Effective Randomness and Continuous Measures

Theodore A. Slaman
(on joint work with Jan Reimann)

University of California, Berkeley

Motivation

Question

For which sequences $X \in 2^\omega$ do there exist (representations of) (continuous) probability measures μ such that X is effectively random for μ?

We have discussed the case of randomness for Lebesgue measure. Now, we turn to other measures.

Representations of Measures

Definition

A *representation* m of a probability measure μ on 2^ω provides rational approximations to each $\mu([\sigma])$ meeting any required accuracy.

Definition

$X \in 2^\omega$ is $(n+1)$-random relative to a representation m of μ if and only if it does not belong to any $m^{(n)}$-presented G_δ set of μ-measure 0. That is to say that X passes every μ-Martin-Löf test relative to $m^{(n)}$.

We will drop the explicit reference to presentations and speak of randomness relative to μ.

The Precise Question

Question

For which sequences $X \in 2^\omega$ do there exist (continuous) probability measures μ such that X is n-random for μ?

Arbitrary Measures

Theorem

For $X \in 2^\omega$, the following are equivalent.

- *X is not recursive.*
- *There is a measure μ such that $\mu(X) = 0$ and X is 1-random relative to μ.*

For each nonrecursive X, there is such a μ which is recursive in X''. The argument is a primitive version of transferring effective randomness. The measure μ is obtained by a compactness argument applied to a space of representations of measures, with little or no control over its properties.

Posner and Robinson

Theorem (Posner and Robinson)

For any nonrecursive X, there is a G such that $X + G \equiv_T G'$.

Relative to G, X has the same Turing degree as the Turing jump of G.

Kučera

Theorem (Kučera)

There is a 1-random set R such that $R \equiv_T 0'$.

Kučera's proof relativizes.

Relative to G, X has the same Turing degree as a random real R.

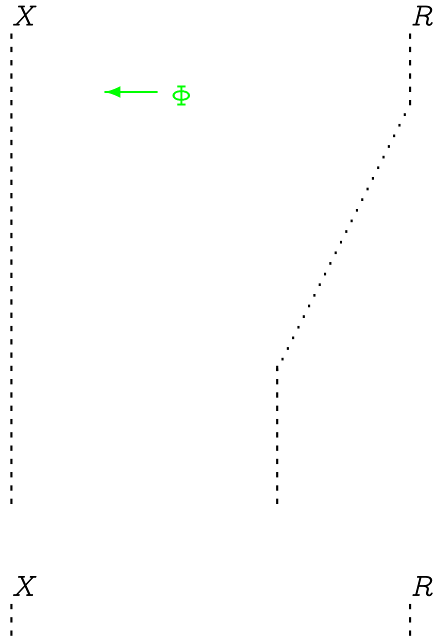
Pushing Randomness from R to X

Let Ψ and Φ be Turing functionals recursive relative to G such that $\Phi(R) = X$ and $\Psi(X) = R$.

If Φ were a homeomorphism with inverse Ψ, then there would be a measure μ obtained by pulling back to Lebesgue measure using $\Phi^{-1} = \Psi$. R's being random would ensure X's being μ-random.

We adapt this paradigm to the partial continuous Φ and Ψ, which are inverses on X and R.

Pushing Randomness from R to X



Pushing Randomness from R to X

For $\sigma \in 2^{<\omega}$, let $Pre(\sigma)$ be the set of minimal elements of

$$\{\tau : \Phi(\tau) = \sigma \text{ and } \Psi(\sigma) \subseteq \tau\}.$$

When X extends σ, $Pre(\sigma)$ is a recursively enumerable antichain of possible initial segments of R.
In the previous slide, τ_1 and τ_2 are elements of $Pre(\sigma)$.

Pushing Randomness from R to X

Let λ denote Lebesgue measure. Consider the following consistency requirements $\mathcal{R}$ on a measure μ.

1. $\mu(U_\sigma) \geq \lambda(\cup\{U_\tau : \tau \in Pre(\sigma)\})$. Thus, μ dominates the measure of pulling back Φ on those strings for which $\Psi(\Phi)$ is consistent with the identity.
2. $\mu(U_\sigma) \leq \lambda(U_{\Psi(\sigma)})$. Thus, μ does not concentrate on reals in the domain of Ψ.

Pushing Randomness from R to X

There is an infinite G-recursive, G-recursively-bounded tree T such that any infinite path in T is a rational representation m of a measure μ satisfying $\mathcal{R}$.

Lemma

Any infinite G-recursive, G-recursively-bounded tree has an infinite path m such that R is random relative to m.

Pushing Randomness from R to X

Fix a path m in T such that R is random relative to m. X's failing an m-recursive Martin-Löf test relative to μ would pull back to R's failing an m-recursive Martin-Löf test relative to λ, an impossibility.

Conclusion: X is μ-random.

Continuous Measures

degree theoretically characterizing relative randomness

It is useful to work with a sequence-based recursion-theoretic characterization of relative continuous randomness.

