Extending Borel's Conjecture From Measure to Dimension

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Abstract

Borel (1919) defined a subset A of $\mathbb{R}$ to have strong measure zero if for every sequence of positive numbers $(\epsilon_i : i \in \omega)$ there is an open cover of A $(U_i : i \in \omega)$ such that for each i, the diameter of U_i is less than ϵ_i. Besicovitch (1956) showed that A has strong measure zero if and only if A has strong dimension zero, which means that for every gauge function f, A is null for its associated measure H^f. We say that $A \subset \mathbb{R}^N$ has strong dimension f if and only if $H^f(A) > 0$ and for every gauge function g of higher order $H^g(A) = 0$. Here, g has higher order than f when $\lim_{t \to 0^+} g(t)/f(t) = 0$. Borel conjectured that a set of strong measure zero must be countable. This conjecture naturally extends to the assertion that a set has strong dimension f if and only if it is σ-finite for H^f. Sierpiński (1928) used the continuum hypothesis to give a counterexample to Borel's conjecture and Besicovitch (1963) did the same for its generalization. Laver (1976) showed that Borel's conjecture is relatively consistent with consistent with *ZFC*, the conventional axioms of set theory including the axiom of choice. We show that its generalization to strong dimension is also relatively consistent with *ZFC*.

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1 Introduction

1.1 Hausdorff and Gauge Dimension

We work in $\mathbb{R}^N$, for some $N \geqslant 1$, in which a basic open set U is a ball of positive diameter. We use $|U|$ to denote the diameter of U.

Definition 1 (see Falconer (2003); Rogers (1998))**.** 1. A *gauge function* is a continuous increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{d \to 0^+} f(d) = 0$.

2. For a set $A \subseteq \mathbb{R}^N$ and a real number $\delta > 0$, a δ-cover of A is a countable set of basic open sets $\{U_i : i \in \omega\}$ such that for all i, U_i has diameter less than δ and such that $A \subseteq \bigcup_i U_i$.

3. Let A be a set and f a gauge function.

   (a) For $\delta > 0$, $H^f_\delta = \inf\left\{\sum f(|U_i|) : \{U_i\} \text{ is a } \delta\text{-cover of } A.\right\}$

   (b) $H^f(A) = \lim_{\delta \to 0^+} H^f_\delta(A)$.

      i. If $f(d) = d^s$, then H^f is the usual s-dimensional Hausdorff measure.

      ii. *Linear measure* refers to H^f when f is the identity function.

Notice, we could use covers consisting of closed balls of positive diameter and achieve the same values for H^f_δ and H^f. We use the term *Hausdorff system* to refer to a collection of sets of basic open sets $\{O_i : i \in \omega\}$ such that for each i, if $U \in O_i$ then $|U| < 1/(i+1)$. In addition, $\{O_i : i \in \omega\}$ covers a set A when for all i, $A \subseteq \bigcup O_i$.

Definition 2. A set A is *σ-finite for H^f* if and only if there is a countable collection $\{A_i : i \in \omega\}$ such that $A = \bigcup\{A_i\}$ and for each $i \in \omega$, $H^f(A_i) < \infty$.

Definition 3. Suppose that g and f are gauge functions. Write $f \prec g$ to indicate that $\lim_{d \to 0^+} g(d)/f(d)$ is equal to 0. In this case, say that g has *higher order* than f.

Definition 4. Suppose that $\{O_i : i \in \omega\}$ is a family of subsets of $\mathcal{N}$. We say that $\{O_i : i \in \omega\}$ is a *Hausdorff system for H^f-size k* whenever $\{O_i : i \in \omega\}$ is a Hausdorff system and for all i, $\sum_{U \in O_i} f(|U|) < k$.

Remark 5. If $f \prec g$ and A is σ-finite for H^f then $H^g(A) = 0$. Remark 5 is a consequence of the following observation: If $\{O_i : i \in \omega\}$ is a Hausdorff system for H^f-size k, then for all $\epsilon > 0$ there is an n such that $\{O_i : i > n\}$ is a Hausdorff system for H^g-size ϵ.

Definition 6. 1. For $A \subseteq \mathbb{R}^N$ and f a gauge function, A has *strong dimension* f if and only if the following conditions hold.

   (a) For all gauge functions h, if $h \prec f$ then $H^h(A) = \infty$. In this case, A is not σ-finite for all such H^h.

   (b) For all gauge functions g, if $f \prec g$ then $H^g(A) = 0$.

