Higher Orders of $\mu$-Randomness

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The Motivating Question

Question (Reimann)

For which sequences $X \in 2^\omega$ does there exist (a presentation of) a measure $\mu$ such that $X$ is 1-random for $\mu$?

It is natural to ask the same question with 1-random replaced by other degrees of randomness and to ask it for measures with particular properties.

We begin with a review of what is known about the 1-random case, which is now understood reasonably well. Then, we will discuss the fragmentary results for $n$-randomness for $\mu$. 
Basic Definitions

Definition
A representation $m$ of a probability measure $\mu$ on $2^\omega$ provides, for each $\sigma \in 2^{<\omega}$ and each $k$, $m_1(\sigma, k)$ and $m_2(\sigma, k)$ such that $m_2(\sigma, k) - m_1(\sigma, k) < 1/2^k$ and $\mu([\sigma]) \in [m_1, m_2](\sigma, k)$.

Definition
$X \in 2^\omega$ is $n$-random relative to a representation $m$ of $\mu$ if and only if $X$ passes every Martin-Löf test relative to $m^{(n-1)}$, in which the measures of the open sets of the test are evaluated using $\mu$.

In what follows, we will speak of an $X$’s being $n$-random for $\mu$ and leave it as understood that this means relative to a representation of $\mu$. Similarly, we will speak of computing relative to $\mu$, taking the Turing jump of $\mu$, and so forth.
1-Randomness
remarks on constructing measures

Definition
For $X$, $Y$, and $Z$ in $2^\omega$, we write $X \equiv_{T,Z} Y$ ($X \equiv_{tt,Z} Y$) to indicate that there are (total) functionals $\Phi$ and $\Psi$ which are recursive in $Z$ such that $\Phi(X) = Y$ and $\Psi(Y) = X$.

Lemma (in the style of of Demuth, Kautz, and Levin-Zvonkin)
For $X$ and $Z$ in $2^\omega$, the following conditions are equivalent.

- There is an $R$ such that $R$ is $n$-random relative to $Z$ and $X \equiv_{tt,Z} R$.
- There is a continuous measure $\mu$ which is recursive in $Z$ such that $X$ is $n$-random for $\mu$. 
## 1-Randomness

**Theorem (Reimann and Slaman)**

For $X \in 2^\omega$, the following are equivalent.
- $X$ is not recursive.
- There is a measure $\mu$ such that $\mu(X) \neq 0$ and $X$ is 1-random relative to $\mu$.

**Theorem**

- (Reimann and Slaman) *If $X$ is not $\Delta^1_1$, then there is a continuous measure $\mu$ such that $X$ is 1-random relative to $\mu$.*
- (Kjos-Hanssen and Montalban) *If $X$ is an element of a countable $\Pi^0_1$ subset of $2^\omega$, then $X$ is not 1-random relative to any continuous $\mu$.***
1-Randomness

remarks on constructing measures

To prove $X$ not recursive implies $X$ is 1-random relative to some $\mu$, we construct the measure as follows.

- Use a theorem of Posner and Robinson to find a $G$ such that $X \oplus G \equiv_T G'$.
- Use Kučera’s Theorem to find an $R$ which is 1-random relative to $G$ for which $G' \equiv_T R \oplus G$.
- Now that $X \equiv_{T,G} R$ for $R$ a 1-random relative to $G$, use a compactness argument on the space of measures to obtain $\mu$.

Note, $\mu$ can be obtained recursively in $X''$. 
1-Randomness
remarks on constructing measures

To prove $X$ not $\Delta^1_1$ implies $X$ is 1-random relative to some continuous $\mu$, we construct the measure as follows.

- Use a theorem of Woodin to find a $G$ such that $X \oplus G \equiv_{tt} G'$.
- Use Kučera’s Theorem again to find an $R$ which is 1-random relative to $G$ for which $G' \equiv_T R \oplus G$.
- Now that $X \equiv_{T,G} R$ for $R$ a 1-random relative to $G$ with a $tt$-reduction in the direction from $X$ to $R$, use the previous compactness argument to obtain a continuous $\mu$.

In Woodin’s construction, finding $G$ from $X$ requires the Turing jump of $\mathcal{O}^X$, the complete $\Pi^1_1(X)$ subset of $\omega$. There are examples in which $X \not\in \Delta^1_1$ yet there is no continuous $\mu$ in $\Delta^1_1(X)$ such that $X$ is 1-random for $\mu$. 
Higher orders of randomness
basic observations

From this point on, we restrict ourselves to continuous measures.

**Fact (Well-known)**

Suppose that $n > 1$ and $X$ is $n$-random for $\mu$.

- $\mu'$ is not recursive in $X$.
- Every function recursive in $X$ is dominated by a function recursive in $\mu'$. 
Higher orders of randomness

\(NR_n\)

**Definition**

Let \(NR_n\) be the set of \(X\)'s for which there is no (continuous) measure \(\mu\) such that \(X\) is \(n\)-random for \(\mu\).

**Theorem**

*For every* \(n\), \(NR_n\) *is countable.*

As in the analysis of relative 1-randomness, we will show that every element of \(NR_n\) is definable. However, and this is a flaw in our method, as \(n\) increases the levels of definability involve unboundedly many iterations of the power set applied to \(\omega\).

Our proof is not sensitive to the value of \(n\), so we take \(n = 2\).
Higher orders of randomness
a cone of Turing degrees disjoint from \( NR_2 \)

**Lemma**

There is a \( B \in 2^\omega \), such that \( X \geq_T B \) implies \( X \not\in NR_2 \).

