Conservation Questions Over $B\Sigma_2$

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First Order Arithmetic

the finite

We start with P^-, the finite set of axioms for the nonnegative part of a discretely ordered ring.

Typically, one extends P^- by starting with a familiar property of the algebra of sets and functions on $\mathbb{N}$ and then adding its definable instances as axioms.

Induction Principles

Induction for Γ (IΓ): For Γ a set of formulas, if A is defined by a formula from Γ and parameters, $0 \in A$, and A is closed under the successor, then A is the set of all numbers.

Base Theory. $P^- + \mathrm{I}\Sigma_1$ is sufficient to prove basic facts about computability. Weaker theories come up in number theoretic and complexity theoretic questions.

Kirby and Paris Equivalences

By results of Kirby and Paris (1977), $I\Sigma_n$ has several equivalent formulations.

- $I\Pi_n$
- The least element principle for Σ_n ($L\Sigma_n$) or for Π_n ($L\Pi_n$)
- Use of recursion define a total function recursively in $0^{(n-1)}$.

Bounding Principles

Bounding for Γ (BΓ): If R is a binary relation defined by a formula from Γ and parameters, then for all a,

$$(\forall x < a)(\exists y)R(x, y) \rightarrow (\exists b)(\forall x < a)(\exists y < b)R(x, y)$$

$B\Sigma_n$ also has equivalent formulations.

- (Kirby and Paris (1977)) BΠ_{n-1}
- (Gandy (unpublished)) LΔ_n
- (Slaman (2004)) IΔ_n, with the hypothesis that exponentiation is total for the case $n = 1$.

Second Order Arithmetic

the countably infinite

Now consider what happens when we enrich the mathematical environment so that we can talk about infinite subsets of $\mathbb{N}$ and their properties.

Base Theory. We will work over the base theory RCA_0, which formally asserts that P^- holds for numbers, that $I\Sigma_1$ holds relative to any set, and that the reals are closed under relative recursive definability. We note that the proofs of the first order equivalences relativize.

Closure Properties

Typical second order principles are expressed as closure properties,

$$(\forall X)(\exists Y)P(X, Y)$$

where P is an arithmetic property. Examples of P include the following.

- (ACA_0) Y is the Turing jump of X.
- (WKL_0) If X codes a tree, then Y an infinite path.
- $(BCT(\Gamma))$ Y decides every open set defined from X by a formula in Γ.
- (RT^2_2) If X is a partition of pairs of numbers into 2 pieces, then Y is an infinite X-homogeneous set.
- (COH) If X is an array of countably many sets X_i, then for all i, either $Y \subseteq^* X_i$ or $Y \subseteq^* \mathbb{N} \setminus X_i$.

Conservation Theorems

One can ask whether properties of the continuum have simply expressed but non-trivial consequences.

Definition

Given two theories $T_1 \subset T_2$ in second order arithmetic and Γ, *T_2 is conservative over T_1 for Γ* iff

$$(\forall \varphi \in \Gamma)[T_2 \vdash \varphi \implies T_1 \vdash \varphi].$$

Adding Reals

There are several conservation theorems with the following analysis. Let T_1 and T_2 be theories in second order arithmetic.

- Show that for every countable model (N, R_1) of T_1, there is a model (N, R_2) obtained by adding subsets of N to (N, R_1) such that $(N, R_2) \models T_2$.
- Conclude that T_2 is Π^1_1-conservative over T_1.

For the conservation theorem, it is sufficient to assume that R_1 has an element of greatest Turing degree.

Examples of Conservation for Induction

For Γ a set of formulas, $I\Gamma$ asserts if A is defined by a formula from Γ and parameters, $0 \in A$, and A is closed under the successor, then A is the set of all numbers.

Examples of Conservation:

- RCA_0 over $P^- + I\Sigma_1$ for arithmetic
- ACA_0 over $RCA_0 + \cup_n I\Sigma_n$ for Π^1_1
- (Harrington) WKL_0 over RCA_0 for Π^1_1
- $BCT(\Sigma_n)$ over RCA_0 for Π^1_1
- (Cholak, Jockusch, and Slaman (2001))
  - COH over RCA_0 for Π^1_1
  - RT^2_2 over $RCA_0 + I\Sigma_2$ for Π^1_1

Induction and Generic Conservation

In the previous examples, the ground model of T_1 is expanded to a model of T_2 either by adding definable sets (RCA_0, ACA_0) or by adding generic sets (BCT, WKL_0, RT^2_2).

