Degree Invariant Functions

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Announcement of the Steel Meeting

"In part, this meeting is organized to honor John Steel on the occasion of his sixtieth birthday. Its focus is on areas of logic related to Steel's work, including descriptive set theory, inner models, recursion theory, reverse mathematics, and philosophy of mathematics."

Martin's Conjecture: Notation

Definition

1. A *cone of reals* is a set $\{y : y \geq_T x\}$, for some x.
2. A property P on D *contains a cone* iff there is a cone of reals all of whose degrees satisfy P.
3. A function $f : 2^\omega \to 2^\omega$ is *degree invariant* iff for x and y, if $x \equiv_T y$ then $f(x) \equiv_T f(y)$.

For functions from D to D, we define order preserving on a cone, constant on a cone, and other notions, similarly.
For degree invariant f and g, $f \geq_M g$ iff $\{x : f(x) \geq_T g(x)\}$ contains a cone.

Martin's Conjecture

Martin has made the following conjecture: Assume $ZF+AD+DC$.

I. If f is degree invariant and not increasing on a cone, then f is constant on a cone.

II. $\leq_M$ prewellorders the set of degree invariant functions which are increasing on a cone. If f has $\leq_M$-rank α, then f' has $\leq_M$-rank $\alpha + 1$, where $f' : x \mapsto f(x)'$ for all x.

Martin's conjecture was prompted by Sacks's question, "Is there a degree invariant solution to Post's problem?"

Regressive Functions

Theorem (Slaman and Steel, 1988)

Assume AD. For any degree invariant function f, if $f(x) <_T x$ on a cone then f is constant on a cone.

Comments on the proof:

- Use determinacy to obtain a pointed perfect tree P and a recursive function Φ_e such that for all $x \in [P]$, $\Phi_e(x) \equiv_T f(x)$.
- Give a recursion theoretic argument, analyzing the rate at which the computations to evaluate Φ_e converge, to show that Φ_e is constant on a cone.

Uniformly Degree Invariant Functions

Definition

- We say that $x \equiv_T y$ *via* (i,j) iff $\Phi_i(x) = y$ and $\Phi_j(y) = x$.
- $f : 2^\omega \to 2^\omega$ is *uniformly degree invariant* iff there is a point perfect set U and a function $t : \omega \times \omega \to \omega \times \omega$ such that

$$\forall x, y \in U[x \equiv_T y \text{ via } (i,j) \implies f(x) \equiv_T f(y) \text{ via } t(i,j)]$$

For example, $x \mapsto x'$ is uniformly degree invariant.

Steel's Theorem

Theorem (Steel 1982, Slaman-Steel 1988)

Martin's conjecture is true when restricted to uniformly degree invariant functions.

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III. Every degree invariant function is uniformly degree invariant.

Comparison with *id*

Proposition (Slaman-Steel 1988)

Assume AD. Suppose that f is uniformly invariant. Then f and id are comparable under $\leq_M$.

Proof

Suppose that $f \not\geq_M id$. Choose a T-pointed perfect tree $P \subseteq 2^{<\omega}$ so that for all $z \in [P]$, $f(z) \not\geq_T z$. Consider the following Identity-Comparison game:

- Play
  - I plays (e, n) followed by x
  - II plays m followed by y
- Winning condition
  - I loses unless $x \in [P]$ and $\Phi_e(x) = m^\frown y$
  - II loses unless $y \geq_T x$
  - If not decided by the above, II wins iff $(n \in f(x) \iff m \in y)$.

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The sequence $(x_m : m \in \omega)$ is uniformly recursive in z, so the sequence $(f(x_m) : m \in \omega)$ is uniformly recursive in $f(z)$. But then, $f(z) \geq_T z$, contradiction to our assumption that $f(z) \not\geq_T s$ on $[P]$.

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Consequently, I cannot have a winning strategy in this game.

