Moduli of Computation

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Modulus of Computation

Definition

Let $f : \omega \to \omega$, denoted $f \in \omega^\omega$, and $X \subseteq \omega$

- f is a *modulus (of computation)* for X iff for every $g \in \omega^\omega$ such that g dominates f point-wise ($g \succeq f$), X is recursive in g.
- X has a *self-modulus* iff X can compute a modulus for itself.

We will also consider point-wise domination for functions with finite domains ($g \in \omega^{<\omega}$). Write $g \succeq f$ to indicate that the domain of g is a subset of the domain of f and for every n in the domain of g, $g(n) \geq f(n)$.

Examples

recursively enumerable sets

Example

If W is recursively enumerable, then W has a self-modulus:

$$f : n \mapsto \text{ the stage at which } W \upharpoonright n \text{ is completely enumerated}$$

By similar means:

- if X is n-REA, then X has a self-modulus
- each of the canonical complete sets in the hyperarithmetic hierarchy has a self-modulus

Examples

Δ^0_2-sets

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- Let $f : n \mapsto s_n$, where s_n is the least stage greater than n such that for all $m \leq n$, $X(m, s_n) = X(m)$.

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  - Clearly, $X \geq_T f$.

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- Let $f : n \mapsto s_n$, where s_n is the least stage greater than n such that for all $m \leq n$, $X(m, s_n) = X(m)$.
  - Clearly, $X \geq_T f$.
  - Given $g \succeq f$ and n, compute $X(n)$ by, (1) finding $s^* > n$ such that for all $m \leq n$ and all s between s^* and $g(s^*)$, $X(m, s) = X(m, s^*)$ and (2) concluding $X(n) = X(n, s^*)$. □

Moduli in Action

Self-moduli are recursion theoretically useful.

- Permitting arguments:
  - construct sets X recursively in given recursively enumerable non-recursive sets W
- Providing a reservoir of examples.
  - If X has a self-modulus, then X is not 2-random relative to any continuous measure.

Finding Sets With Moduli

uniformity

Proposition

Suppose that X has a modulus f. There is a Turing functional Φ and an $f^* \succeq f$ such that for every g, if $g \succeq f^*$ then $\Phi(g) = X$.

Finding Sets With Moduli

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- By Σ^1_2-absoluteness, f is a modulus for X in $V[G]$, so there is a Φ_0 and an f^* such that $(\emptyset, f^*) \Vdash \Phi_0(g) = X$.

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- If $g \succeq f^*$, then for each n, g can compute $X(n)$ by finding a g_0 such that $g_0 \succeq g$ and $\Phi_0(n, g_0)$ converges. □

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We say that Φ is the uniform index for X.

Finding Sets With Moduli

countability

Corollary

There are only countably many sets with moduli.

Proof

An X with a modulus is determined by its uniform index Φ and there are only countably many Φ's. $\square$

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Proof

Suppose that X has a modulus and that Φ is the uniform index for X. Then $X(n) = i$ has a Σ^1_1 description as follows.

$$X(n) = i \iff (\exists f^*)(\forall g_0 \in \omega^{<\omega}) \left[\begin{array}{c} (g_0 \succeq f^* \wedge \Phi(n, g_0)\downarrow) \\ \implies \Phi(n, g_0) = i \end{array} \right]$$

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$X(n) \neq i$ is also Σ^1_1.

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Hence, X is Δ^1_1. □

Working From Sets With Self-Moduli

Every Δ^0_2 set has a self-modulus, hence there are a variety of examples.

- 1-generic
- 1-random
- complete extensions of Peano Arithmetic
- of minimal Turing degree

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What about examples which are not Δ^0_2?

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Suppose that X has a self-modulus. Then either X is Δ^0_2 or X can compute a 1-generic set G.

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Let $g^* \in \omega^\omega \leq_T 0'$ map n to the least s such that for all $p \in 2^n$ and all $e \leq n$,

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Any function not eventually dominated by g^* can compute a 1-generic set, details in next frame.

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Let f be the self-modulus for X. If f is eventually dominated by g^*, then X is Δ^0_2. Otherwise, X computes a 1-generic set. □

Working From Sets With Self-Moduli

building a 1-generic

Suppose that f dominates g^*. Compute a set G from f by recursion where $G(s)$ is defined at stage s so as to move toward the condition meeting the highest priority Σ^0_1 set visible in $f(s)$ steps.

