Effective Randomness for Continuous Measures

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Motivation

Question

For which sequences $X \in 2^\omega$ do there exist (representations of) continuous probability measures $\mu$ such that $X$ is effectively random for $\mu$?
Representations of Measures

**Definition**

A *representation* $m$ of a probability measure $\mu$ on $2^\omega$ provides, for each $\sigma \in 2^{<\omega}$ and each $k$, rational numbers $m_1(\sigma, k)$ and $m_2(\sigma, k)$ such that $m_2(\sigma, k) - m_1(\sigma, k) < 1/2^k$ and $\mu([\sigma]) \in [m_1, m_2](\sigma, k)$. That is, $m$ provides rational approximations to $\mu([\sigma])$ meeting any required accuracy.

**Definition**

$X \in 2^\omega$ is $n$-random relative to a representation $m$ of $\mu$ if and only if $X$ passes every Martin-Löf test relative to $m^{(n-1)}$ (the $n - 1$st Turing jump of $m$), in which the measures of the open sets of the test are evaluated using $\mu$. 
We will speak of an $X$’s being $n$-random for $\mu$ and mean relative to a representation of $\mu$.

We may include a real parameter $Z$ and speak of $X$’s being random relative to $\mu$ and $Z$.

We will speak of a set’s being recursive in $\mu$, the jump of $\mu$, etc.
Continuous Measures

degree theoretically characterizing relative randomness

It what follows, we will work not with representations of measures which might make $X$ random, but rather with the random content of $X$ itself.

It is useful to work with a degree theoretic characterization of continuous relative randomness.
Continuous Measures

degree theoretically characterizing relative randomness

Definition

- For $X$, $Y$, and $Z$ in $2^\omega$, we write $X \equiv_{T,Z} Y$ to indicate that there are Turing reductions $\Phi$ and $\Psi$ which are recursive in $Z$ such that $\Phi(X) = Y$ and $\Psi(Y) = X$.
- When $\Phi$ and $\Psi$ are total, we write $X \equiv_{tt,Z} Y$.

Turing reductions correspond to continuous functions defined on subsets of $2^\omega$. Truth-table ($tt$) reductions correspond to continuous functions defined on all of $2^\omega$. 
Proposition

For $X$ and $Z$ in $2^\omega$, the following conditions are equivalent.

- There is a continuous measure $\mu$ which is recursive in $Z$ such that $X$ is $n$-random for $\mu$ and $Z$.
- There is a continuous dyadic measure $\mu$ which is recursive in $Z$ such that $X$ is $n$-random for $\mu$ and $Z$.
- There is an $R$ such that $R$ is $n$-random relative to $Z$ and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that $f$ is recursive in $Z$ and $f(R) = X$.
- There is an $R$ such that $R$ is $n$-random relative to $Z$ and $X \equiv_{tt,Z} R$. 
Suppose that $n \geq 2$, $Y \in 2^\omega$, and $X$ is $n$-random relative to $\mu$.

- If $i$ is less than $n$, $Y$ is recursive in $(X \oplus \mu)$ and recursive in $\mu^{(i)}$, then $Y$ is recursive in $\mu$.
- If $Y$ is recursive in $X \oplus \mu$ and not recursive in $\mu$, then $Y$ is $(n - 2)$-random for some continuous measure $\mu_Y$ recursive in $\mu''$ (relative to $\mu''$). (Apply a theorem of Demuth.)

In general, using arithmetic definitions with fewer than $n$ quantifiers, $n$-random reals do not accelerate arithmetic definability and nontrivially define only relatively random reals.
Randomness and Well-Foundedness

**Definition**

A linear order \( \prec \) on \( \omega \) is *well-founded* iff every non-empty subset of \( \omega \) has a least element.

As with arithmetic definability, random reals cannot accelerate the calculation of well-foundedness.

**Theorem**

*Suppose that* \( X \) *is 5-random relative to* \( \mu \), \( \prec \) *is recursive in* \( \mu \), *and* \( I \) *is the largest initial segment of* \( \prec \) *which is well-founded. If* \( I \) *is recursive in* \( X \oplus \mu \), *then* \( I \) *is recursive in* \( \mu \).
Randomness and Well-Foundedness

Proof

Suppose $I \leq_T X \oplus \mu$ and $I \not\leq_T \mu$. Then, there is a continuous $\mu_I$ recursive in $\mu''$ such that $I$ is 3-random for $\mu_I$ relative to $\mu''$.