2. A has *strong dimension zero* when for all gauge functions g, $H^g(A) = 0$.

There are three fundamental sizes of a set A with respect to a gauge function f: $H^f(A) = 0$, $H^f(A) > 0$ and A is σ-finite for H^f, or A is not σ-finite for H^f. We study the prospects for A to have strong dimension f for each of these cases.

Case 1: $H^f(A) = 0$. This case was settled by Besicovitch.

Theorem 7 (Besicovitch (1956))**.** *If f is a gauge function, $A \subseteq \mathbb{R}^N$ and $H^f(A) = 0$ then there is a gauge function h such that $h \prec f$ and $H^h(A) = 0$.*

Consequently, if $H^f(A) = 0$, then A does not have strong dimension f.

Case 2: $H^f(A) > 0$ and A is σ-finite for H^f. This case is settled by Remark 5: If f is a gauge function, $A \subseteq \mathbb{R}^N$, $H^f(A) > 0$ and A is σ-finite for H^f then A has strong dimension f.

Case 3: A is not σ-finite for H^g. Meta-mathematical considerations appear in the final case.

Besicovitch settled the matter for analytic sets:

Theorem 8 (Besicovitch (1956), Theorem 7)**.** *Suppose that A is an analytic set, f is a gauge function, and A is not σ-finite for H^f. Then there is a gauge function g such that $f \prec g$ and A is not σ-finite for $H^g(A)$.*

Besicovitch also showed that the restriction to analytic sets A in Theorem 8 cannot be unconditionally removed.

Theorem 9 (Besicovitch (1963))**.** *Assume the Continuum Hypothesis (CH).*[*] *Then, there is a set $A \subset \mathbb{R}^2$ such that A is not σ-finite for linear measure and A has strong linear dimension.*

[*]Besicovitch (1963) does not explicitly reference the *CH*, but uses an uncountable set concentrated on the rationals, the existence of which is a consequence of the *CH*.

In Theorem 24 of Section 3, we show that it is relatively consistent with ZFC, the conventional axioms of set theory including the axiom of choice, that for all gauge functions f and all sets A, if A is not σ-finite for H^f then A does not have strong dimension f. Consequently, ZFC does not settle the question of whether there are a gauge function f and a set A such that A is non-σ-finite for H^f and of strong dimension f.

1.2 Besicovitch's Example and Sets of Strong Measure Zero

Definition 10 (Borel (1919))**.** A set $A \subseteq \mathbb{R}$ has *strong measure zero* if and only if for every sequence $(\epsilon_i : i \in \omega)$ of positive real numbers, there exists a sequence of intervals $(I_i : i \in \omega)$ such that, first, for each i, the length of I_i is less than or equal to ϵ_i and, second, $A \subseteq \bigcup\{I_i : i \in \omega\}$.

Besicovitch noted that the sets of strong measure zero have a distinguished place in the context of gauge dimension.

Theorem 11 (Besicovitch (1956))**.** *For any set $A \subset \mathbb{R}$, A has strong measure zero if and only if A has strong dimension zero, that is for every gauge function f, $H^f(A) = 0$.*

Further, strong measure zero sets are implicit in Besicovitch's Theorem 9, which we make explicit in the following.

Theorem 12 (in the style of Besicovitch (1963))**.** *If there is an uncountable set of strong measure zero, then there is a set $A \subset \mathbb{R}^2$ such that A is not σ-finite for linear measure and A has strong linear dimension.*

Proof. Suppose that $A_0 \subset \mathbb{R}$ has strong measure zero. Let A be the set of (x, y) such that $y \in A_0$ and $x \in [0, 1]$. Since A is an uncountable disjoint union of sets of positive linear measure, the horizontal line segments associated with the elements of A_0, A not σ-finite for linear measure (see Rogers, 1998, page 123, Theorem 58).

Next, let g be a super-linear gauge function, that is $\lim_{d \to 0} g(d)/d = 0$, and let δ be greater than 0. Since we may use covers by closed sets to calculate gauge measures and a closed square with sides of length ℓ can be circumscribed by a closed circle of radius $\ell\sqrt{2}$, it is enough to show that A can be covered by a collection of closed squares with side lengths ℓ_i and hence diameter lengths $\ell_i\sqrt{2}$ so that $\sum_{i \in \omega} g(\ell_i\sqrt{2}) < \delta$.

