Effective Randomness for Continuous Measures

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Motivation

Question

For which sequences $X \in 2^\omega$ do there exist (representations of) continuous probability measures μ such that X is effectively random for μ?

Representations of Measures

Definition

A *representation* m of a probability measure μ on 2^ω provides, for each $\sigma \in 2^{<\omega}$ and each k, rational numbers $m_1(\sigma, k)$ and $m_2(\sigma, k)$ such that $m_2(\sigma, k) - m_1(\sigma, k) < 1/2^k$ and $\mu([\sigma]) \in [m_1, m_2](\sigma, k)$. That is, m provides rational approximations to $\mu([\sigma])$ meeting any required accuracy.

Definition

$X \in 2^\omega$ is n-random relative to a representation m of μ if and only if X passes every Martin-Löf test relative to $m^{(n-1)}$ (the $n-1$st Turing jump of m), in which the measures of the open sets of the test are evaluated using μ.

Representations of Measures

notational conventions

- We will speak of an X's being n-random for μ and mean relative to a representation of μ.
- We may include a real parameter Z and speak of X's being random relative to μ and Z.
- We will speak of a set's being recursive in μ, the jump of μ, etc.

Continuous Measures

degree theoretically characterizing relative randomness

It what follows, we will work not with representations of measures which might make X random, but rather with the random content of X itself.

It is useful to work with a degree theoretic characterization of continuous relative randomness.

Continuous Measures

degree theoretically characterizing relative randomness

Definition

- For X, Y, and Z in 2^ω, we write $X \equiv_{T,Z} Y$ to indicate that there are Turing reductions Φ and Ψ which are recursive in Z such that $\Phi(X) = Y$ and $\Psi(Y) = X$.
- When Φ and Ψ are total, we write $X \equiv_{tt,Z} Y$.

Turing reductions correspond to continuous functions defined on subsets of 2^ω. Truth-table (tt) reductions correspond to continuous functions defined on all of 2^ω.

Continuous Measures

degree theoretically characterizing relative randomness

Proposition

For X and Z in 2^ω, the following conditions are equivalent.

- *There is a continuous measure μ which is recursive in Z such that X is n-random for μ and Z.*
- *There is a continuous dyadic measure μ which is recursive in Z such that X is n-random for μ and Z.*
- *There is an R such that R is n-random relative to Z and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that f is recursive in Z and $f(R) = X$.*
- *There is an R such that R is n-random relative to Z and $X \equiv_{tt,Z} R$.*

Effectively Random Reals

Suppose that $n \geq 2$, $Y \in 2^\omega$, and X is n-random relative to μ.

- If i is less than n, Y is recursive in $(X \oplus \mu)$ and recursive in $\mu^{(i)}$, then Y is recursive in μ.
- If Y is recursive in $X \oplus \mu$ and not recursive in μ, then Y is $(n-2)$-random for some continuous measure μ_Y recursive in μ'' (relative to μ''). (Apply a theorem of Demuth.)

In general, using arithmetic definitions with fewer than n quantifiers, n-random reals do not accelerate arithmetic definability and nontrivially define only relatively random reals.

Randomness and Well-Foundedness

Definition

A linear order $\prec$ on ω is *well-founded* iff every non-empty subset of ω has a least element.

As with arithmetic definability, random reals cannot accelerate the calculation of well-foundedness.

Theorem

Suppose that X is 5-random relative to μ, $\prec$ is recursive in μ, and I is the largest initial segment of $\prec$ which is well-founded. If I is recursive in $X \oplus \mu$, then I is recursive in μ.

Randomness and Well-Foundedness

Proof

Suppose $I \leq_T X \oplus \mu$ and $I \not\leq_T \mu$. Then, there is a continuous μ_I recursive in μ'' such that I is 3-random for μ_I relative to μ''.

For $a \in \omega$, let $\mathcal{I}(a)$ be the set of X's such that X is an initial segment of $\prec$ and all of X's elements are bounded by a. Note that $\mathcal{I}(a)$ is $\Pi^0_1(\mu)$. Hence, there is a μ''-effective procedure to go from a to a sequence $\mathcal{U}(a) = (U_n(a) : n \in \omega)$ of clopen sets such that if $\mu_I(\mathcal{I}(a)) = 0$ then $\mathcal{U}(a)$ is a μ_I-Martin-Löf test relative to μ''.