For $a \in \omega$, let $\mathcal{I}(a)$ be the set of $X$’s such that $X$ is an initial segment of $\prec$ and all of $X$’s elements are bounded by $a$. Note that $\mathcal{I}(a)$ is $\Pi^0_1(\mu)$. Hence, there is a $\mu''$-effective procedure to go from $a$ to a sequence $\mathcal{U}(a) = (U_n(a) : n \in \omega)$ of clopen sets such that if $\mu_I(\mathcal{I}(a)) = 0$ then $\mathcal{U}(a)$ is a $\mu_I$-Martin-Löf test relative to $\mu''$.

- If $a \in I$, then $\mathcal{I}(a)$ is countable and $\mathcal{U}(a)$ is a $\mu_I$-Martin-Löf test defined relative to $\mu''$.
- If $a \not\in I$, then $I \in \mathcal{I}(a)$, $I$ is 3-random for $\mu_I$ relative to $\mu''$, and so $\mathcal{U}(a)$ is not a $\mu_I$-Martin-Löf test.

Thus, $I$ is $\Pi^0_2$ relative to $\mu''$, contradiction to $I$’s being 3-random for $\mu_I$ relative to $\mu''$. 

Definition

Let $NCR_n$ be the set of $X$’s for which there is no continuous measure $\mu$ such that $X$ is $n$-random for $\mu$.

By specialized arguments:

- (Reimann and Slaman) $NCR_1 \subseteq \Delta_1^1$
- (Kjos-Hanssen and Montalban) $NCR_1$ is cofinal in the Turing degrees of the $\Delta_1^1$ sets
Higher Orders of Randomness

$NCR_n$

For $NCR_n$, we have the following generalization of $NCR_1 \subset \Delta_1^1$.

**Theorem**

*For every $n$, $NCR_n$ is countable.*

Features of the proof:

- Applies Martin’s theorem that all arithmetic games on $2^\omega$ are determined.
- Concludes that the elements of $NCR_n$ are definable. They belong to the least initial segment of Gödel’s universe of constructible sets $L_\alpha$ such that

$$L_\alpha \models ZFC^- + \text{there are } n \text{ iterates of the power set of } \omega,$$

where $ZFC^-$ is Zermelo-Frankel set theory without the power set axiom.
In the following sense, these features of the proof are necessary.

**Theorem**

*For every $k$, the statement*

$\text{For every } n, \text{NCR}_n \text{ is countable}$

*cannot be proven in*

$\text{ZFC}^- + \text{There are } k \text{ many iterates of the power set of } \omega.$

We will sketch the proof for $k = 0$ and indicate how to adapt it for $k > 0.$
Gödel’s $L$

<table>
<thead>
<tr>
<th>Definition</th>
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<tbody>
<tr>
<td>Gödel’s hierarchy of constructible sets $L$ is defined by the following recursion.</td>
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<tr>
<td>$L_0 = \emptyset$</td>
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<tr>
<td>$L_{\alpha + 1} = \text{Def}(L_\alpha)$, the set of subsets of $L_\alpha$ which are first order definable in parameters over $L_\alpha$.</td>
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<tr>
<td>$L_\lambda = \bigcup_{\alpha &lt; \lambda} L_\alpha$.</td>
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We focus on the least ordinal $\lambda$ such that $L_\lambda \models ZFC^-$. We show that there is an $n$ such that $NCR_n$ is cofinal in the Turing degrees of $L_\lambda$. 
About $L_\lambda$

Let $LOR$ be the set of limit ordinals. Note that $LOR$ is cofinal in $\lambda$.

- For any $\beta < \lambda$ with $\beta \in LOR$, there is an $X \subset \omega$ such that $X \in \text{Def}(L_\beta) \setminus L_\beta$.

- (Putnam and Enderton) For any $\beta < \lambda$ with $\beta \in LOR$, there is an $E \subset \omega \times \omega$ such that $E \in L_{\beta+3}$ and $(\omega, E)$ is isomorphic to $(L_\beta, \epsilon)$. $E$ is obtained by observing that Gödel’s Condensation Theorem implies that $L_\beta$ is the Skolem hull of the parameters which define the previous $X$ in $L_\beta$.

- (Jensen) For any $\beta < \lambda$ with $\beta \in LOR$, there is a canonical set $M_\beta \in L_{\beta+3} \cap 2^\omega$, called the master code for $L_\beta$, such that $M_\beta$ is the elementary diagram of a canonical counting of $L_\beta$. 
If $\alpha < \beta < \lambda$ and $\alpha, \beta \in LOR$, then all of $X$, $Y$, $M_\alpha$, and the isomorphism between $L_\alpha$ and $M_\alpha$’s representation of $L_\alpha$ mentioned earlier are elements of $L_\beta$.

For every $X \in 2^\omega \cap L_\lambda$, there is a $\beta \in LOR$ such that $\beta < \lambda$ and $X$ is recursive in some $M_\beta$. Hence, the set $\{M_\beta : \beta < \lambda\}$ is not countable in $L_\lambda$.

