Effective Randomness and Continuous Measures

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Review

Let NCR_n be the set of $X \in 2^\omega$ such that there is no (representation of a) continuous measure μ on 2^ω such that X is n-random relative to μ.

Last time, we showed that for every n, NCR_n is a countable set. The proof invoked Borel Determinacy in a detailed way.

In this chapter, we will give some examples of elements of NCR_n, for various n and show that the use of meta-mathematics in the form of iterations of the power set cannot be eliminiated from the above countability theorem.

NCR_1

The case for NCR_1 is the best understood.

Theorem

Every element of NCR_1 is hyperarithmetic.

Proof.

We optimize the proof of the general theorem so that we can obtain relative 2-randomness by applying only determinacy for game whose strategies are recursive in $\mathcal{O}$. Then we apply a join theorem of Woodin's, that for every non-hyperarithmetic real X there is a G such that $X + G \geq_T \mathcal{O}^G$. The conclusion follows. $\square$

NCR_1

The previous theorem is optimal.

Theorem (Kjos-Hanssen and Montalbán)

If X is an element of a countable Π^0_1 class, then X belongs to NCR_1.

Thus, by an early theorem of Kreisel, there is no hyperarithmetic upper bound on NCR_1.

However, the Kjos-Hanssen and Montalbán result does not characterize NCR_1.

Theorem

There is an X in NCR_1 which is not in any countable Π^0_1-class.

Other Examples

A sequence of results leads to the following.

Theorem (Reimann and Slaman; Barmpalias, Greenberg, Montalbán, and Slaman)

If X is recursive in an incomplete recursively enumerable set, then $X \in NCR_1$.

Consequently, NCR_1 contains elements of the following types.

- 1-generic
- packing dimension 1
- recursively enumerable
- of minimal Turing degree

NCR_n and the necessity of power sets

The proof that all the NCR_n are countable invoked infinitely many iterates of the power set in the form of Borel Determinacy. By work of H. Friedman, these are necessary in the proof of Borel Determinacy.

We will show that the infinitely many iterates of the power set cannot be removed from the analysis of relative randomness.

Necessity of power sets

Our analysis has some parallels to Friedman's, in that both proofs use initial segments of Gödel's L to exhibit models in which NCR_n is not countable or some Borel game is not determined, respectively.

However, the two analyses diverge early on. Where Friedman's argument could invoke a game which spoke directly about closure points in L, we must show that new elements of NCR_n are constructed cofinally within the reals of those closure points and hence that those closure points are insufficient to prove its countability.

We need to recall a few facts about random sequences from the first lecture, and to remind ourselves of few more facts from set theory.

Necessity of power sets

A connection between failure of randomness and definability

In the first lecture, we gave a precise account of the following heuristic principles.

- If X is μ-random then $\mu + X$ cannot accelerate the definitions of μ-definable sets.
- If X is μ-random then $\mu + X$ cannot accelerate the definition of the well-founded part of a μ-definable linear order.

Necessity of power sets

A connection between failure of randomness and definability

Example

For all k, $0^{(k)}$ is not 2-random relative to any μ.

Proof.

- Say $0^{(k)}$ is 2-random relative to μ.
- $0'$ is recursively enumerable in μ and recursive in the supposedly 2-random $0^{(k)}$. Thus, $0'$ is recursive in μ and, thereby, $0^{(2)}$ is recursively enumerable in μ.
- Use induction to conclude $0^{(k)}$ is recursive in μ, a contradiction.

$\square$

Necessity of power sets

a little more about set theory

Definition

Gödel's hierarchy of constructible sets L is defined by the following recursion.

- $L_0 = \emptyset$
- $L_{\alpha+1} = Def(L_\alpha)$, the set of subsets of L_α which are first order definable in parameters over L_α.
- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.

Necessity of power sets

a little more about set theory

We focus on the least ordinal λ such that L_λ satisfies ZFC^-.

- For $\beta < \lambda$, L_β is a countable structure obtained by iterating first order definability over smaller L_α's and taking direct limits.
- There is a sequence $M_\beta \in 2^\omega \bigcap L_\lambda$, for $\beta < \lambda$, of representations these countable structures. These are a particular form of Jensen's *Master Codes*.
  - M_β is obtained from smaller M_α's by iterating the Turing jump and taking arithmetically definable direct limits.
  - Every $X \in 2^\omega \bigcap L_\lambda$ is recursive in some M_β.

Theorem

There is an n such that for all $\beta \in LOR$, if $\beta < \lambda$ then there is no continuous measure μ such that M_β is n-random relative to μ.

Corollary

ZFC^- does not prove the Co-countability Theorem.

Master Codes and Effective Randomness

failures of continuous randomness—outline of proof

Suppose that M_β were n-random relative to μ.

- Let $\mathcal{M}$ be the sequence of possible Master Codes which are recursive in μ.
  - The well-founded part of $\mathcal{M}$ is of the form $\mathcal{M}_{<\gamma} = (M_\alpha : \alpha < \gamma)$ for some $\gamma \leq \beta$.
  - $\mathcal{M}_{<\gamma}$ is uniformly arithmetically definable from M_β and hence from μ. (By increasing the degree of randomness, we can assume that $\mathcal{M}_{<\gamma}$ is recursive in μ.)
- M_γ is obtained by iterating uniformly arithmetically definable operations on $\mathcal{M}_{<\gamma}$.
- The results at each step and M_γ itself are recursive in M_β.
- The results at each step and M_γ itself are recursive in μ.
- M_γ is in the well-founded part of $\mathcal{M}$. Contradiction.

The higher iterates of the power set make it more complicated formulate the notion of pseudo-Master Code, to define $\mathcal{M}$, and to define the process of going from $\mathcal{M}_{<\gamma}$ to M_γ.

Consequently, the failure of randomness for the M_β's for these models is more complicated to describe. By which we mean that they can exhibit relative k-randomness for $k < n$, but not relative n-randomness.

Even so, for each n, the first initial segment of L satisfying ZFC^- and there are n iterates of the power set of $\mathbb{R}$ does not satisfy the Co-countablity Theorem.

Category vs Measure

One can ask these questions about relative genericity in place of relative randomness.

Definition

X is relatively n-generic iff there is a perfect tree T such that X is n-generic relative to T as a path through T.

B. Anderson reworked the previous arguments, with properties of genericity in place of randomness.

Theorem (Anderson)

- *For every n, the set of never relatively generic reals is countable.*
- *The co-countability theorem for genericity cannot be proven using only finitely many iterates of the power set of $\mathbb{R}$.*

Category vs Measure

Remark

The two co-countability theorems for measure and category are provably equivalent over a much weaker base theory (such as second order arithmetic). However, the only known proof of the equivalence is to use one to show the existence of the strategies needed in the application of determinacy to establish the other.

(There are many sub-plots in this story.)

Final Comments

For each n, NCR_n is a countable Π^1_1 subset of 2^ω. A reasonably direct argument shows that it is not Δ^1_1, so any clear understanding of it should be as a Π^1_1-set.

Question

Is there a natural Π^1_1-norm on NCR_n which explains the observed connections between definability and failure of randomness?

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