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For $n \in \omega$, use the effective form of the recursion theorem to choose e_n such that $\Phi_{e_n}(z)$ is equal to II's play according to σ against $(e_n, n)^\frown z$. Let m_n be $\sigma((e_n, n))$. Then, for each n,

$$n \in f(z) \iff m_n \in \sigma((e_n, n)^\frown z).$$

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The right-hand side above is uniformly recursive in z, so $z \geq_T f(z)$.

Conclusion: $id \geq_M f$.

Nonuniformity

Question

Is there a scheme to organize degree invariant functions which are not uniformly degree invariant?

Enter Borel Equivalence Relations

Definition

Suppose E and F are equivalence relations on Polish spaces X and Y.

- E Borel reduces to F iff there is a Borel function $f : X \to Y$ such that for all $x, y \in X$,

$$x E y \iff f(x) F f(y).$$

- We write $E \leq_B F$ to denote E Borel reduces to F. $\sim_B$ and $<_B$ are defined similarly.

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Equivalence classes replace Turing degrees and Borel reductions are degree-invariant functions which happen to be injective on degree.

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The elements of G replace the Turing reductions.

A Universal Countable Borel Equivalence Relation

Theorem (Dougherty, Jackson, and Kechris 1994)

Let F_2 denote the free group on two generators. Let $E(F_2, 2)$ be the equivalence relation induced by the shift action of F_2 on the space of subsets of F_2. Then $E(F_2, 2)$ is universal in the sense that every countable Borel equivalence relation is Borel reducible to it.

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Even better, $E(F_2, 2)$ is *uniformly universal*. If E is generated by the action of G, then there is a reduction f from E to $E(F_2, 2)$ and a $t : G \to F_2$ such that for all x and y,

$$g \cdot x = y \implies t(g) \cdot f(x) = f(y).$$

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Remark

- If yes, then there are many Turing degree invariant functions, including a Borel pairing function, and Martin's Conjecture must fail.
- Whether $E(F_2, 2)$ reduces to $\equiv_T$ clarifies the technical issues in studying T-degree invariance.

Turing equivalence is not uniformly universal

Theorem (Montalbán, Reimann, and Slaman)

There do not exist Borel $f : 2^{F_2} \to 2^\omega$ and $t : F_2 \to \omega$ such that for all $x, y \in 2^{F_2}$,

$$x \, E(F_2, 2) \, y \text{ via } g \implies f(x) \equiv_T f(y) \text{ via } t(g).$$

Outline of the proof

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1. Exhibit a rich group G_T whose elements act locally like Turing functionals and which induces $\equiv_T$.
2. Reduce $\equiv_T$ to $E(F_2, 2)$ uniformly in G_T. Composing this reduction with f gives a Turing degree invariant map which is uniformly invariant in terms of G_T.
3. Adapt the identity-comparison game so that the indices e used by player I naturally translate to actions of elements of G_T. Conclude that f is $\leq_M$-comparable with id.
4. Get a contradiction

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Continue the proof from the point of having established that for any f and t as above f is $\leq_M$-comparable with the identity.

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For such an f, if $id >_M f$, then we can apply the known case of Martin's conjecture: f is constant on a cone. Impossible.

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Continue the proof from the point of having established that for any f and t as above f is $\leq_M$-comparable with the identity.

For such an f, if $id >_M f$, then we can apply the known case of Martin's conjecture: f is constant on a cone. Impossible.

Hence, any such f has $f \geq_M id$.

- On a cone of z's, $f(z) \geq_T z$.
- The set of y such that there is a z recursive in y with $y = f(z)$ is Borel and co-final; hence the range of any such f contains a cone.

Finally, by $E(F_2, 2)$'s being uniformly universal, we can find two such fs and ts with disjoint ranges. Impossible, since both ranges must contain a cone.

Another Example

Theorem (Slaman-Steel)

Arithmetic equivalence for subsets of ω is (uniformly) universal.

The Steel games do not result in a contradiction for arithmetic equivalence; uniformly specifying infinitely many arithmetic properties does not specify an arithmetic set.

The Question of Uniformity

Question

What is the role of uniformity in the theory of Borel equivalence relations?