We will show that there is an $n$ such that

$$\{M_\beta : \beta < \lambda\} \subset NCR_n.$$
There is an arithmetic formula $\varphi$ as follows.

- For every $\beta$ in $LOR$ less than $\lambda$, $\varphi(M_\beta)$.
- For every $M$ and $N$ satisfying $\varphi$, either one belongs to the structure coded by the other and embeds its coded structure as an initial segment of the other’s, or there is a $\Pi^0_3(M \oplus N)$ set which exhibits a failure of well-foundedness in one of their coded structures.

In other words, there is an arithmetic $\varphi$ specifying a collection of pseudo-master codes and an arithmetic method to linearly order the apparently well-founded models they code.
About the Master Codes
obtaining $M_\beta$ by iterated relative definability.

In the previous frames, we defined $L$ by iterating first order definability from parameters and taking unions. This iteration is reflected by the master codes.

- For $\alpha \in LOR$, $M_{\alpha+\omega}$ can be defined from $M_{\alpha}$ by iterating $\Sigma^0_1$-relative definability and taking uniformly arithmetic limits.

- For a limit $\gamma \in LOR$, $M_{\gamma}$ can be defined from the sequence of smaller $M_{\alpha}$’s by taking a uniformly arithmetic limit and then iterating $\Sigma^0_1$-relative definability.
Theorem

There is an $n$ such that for all $\beta \in LOR$, if $\beta < \lambda$ then $M_\beta \in NCR_n$.

Corollary

There is an $n$ such that $ZFC^- \not\vdash "NCR_n \text{ is countable}"$. 
Master Codes and Effective Randomness
failures of continuous randomness (proof)

Let \( n \) be larger by 10 than the complexity of any of the arithmetic operations needed for the following:

- recognition and comparison of pseudo-master codes less than \( \lambda \),
- recognition of those pseudo-master codes recursive in \( \mu \) whose non-well-foundedness is witnessed by a failure of comparison with other pseudo-master codes recursive in \( \mu \)
- iteration to obtain \( M_\gamma \), with \( \gamma \in LOR \) less than \( \lambda \), from the set of \( M_\alpha \)'s, with \( \alpha \in LOR \) and \( \alpha < \gamma \).
For a contradiction, assume that $\beta < \lambda$ and that $M_\beta$ is $n$-random for the continuous measure $\mu$.

Let $\mathcal{M}^*$ be the set of $M$ such that

- $M \leq_T \mu$;
- $\varphi(M)$, so $M$ is a pseudo-master code below $\lambda$;
- no comparison between pseudo-master codes recursive in $\mu$ shows that $M$ is not well-founded.

Let $\prec$ be the ordering on $\mathcal{M}^*$ induced by inclusion between the coded models. By a choice of $n$, $\prec$ is recursive in $\mu^{(n-10)}$. 
By comparing $M_\beta$ to the elements of $M^*$, the well-founded initial segment $M$ of $\prec$ is recursive in $(M_\beta \oplus \mu)^{(n-7)}$ (a crude estimate).

Since 5-random reals do not accelerate the calculation of well-foundedness, $M$ is recursive in $\mu^{(n-7)}$.

Now note, $M$ is the set of genuine (well-founded) master codes which are recursive in $\mu$. 
Let $\gamma$ be the least limit ordinal such that $M_\gamma$ is not recursive in $\mu$ (i.e. $M_\gamma \not\in M$). $M_\gamma$ is obtained by iterating low-level arithmetic operations over $M$. Here low-level is contained in $\Sigma^0_6(\mu^{(n-7)})$.

The results of these iterative steps are recursive in $M_\beta$, since $M_\beta$ is the master code for $L_\beta$ and these operations are definable in $L_\beta$.

Since $M_\beta$ cannot accelerate $\Sigma^0_6$-definability over $\mu^{(n-7)}$, the result of each iterative step needed to obtain $M_\gamma$ from $M$ is recursive in $\mu^{(n-7)}$. In particular, $M_\gamma$ is recursive in $\mu^{(n-7)}$.

Now, $M_\gamma \leq_T M_\beta$ and again $M_\beta$ cannot accelerate $\Sigma^0_{n-6}$-definability relative to $\mu$. So, $M_\gamma$ is recursive in $\mu$. This contradiction completes the proof.
We can apply the previous argument to the case in which there are finitely many uncountable cardinals.

- More cardinals make for a more complicated collection $\mathcal{M}$ of master codes and a more complicated comparison between coded models.
- This greater arithmetic complexity requires more randomness of $M_\beta$ in order to conclude that $\mathcal{M}$ is arithmetic in $\mu$ and reach a contradiction.
Finis