem

Lemma

Suppose that $X \notin L_\beta$. Then there is a G such that

- *$L_\beta[G]$ is a model of ZFC.*
- *Every element of $2^\omega \bigcap L_\beta[G]$ is recursive in $X \oplus G$.*

Proof.

Use Kumabe–Slaman forcing P to generically extend L_β. This forcing builds a functional Φ_G by finite approximation. The definability of forcing and compactness show that if $D \in L_\beta$ is dense and $p \in P$, then there is a q in D extending p such that q makes no additional commitments about $\Phi_G(X)$.

Higher orders of randomness

a join theorem

Thus, for each term τ in the forcing language and each $n \in \omega$, it is possible to decide $n \in \tau$ and then extend our commitment on $\Phi_G(X)$ to record this decision.

We construct G in ω-many steps so that G is P-generic for L_β and so that $\Phi_G(X)$ records what is forced during our construction. $\square$

Higher orders of randomness

Kumabe-Slaman forcing in detail

- The elements p of the forcing partial order P are pairs $(\Phi_p, \overrightarrow{X}_p)$ in which Φ_p is a finite use-monotone functional and $\overrightarrow{X}_p$ is a finite subset of 2^ω.
- If p and q are elements of P, then $p \geqslant q$ if and only if
  - $\Phi_p \subseteq \Phi_q$ and for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
  - $\overrightarrow{X}_p \subseteq \overrightarrow{X}_q$,
  - for every x, y, and $X \in \overrightarrow{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

Higher orders of randomness

$NR_2 \subseteq L_\beta$.

Corollary

$NR_2 \subseteq L_\beta$. Hence, NR_2 is countable.

Proof

Suppose $X \notin L_\beta$ and apply the previous lemma to obtain a G such that $L_\beta[G]$ is a model of ZFC and every element of $2^\omega \bigcap L_\beta[G]$ is recursive in $X \oplus G$.

Relativize the discussion of NR_2 to G. Relative to G, X belongs to every cone with base in $L_\beta[G]$. In particular, X belongs to the cone avoiding NR_2 relative to G.

Thus, there is a continous measure μ such that X is 2-random for μ relative to G.

But then, X is 2-random for a continuous μ, as required.

Higher orders of randomness

obtaining μ from X

Given $X \notin L_\beta$, we showed that there is a continuous μ such that X is 2-random for μ. We can define such a μ using X and a presentation of the elementary diagram of L_β as a countable model.

Question

Is it provable in analysis that for all k, NR_k is countable?

Higher orders of randomness

a complicated member of NR_3

Theorem

$\mathcal{O}$, the complete Π^1_1 subset of ω, is an element of NR_3.

Proof.

One representation of $\mathcal{O}$ is the following.

$$\mathcal{O} = \{e : \text{The } e\text{th recursive subtree } T_e \text{ of } \omega^{<\omega} \text{ is well-founded.}\}$$

For a contradiction, suppose that μ is given so that $\mathcal{O}$ is 3-random for μ.

Higher orders of randomness

a complicated member of NR_4

Then, every function recursive in $\mathcal{O}$ is dominated by one recursive in μ'.

Hence, μ' computes a uniform family of functions $(f_e : e \in \omega)$ such that for each e, f_e dominates the left-most infinite path in T_e.

Then, for each e, compactness implies that the following conditions are equivalent.

- T_e is well-founded.
- The subtree of T_e to the left of f_e is finite.

The second condition is $\Pi^0_1(\mu')$. But, no 3-random for μ can be $\Pi^0_2(\mu)$. □

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