Definition

- For X, Y, and Z in 2^ω, we write $X \equiv_{T,Z} Y$ to indicate that there are Turing reductions (i.e. representations of continuous functions) Φ and Ψ which are recursive in Z such that $\Phi(X) = Y$ and $\Psi(Y) = X$.
- When Φ and Ψ have domain 2^ω, we write $X \equiv_{tt,Z} Y$.

Turing reductions correspond to continuous functions defined on subsets of 2^ω. Truth-table (tt) reductions correspond to continuous functions defined on all of 2^ω.

Continuous Measures

degree theoretically characterizing relative randomness

Proposition

For X and Z in 2^ω, the following conditions are equivalent.

- *There is a continuous measure μ which is recursive in Z such that X is n-random for μ and Z.*
- *There is a continuous dyadic measure μ which is recursive in Z such that X is n-random for μ and Z.*
- *There is an R such that R is n-random relative to Z and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that f is recursive in Z and $f(R) = X$.*
- *There is an R such that R is n-random relative to Z and $X \equiv_{tt,Z} R$.*

Constructing continuous measures

In order to conclude that X is n-random relative to some continuous measure, it is equivalent to find a Z relative to which X is tt-equivalent to some n-random sequence R. It is sufficient to find a Z relative to which X is T-equivalent to some $n+1$-random sequence R.

Theorem (Martin, Borel Determinacy)

Suppose that $\mathcal{B}$ is a Borel subset of 2^ω and that for every A there is a Y such that $Y \geq_T A$ and $Y \in \mathcal{B}$. There is a $B \in 2^\omega$ such that for every $X \geq_T B$ there is a Y such that $Y \equiv_T X$ and $Y \in \mathcal{B}$.

Corollary

For any $n \in \mathbb{N}$, there is a B such that for all $X \geq_T B$, there is a continuous measure μ such that X is n-random relative to μ.

Constructing continuous measures

the first interesting turn

- In Martin's proof, the later $\mathcal{B}$ appears in the Borel hierarchy, the more iterates of the power set of $\mathbb{R}$ are used in producing the B such that $X's$ computing B have $\equiv_T$-equivalents in $\mathcal{B}$.
- Martin's proof implies that if G is a real parameter used to define a Borel game $\mathcal{B}$, then the B for that game belongs to the smallest countable model of a sufficiently large subset of ZFC, the axioms of set theory.

Constructing continuous measures

Fix n and let $L_{\lambda(n)}$ be the smallest countable model satisfying ZFC^{-}, set theory without the power set axiom, and the existence of n-iterates of the power set applied to $\mathbb{R}$, which is sufficient to produce a cone of relatively $(n+1)$-random degrees.

Theorem (Reimann and Slaman, Woodin)

Suppose that $X \notin L_{\lambda(n)}$. Then there is a G such that

- *$L_{\lambda(n)}[G]$ is a model of ZFC^{-} and the existence of n-iterates of the power set applied to $\mathbb{R}$.*
- *Every element of $2^{\omega} \bigcap L_{\lambda(n)}[G]$ is recursive in $X + G$.*

Consequently, if $X \notin L_{\lambda(n)}$, then relative to G, X is in the set of relatively $(n+1)$-random degrees.

Constructing continuous measures

Theorem

For any X which is not in $L_{\lambda(n)}$, there is a continuous measure μ such that X is n-random relative to μ.

Proof.

Relative to G, X is in the set of relatively $(n+1)$-random degrees. Then, X is tt-equivalent to an n-random relative to G and some Z. Hence, there is a continuous measure for which X is n-random. □

Theorem (Co-countability)

For all n, for all but countably many $X \in 2^\omega$ there is a continuous measure μ such that X is n-random relative to μ.

Proof of the join theorem

Kumabe–Slaman forcing in detail

Use Kumabe–Slaman forcing P to generically extend L_β. This forcing builds a functional Φ_G by finite approximation.

- Ignoring some technical points, the elements p of the forcing partial order P are pairs $(\Phi_p, \overrightarrow{Y}_p)$ in which Φ_p is a finite Turing functional and $\overrightarrow{Y}_p$ is a finite subset of 2^ω.
- If p and q are elements of P, then $p \geq q$ if and only if
  - $\Phi_p \subseteq \Phi_q$.
  - $\overrightarrow{Y}_p \subseteq \overrightarrow{Y}_q$,
  - for every x, y, and $Y \in \overrightarrow{Y}_p$, if $\Phi_q(x, Y) = y$ then $\Phi_p(x, Y) = y$.

Higher orders of randomness

a join theorem

The definability of forcing and compactness show that if $D \in L_\beta$ is dense and $p \in P$, then there is a q in D extending p such that q makes no additional commitments about $\Phi_G(X)$.

Thus, for each term τ in the forcing language and each $n \in \omega$, it is possible to decide $n \in \tau$ and then extend our commitment on $\Phi_G(X)$ to record this decision.

We construct G in ω-many steps so that G is P-generic for L_β and so that $\Phi_G(X)$ records what is forced during our construction. $\square$