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$G \upharpoonright s$ defined by stage $s-1$

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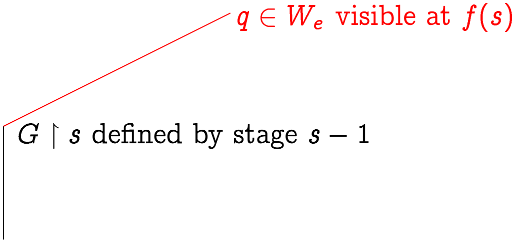

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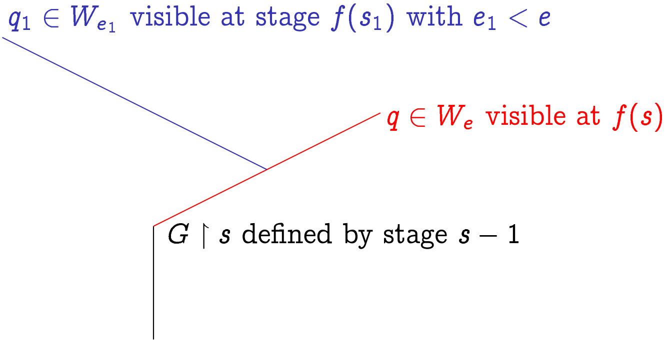

Non-iterative example

Thus far, we have obtained sets X with self-moduli using approximations to Skolem functions for Σ^0_1 or Δ^0_2 definability, possibly iterated in a degree increasing manner.

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However, not every set with a self-modulus is of this type.

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However, not every set with a self-modulus is of this type.

Theorem

There is a non-recursive set X with a self-modulus such that X does not compute any non-recursive Δ^0_2-set.

Non-iterative example

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We build a Δ^0_3 function f and a partial recursive functional Γ to satisfy the following requirements.

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We build a Δ_3^0 function f and a partial recursive functional Γ to satisfy the following requirements.

- If $g \succeq f$, then $\Gamma(g) = f$.
- For each Φ and Ψ, either there is an n such that $\Phi(n, f) \neq \lim_{s\to\infty} \Psi(n, s)$ or $\Phi(f)$ is recursive.

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Building f and Γ

We construct a recursive sequence $f_s \in \omega^{<\omega}$ and let f be the limit infimum this sequence.

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We simultaneously enumerate the functional Γ as a set of pairs $(p, q) \in \omega^{<\omega} \times \omega^{<\omega}$. Here, we mean that if $(p, q) \in \Gamma$ and $p \subset h$, then $q \subset \Gamma(h)$. During stage s, we enumerate pairs (p, f_s) with a overarching requirement that if $g \succeq f$ then $\Gamma(g) = f$.

Non-iterative example

Building f and Γ

For every f_s, we maintain the possibility of later defining f_t so that $f_s \subset f_t$. For this, we need that if $p \succeq f_t$ then $f_s \subseteq \Gamma(p)$, which we arrange as in the following picture.

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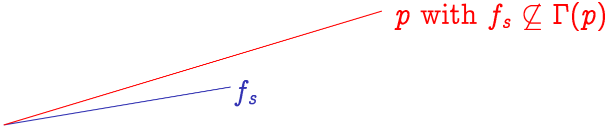

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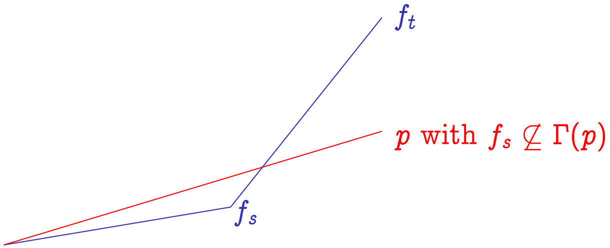

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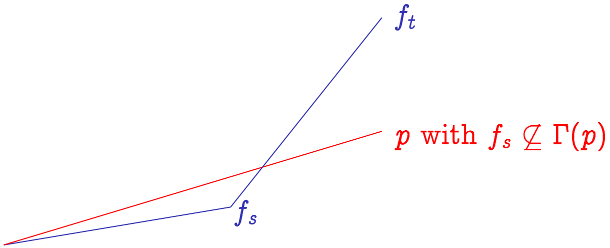

We make moot the computations (p, q) enumerated into Γ during the interval (s, t) and acquire the obligation that $p \not\succeq f$.