**Proof**

A Borel subset of \( \neg NR_2 \). Suppose \( Z \in 2^\omega \), \( R \) is 3-random relative to \( Z \), and \( X \equiv_{T,Z} R \). Then, \( X \equiv_{tt,Z'} R \), \( R \) is 2-random relative to \( Z' \), and so \( X \) is 2-random relative to some continuous measure.

\( \neg NR_2 \) contains a cone in \( \mathcal{D} \). By the above, \( \neg NR_2 \) contains the cofinal and degree-invariant set

\[
\{ Y : \exists Z \exists R ( R \text{ is 3-random in } Z \text{ and } Y \equiv_T Z \oplus R ) \}.
\]

This set is clearly cofinal in \( \mathcal{D} \). By Borel Determinacy, it contains a cone in \( \mathcal{D} \).
Higher orders of randomness
an observation about Borel Determinacy

- Martin’s proof of Borel Determinacy starts with a description of a Borel game and produces a winning strategy for one of the players.
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the strategy.
- The absoluteness of $\Pi_1^1$ sentences between well-founded models and the direct nature of Martin’s proof imply that if $G$ is a real parameter used to define a Borel game, then the winning strategy for that game belongs to the smallest $L_\beta[G]$ such that $L_\beta[G]$ is a model of a sufficiently large subset of ZFC.

To keep things simple, we will work with models of ZFC and assume that there is a well-founded model of ZFC. Let $L_\beta$ be the smallest well-founded model of ZFC. Note, $L_\beta$ is countable.
### Lemma

*Suppose that $X \not\in L_\beta$. Then there is a $G$ such that*

- $L_\beta[G]$ is a model of ZFC.
- Every element of $2^\omega \cap L_\beta[G]$ is recursive in $X \oplus G$.

### Proof.

Use Kumabe–Slaman forcing $P$ to generically extend $L_\beta$. This forcing builds a functional $\Phi_G$ by finite approximation. The definability of forcing and compactness show that if $D \in L_\beta$ is dense and $p \in P$, then there is a $q$ in $D$ extending $p$ such that $q$ makes no additional commitments about $\Phi_G(X)$. 
Thus, for each term $\tau$ in the forcing language and each $n \in \omega$, it is possible to decide $n \in \tau$ and then extend our commitment on $\Phi_G(X)$ to record this decision.

We construct $G$ in $\omega$-many steps so that $G$ is $P$-generic for $L_\beta$ and so that $\Phi_G(X)$ records what is forced during our construction.
The elements $p$ of the forcing partial order $P$ are pairs $(\Phi_p, \overrightarrow{X}_p)$ in which $\Phi_p$ is a finite use-monotone functional and $\overrightarrow{X}_p$ is a finite subset of $2^\omega$.

If $p$ and $q$ are elements of $P$, then $p \geq q$ if and only if

1. $\Phi_p \subseteq \Phi_q$ and for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of $\sigma_q$ is greater than the length $\sigma_p$,
2. $\overrightarrow{X}_p \subseteq \overrightarrow{X}_q$,
3. for every $x$, $y$, and $X \in \overrightarrow{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$. 
Higher orders of randomness

$NR_2 \subseteq L_\beta$.

Corollary

$NR_2 \subseteq L_\beta$. Hence, $NR_2$ is countable.

Proof

Suppose $X \not\in L_\beta$ and apply the previous lemma to obtain a $G$ such that $L_\beta[G]$ is a model of ZFC and every element of $2^\omega \cap L_\beta[G]$ is recursive in $X \oplus G$.

Relativize the discussion of $NR_2$ to $G$. Relative to $G$, $X$ belongs to every cone with base in $L_\beta[G]$. In particular, $X$ belongs to the cone avoiding $NR_2$ relative to $G$.

Thus, there is a continuous measure $\mu$ such that $X$ is 2-random for $\mu$ relative to $G$.

But then, $X$ is 2-random for a continuous $\mu$, as required.
Given $X \not\in L_\beta$, we showed that there is a continuous $\mu$ such that $X$ is 2-random for $\mu$. We can define such a $\mu$ using $X$ and a presentation of the elementary diagram of $L_\beta$ as a countable model.

**Question**

*Is it provable in analysis that for all $k$, $NR_k$ is countable?*
Theorem

\( \emptyset, \text{ the complete } \Pi^1_1 \text{ subset of } \omega, \text{ is an element of } NR_3. \)

Proof.

One representation of \( \emptyset \) is the following.

\[ \emptyset = \{ e : \text{The eth recursive subtree } T_e \text{ of } \omega^{<\omega} \text{ is well-founded.} \} \]

For a contradiction, suppose that \( \mu \) is given so that \( \emptyset \) is 3-random for \( \mu \).
Then, every function recursive in $\emptyset$ is dominated by one recursive in $\mu'$.

Hence, $\mu'$ computes a uniform family of functions $(f_e : e \in \omega)$ such that for each $e$, $f_e$ dominates the left-most infinite path in $T_e$.

Then, for each $e$, compactness implies that the following conditions are equivalent.

- $T_e$ is well-founded.
- The subtree of $T_e$ to the left of $f_e$ is finite.

The second condition is $\Pi^0_1(\mu')$. But, no 3-random for $\mu$ can be $\Pi^0_2(\mu)$.  


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