- If $a \in I$, then $\mathcal{I}(a)$ is countable and $\mathcal{U}(a)$ is a μ_I-Martin-Löf test defined relative to μ''.
- If $a \notin I$, then $I \in \mathcal{I}(a)$, I is 3-random for μ_I relative to μ'', and so $\mathcal{U}(a)$ is not a μ_I-Martin-Löf test.

Thus, I is Π^0_2 relative to μ'', contradiction to I's being 3-random for μ_I relative to μ''.

Definition

Let NCR_n be the set of X's for which there is no continuous measure μ such that X is n-random for μ.

By specialized arguments:

- (Reimann and Slaman) $NCR_1 \subset \Delta^1_1$
- (Kjos-Hanssen and Montalban) NCR_1 is cofinal in the Turing degrees of the Δ^1_1 sets

Higher Orders of Randomness

NCR_n

For NCR_n, we have the following generalization of $NCR_1 \subset \Delta^1_1$.

Theorem

For every n, NCR_n is countable.

Features of the proof:

- Applies Martin's theorem that all arithmetic games on 2^ω are determined.
- Concludes that the elements of NCR_n are definable. They belong to the least initial segment of Gödel's universe of constructible sets L_α such that

$$L_\alpha \models ZFC^- + \text{there are } n \text{ iterates of the power set of } \omega,$$

  where ZFC^- is Zermelo-Frankel set theory without the power set axiom.

In the following sense, these features of the proof are necessary.

Theorem

For every k, the statement

For every n, NCR_n is countable

cannot be proven in

$$ZFC^- + \textit{There are } k \textit{ many iterates of the power set of } \omega.$$

We will sketch the proof for $k = 0$ and indicate how to adapt it for $k > 0$.

Gödel's L

Definition

Gödel's hierarchy of constructible sets L is defined by the following recursion.

- $L_0 = \emptyset$
- $L_{\alpha+1} = Def(L_\alpha)$, the set of subsets of L_α which are first order definable in parameters over L_α.
- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.

We focus on the least ordinal λ such that $L_\lambda \models ZFC^-$. We show that there is an n such that NCR_n is cofinal in the Turing degrees of L_λ.

About L_λ

Let LOR be the set of limit ordinals. Note that LOR is cofinal in λ.

- For any $\beta < \lambda$ with $\beta \in LOR$, there is an $X \subset \omega$ such that $X \in \mathit{Def}(L_\beta) \setminus L_\beta$.
- (Putnam and Enderton) For any $\beta < \lambda$ with $\beta \in LOR$, there is an $E \subset \omega \times \omega$ such that $E \in L_{\beta+3}$ and (ω, E) is isomorphic to (L_β, ϵ). E is obtained by observing that Gödel's Condensation Theorem implies that L_β is the Skolem hull of the parameters which define the previous X in L_β.
- (Jensen) For any $\beta < \lambda$ with $\beta \in LOR$, there is a canonical set $M_\beta \in L_{\beta+3} \cap 2^\omega$, called the *master code* for L_β, such that M_β is the elementary diagram of a canonical counting of L_β.

About the Master Codes

If $\alpha < \beta < \lambda$ and $\alpha, \beta \in LOR$, then all of X, Y, M_α, and the isomorphism between L_α and M_α's representation of L_α mentioned earlier are elements of L_β.

For every $X \in 2^\omega \cap L_\lambda$, there is a $\beta \in LOR$ such that $\beta < \lambda$ and X is recursive in some M_β. Hence, the set $\{M_\beta : \beta < \lambda\}$ is not countable in L_λ.

We will show that there is an n such that

$$\{M_\beta : \beta < \lambda\} \subset NCR_n.$$

About the Master Codes

recognition and comparison

There is an arithmetic formula φ as follows.