Let $(n_i : i \in \omega)$ be a sequence of positive integers such that for each i, $g(\sqrt{2}/n_i) < (\delta/2^{i+1})/n_i$. Let $(I_i : i \in \omega)$ be sequence of intervals such that $A_0 \subseteq \bigcup\{I_i : i \in \omega\}$ and for all i, $|I_i| = 1/n_i$. Let c_i be the center of I_i and let $\{B_{i,j} : 1 \leqslant j \leqslant n_i\}$ be the pairwise adjacent closed squares with sides of length $1/n_i$, sides parallel to the axes and centers on the line $y = c_i$ so that $\bigcup_{1\leqslant j\leqslant n_i} B_{i,j}$ covers all of the horizon line segments in A with y-components covered by I_i. Then, $\{B_{i,j} : i \in \omega \text{ and } 1 \leqslant j \leqslant n_i\}$ covers A. Further,

$$\sum_{i\in\omega, 1\leqslant j\leqslant n_i} g(|B_{i,j}|) \leqslant \sum_{i\in\omega} g(\sqrt{2}/n_i)n_i \leqslant \sum_{i\in\omega} (\delta/n_i 2^{i+1})n_i = \delta,$$

as required. □

Erdős, Kunen, and Mauldin (1981) showed that in Gödel's universe of constructible sets there is a co-analytic set that has strong measure zero. It follows that if every set is constructible then there is a co-analytic set A which is not σ-finite for linear measure and has strong dimension linear.

1.3 The Borel Conjecture

Borel (1919) conjectured that all strong measure zero sets are countable, which if true would void the construction in the proof of Theorem 12. This conjecture naturally extends to the assertion that a set has strong dimension f if and only if it is σ-finite for H^f. By Theorem 12, this assertion for strong dimension implies Borel's conjecture so it is indeed an extension.

Sierpiński (1928) used the *CH* to construct an uncountable set of strong measure zero, hence a counter example to Borel's conjecture. Again by Theorem 12, if the *CH* holds, then there is a set $A \subset \mathbb{R}^2$ such that A is not σ-finite for linear measure and A has strong linear dimension, which is Theorem 9.

Laver (1976) established the relative consistency of Borel's conjecture:

Theorem 13 (Laver (1976))**.** *If ZFC is consistent, then so is ZFC together with Borel's conjecture.*

Since a counter example to the Borel Conjecture yields a set of strong linear dimension which is not σ-finite for linear measure, Laver's model is the natural context in which to analyze the case of strong dimension f for a set which is not σ-finite for a gauge measure H^f.

2 Laver's Model

In this section and the next one, we conform to the notation of Laver (1976) to the extent that it is possible. Similarly, our summary closely follows the original development.

2.1 Laver Forcing

We summarize the properties of Laver forcing. Most importantly, we review how Laver forcing admits fusion arguments. We do make one systematic change in notation: for forcing conditions p and q, $p \geqslant q$ indicates that q is a stronger condition than p.

We let $\mathcal{M}$ denote a countable transitive model of $ZFC+CH$. CH is not necessary for the basic definitions and properties, but it will be essential for our application. We view the following as taking place in $\mathcal{M}$.

The Laver partial ordering $\mathcal{J}$ consists of a collection of subtrees T of $\omega^{<\omega}$ for which there is a member $T\langle 0\rangle$ of T, the stem of T, such that every element of T is compatible with $T\langle 0\rangle$ and for all $\sigma \in T$, if σ is equal to $T\langle 0\rangle$ or extends $T\langle 0\rangle$ then there are infinitely many i such that $\sigma * i$, the extension of σ obtained by appending i, belongs to T. When $T \in \mathcal{J}$ and $T\langle 0\rangle \subseteq \sigma \in T$, T_σ is the subtree of T consisting of those sequences in T which are compatible with σ. Note that σ is the stem of T_σ. For T_1 and T_2 in $\mathcal{J}$, $T_1 \geqslant T_2$ when $T_2 \subseteq T_1$. In this case, say that T_2 is stronger than T_1. We will identify a $\mathcal{J}$-generic filter with the unique element $G \in \omega^\omega$ which is an infinite path through all of the trees in that filter. We refer to G as a Laver generic real.

We fix an enumeration of $\omega^{<\omega}$, $\sigma_1, \sigma_2, \dots$ so that, first, if $\sigma_i \subsetneq \sigma_j$ then $i < j$ and, second, if $n < m$, $\sigma_i = \sigma * n$ and $\sigma_j = \sigma * m$ then $i < j$. For any $T \in \mathcal{J}$, this provides an enumeration of $\{\sigma : \sigma \in T \text{ and } T\langle 0\rangle \subseteq \sigma\}$ under the natural isomorphism between this set and $\omega^{<\omega}$. The strings $\{T\langle 0\rangle, \dots, T\langle n\rangle\}$ determine a maximal antichain $\{T_i : i \leqslant n\}$ below T by letting T_i be the union of all of the T_σ's such that σ is an immediate successor of $T\langle i\rangle$ in T and σ is not $T\langle j\rangle$ for any j less than or equal to n. For S and T in $\mathcal{J}$, $S \geqslant^n T$ means that $S \geqslant T$ and, for all i less than or equal to n, $S\langle i\rangle = T\langle i\rangle$.