Forcing is well suited to preserve induction:

In (N, R_1), given a formula $\varphi(n, G)$, a condition p an $n \in N$, define a sequence of conditions q_i extending p such that q_i decides $\varphi(i, G)$ and $q_i > q_{i+1}$. Use induction in (N, R_1) to show that either

- $q_a \Vdash (\forall i \leq a)\varphi(i, G)$
- or there is a least j such that $q_j \Vdash \neg\varphi(j, G)$.

There is an art is to choosing the forcing partial order appropriately.

Examples of Conservation for Bounding

$\mathrm{B}\Sigma_2$ asserts, for all X and all $\varphi \in \Sigma^0_2$,

$$(\forall x < a)(\exists y)\varphi(x, y, X) \rightarrow (\exists b)(\forall x < a)(\exists y < b)\varphi(x, y, X)$$

Examples of Conservation:

- RCA_0 over $P^- + \mathrm{B}\Sigma_2$ for arithmetic
  - Adjoin the recursive sets to a model of $P^- + \mathrm{B}\Sigma_2$: no new instances of $\mathrm{B}\Sigma_2$.
- (Hajek (1994)) WKL_0 over $RCA_0 + \mathrm{B}\Sigma_2$ for Π^1_1
  - Add super-low paths to infinite trees: no new instances of $\mathrm{B}\Sigma_2$.
- (Chong Chi Tat, Slaman, and Yang Yue) COH over $RCA_0 + \mathrm{B}\Sigma_2$ for Π^1_1
  - As described in the following.

Bounding and Generic Conservation

Bounding principles do not mix well with forcing.

Proposition (Hirschfeldt, Shore, and Slaman (2009))

If $(N, R_1) \models B\Sigma_2 + \neg I\Sigma_2$ *and* (N, R_2) *includes a set sufficiently Cohen generic over* (N, R_1)*, then* $(N, R_2) \models \neg B\Sigma_2$

Here is what goes wrong:

In the analysis of forcing, $p \Vdash \forall x < a \exists y \varphi(x, y, G)$ yields $\forall x < a \exists q_x < p \exists y (q_x \Vdash \varphi(x, y, G))$.

$p \Vdash \forall x < a \exists y \varphi(x, y, G)$ does not yield one q fixing a y for each $x < a$. *The* q_x*'s need not be compatible.*

COH

COH asserts that for any array of sets $(X_i : i \in \omega)$, there is a Y such that for all i, either $Y \subseteq^* X_i$ or $Y \subseteq^* \mathbb{N} \setminus X_i$.

Mathias Forcing is the natural way to satisfy *COH* generically.

- Conditions $p = (G_p, Y_p)$ consist of a finite set g_p and an infinite set Y_p, written as a finite intersection of X_i's and their complements.
- $p > q$ iff
  - $g_q \setminus g_p \subset Y_p$ and
  - $Y_q \subseteq Y_p$.

Notice, if all the X_i's are ω, then Mathias forcing is the same as Cohen forcing. Interpreting it in a model of $\mathrm{B}\Sigma_2 + \neg\mathrm{I}\Sigma_2$ will yield generic sets satisfying $\neg\mathrm{B}\Sigma_2$.

Extending to Ensure COH and Preserve $\mathrm{B}\Sigma_2$

- Start with a countable $(N, R_1) \models RCA_0$ in which there is a greatest Turing degree X and that X computes a sequence of sets $(X_i : i \in N)$.
- Working in (N, R_1), define a tree T such that
  - T is recursive in X' and
  - any path P in T is sufficiently Mathias generic for the sequence $(X_i : i \in N)$ that it computes a set G_P such that
    - G_P is $(X_i : i \in N)$-cohesive and
    - $(X + G_P)' <_{tt} X' + P$.
- Use the Jockusch-Soare analysis of forcing with Π^0_1-classes to show that there is a path P in T with $\mathrm{B}\Sigma_1(X' + P)$.
- Conclude $\mathrm{B}\Sigma_2(X + G_P)$.

A Challenge

Question (Cholak, Jockusch, Slaman)

Is RT^2_2 conservative over $B\Sigma_2$ for Π^1_1?