- For every β in LOR less than λ, $\varphi(M_\beta)$.
- For every M and N satisfying φ, either one belongs to the structure coded by the other and embeds its coded structure as an initial segment of the other's, or there is a $\Pi^0_3(M \oplus N)$ set which exhibits a failure of well-foundedness in one of their coded structures.

In other words, there is an arithmetic φ specifying a collection of pseudo-master codes and an arithmetic method to linearly order the apparently well-founded models they code.

About the Master Codes

obtaining M_β by iterated relative definability.

In the previous frames, we defined L by iterating first order definability from parameters and taking unions. This iteration is reflected by the master codes.

- For $\alpha \in LOR$, $M_{\alpha+\omega}$ can be defined from M_α by iterating Σ^0_1-relative definability and taking uniformly arithmetic limits.
- For a limit $\gamma \in LOR$, M_γ can be defined from the sequence of smaller M_α's by taking a uniformly arithmetic limit and then iterating Σ^0_1-relative definability.

Theorem

There is an n such that for all $\beta \in LOR$, if $\beta < \lambda$ then $M_\beta \in NCR_n$.

Corollary

There is an n such that $ZFC^- \nvdash$ "NCR_n is countable."

Master Codes and Effective Randomness

failures of continuous randomness (proof)

Let n be larger by 10 than the complexity of any of the arithmetic operations needed for the following:

- recognition and comparison of pseudo-master codes less than λ,
- recognition of those pseudo-master codes recursive in μ whose non-well-foundedness is witnessed by a failure of comparison with other pseudo-master codes recursive in μ
- iteration to obtain M_γ, with $\gamma \in LOR$ less than λ, from the set of M_α's, with $\alpha \in LOR$ and $\alpha < \gamma$.

For a contradiction, assume that $\beta < \lambda$ and that M_β is n-random for the continuous measure μ.

Let $\mathfrak{M}^*$ be the set of M such that

- $M \leq_T \mu$;
- $\varphi(M)$, so M is a pseudo-master code below λ;
- no comparison between pseudo-master codes recursive in μ shows that M is not well-founded.

Let $\prec$ be the ordering on $\mathfrak{M}^*$ induced by inclusion between the coded models. By a choice of n, $\prec$ is recursive in $\mu^{(n-10)}$.

By comparing M_β to the elements of $\mathfrak{M}^*$, the well-founded initial segment $\mathfrak{M}$ of $\prec$ is recursive in $(\mathfrak{M}_\beta \oplus \mu)^{(n-7)}$ (a crude estimate).

Since 5-random reals do not accelerate the calculation of well-foundedness, $\mathfrak{M}$ is recursive in $\mu^{(n-7)}$.

Now note, $\mathfrak{M}$ is the set of genuine (well-founded) master codes which are recursive in μ.

Master Codes and Effective Randomness

failures of continuous randomness (proof)

Let γ be the least limit ordinal such that M_γ is not recursive in μ (i.e. $M_\gamma \notin \mathfrak{M}$). M_γ is obtained by iterating low-level arithmetic operations over $\mathfrak{M}$. Here low-level is contained in $\Sigma^0_6(\mu^{(n-7)})$.

The results of these iterative steps are recursive in M_β, since M_β is the master code for L_β and these operations are definable in L_β.

Since M_β cannot accelerate Σ^0_6-definability over $\mu^{(n-7)}$, the result of each iterative step needed to obtain M_γ from $\mathfrak{M}$ is recursive in $\mu^{(n-7)}$. In particular, M_γ is recursive in $\mu^{(n-7)}$.

Now, $M_\gamma \leq_T M_\beta$ and again M_β cannot accelerate Σ^0_{n-6}-definability relative to μ. So, M_γ is recursive in μ. This contradiction completes the proof.

More Cardinals and More Randomness

We can apply the previous argument to the case in which there are finitely many uncountable cardinals.

- More cardinals make for a more complicated collection $\mathfrak{M}$ of master codes and a more complicated comparison between coded models.
- This greater arithmetic complexity requires more randomness of M_β in order to conclude that $\mathfrak{M}$ is arithmetic in μ and reach a contradiction.

Finis