Non-iterative example

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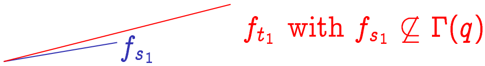

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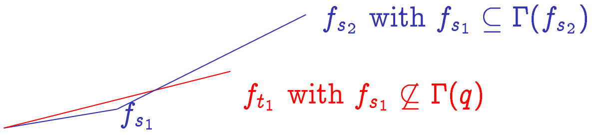

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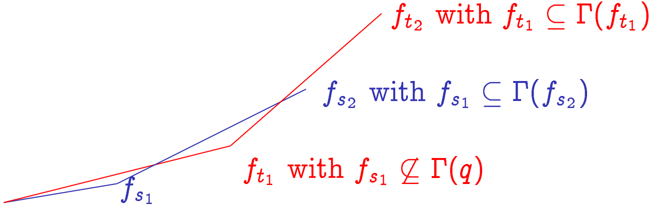

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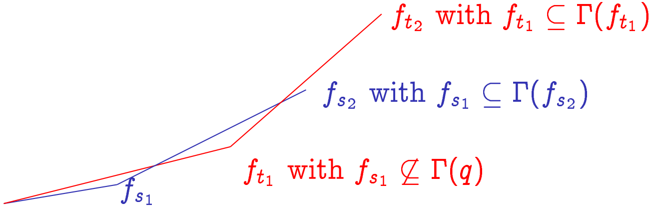

To meet the requirement that Γ is total on every g such that $g \succeq f$, we ensure that any two strings which are extended by infinitely many of the f_s are compatible.

Non-iterative example

$\Phi(n, f) \neq \lim_{s\to\infty} \Psi(n, s)$ or $\Phi(f)$ is recursive

Ensuring that either $\Phi(n, f) \neq \lim_{s\to\infty} \Psi(n, s)$ or $\Phi(f)$ is recursive requires a Π^0_2-strategy. A priori, some aspects of the construction should be infinitary or f itself would be Δ^0_2.

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  - The strategy cannot simply alternate between the conditions in the split and ensure that $\Phi(m, f) \neq \lim_{s\to\infty} \Psi(m, s)$.

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- Fix an initial condition q.
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  - The strategy cannot simply alternate between the conditions in the split and ensure that $\Phi(m, f) \neq \lim_{s\to\infty} \Psi(m, s)$.
- The strategy alternates between one of the conditions in the split and conditions extending $q * n$, where $n > 0$. Based on the behavior of Ψ, it settles upon an n and an r extending $q * n$ so that $\Phi(m, r) \neq \lim_{s\to\infty} \Psi(m, s)$ and it returns to conditions extending r infinitely often.

Non-iterative example

question

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Suppose that H is Δ^1_1. Does there exist an X such that X has a self-modulus and such that every set that is recursive in both X and H is recursive?

Non-iterative example

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$f \in \omega^\omega$ is a *modulus of 1-genericity* iff for every $h \in \omega^\omega$, if $h \succeq f$ then there is a 1-generic set recursive in h.

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Example

The function $g^* \in \omega^\omega \leq_T 0'$ (encountered earlier) mapping n to the least s such that for all $p \in 2^n$ and all $e \leq n$,

$$(\exists q \supseteq p)[q \in W_e] \implies (\exists q \supseteq p)[|q| < s \wedge q \in W_{e,s}]$$

is a modulus of 1-genericity. In fact, if g^* does not eventually dominate h, then there is a 1-generic set recursive in h. Hence, any such h is a modulus of 1-genericity.

Moduli of 1-Genericity

other examples

Example

If G is 2-generic, then G computes a modulus of 1-genericity. The function mapping n to the nth element of G is not eventually dominated by the Δ^0_2 function g^*.

Example

There is an f such that f is not dominated by any recursive function and f is not a modulus of 1-genericity. Consider the self-modulus of a Δ^0_2 set of minimal Turing degree.

Moduli of 1-Genericity

a 1-generic example

Theorem

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Note, since Δ^0_2 sets have self-moduli, G cannot be Δ^0_2.

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Note, since Δ^0_2 sets have self-moduli, G cannot be Δ^0_2.

We construct G as a limit infimum in the context of a (more involved) full-approximation priority argument like the previous one.

Finis