$\mathcal{P}_\alpha$ is the α-length iteration of forcing with $\mathcal{J}$ using countable support, defined along with its order $\geqslant_\alpha$, greatest element o_α and forcing relation $\Vdash_\alpha$. by induction on α:

1. $\mathcal{P}_1$ is the set of all functions from $1 = \{0\}$ into $\mathcal{J}$ ordered by $p \geqslant_1 q$ if and only if $p(0) \geqslant q(0)$ in $\mathcal{J}$.

2.
$$\mathcal{P}_{\alpha+1} = \left\{ p : \begin{array}{l} \text{dom } p = \alpha+1,\ p{\restriction}\alpha \in \mathcal{P}_\alpha, \text{ and} \\ p(\alpha) \text{ is a canonical term for a member of } \mathcal{J} \text{ in } \mathcal{M}[G_\alpha] \end{array} \right\}.$$

   For p and q in $\mathcal{P}_{\alpha+1}$, $p \geqslant_{\alpha+1} q$ if and only if $p{\restriction}\alpha \geqslant_\alpha q{\restriction}\alpha$ and $q{\restriction}\alpha \Vdash_\alpha p(\alpha) \geqslant q(\alpha)$.

3. When α is a limit ordinal, $\mathcal{P}_\alpha$ is the set of all p with domain α such that, first, if $1 \leqslant \beta < \alpha$, then $p \restriction \beta \in \mathcal{P}_\beta$ and, second, for all but countably many β with $1 \leqslant \beta < \alpha$, $\Vdash_\beta p(\beta) = 0$. Here, we use 0 to denote the greatest element of $\mathcal{J}$, namely $\omega^{<\omega}$.

When $1 \leqslant \alpha < \beta \leqslant \omega_2$, $\mathcal{P}^{\alpha\beta}$ is the set of functions f with domain $[\alpha, \beta)$ such that $o_\alpha \cup f$ is an element of $\mathcal{P}_\beta$. For p in $\mathcal{P}_\alpha$ or in $\mathcal{P}^{\alpha\beta}$, the support of p is the set of γ in the domain of p such that $\nVdash_\gamma p(\gamma) = 0$. G_α refers to a set which is generic over the understood model $\mathcal{M}$ for the forcing $\mathcal{P}_\alpha$. $\mathcal{P}_\alpha$ is naturally embedded in P_β, and we can write $G_\alpha = \{p \restriction \alpha : p \in G_\beta\}$. Similarly, $\mathcal{M}[G_\beta] = \mathcal{M}[G_\alpha][G^{\alpha\beta}]$, with $G^{\alpha\beta}$ generic over $\mathcal{M}[G_\alpha]$ for $\mathcal{P}^{\alpha\beta}$.

Theorem 14 (Laver (1976))**.** *Let $\mathcal{M}$ be a countable model of ZFC + CH and G_{ω_2} be $\mathcal{P}_{\omega_2}$-generic over $\mathcal{M}$.*

1. *$\mathcal{P}_{\omega_2}$ has the ω_2-chain condition.*
2. *For all $\alpha \leqslant \omega_2$, all cardinals in $\mathcal{M}$ are preserved in $\mathcal{M}[G_\alpha]$.*
3. *For all $\alpha < \omega_2$, $\mathcal{M}[G_{\omega_2}] = \mathcal{M}[G_\alpha][G^*_{\omega_2}]$, where $G^*_{\omega_2}$ is $(\mathcal{P}_{\omega_2})^{\mathcal{M}[G_\alpha]}$-generic over $\mathcal{M}[G_\alpha]$. Here, $(\mathcal{P}_{\omega_2})^{\mathcal{M}[G_\alpha]}$ is $\mathcal{P}_{\omega_2}$ as defined in $\mathcal{M}[G_\alpha]$.*

Going forward, we will omit the subscripts when the forcing partial order is clear from context.

2.2 Fusion Sequences

Suppose that $1 \leqslant \alpha \leqslant \omega_2$, F is a finite subset of α and $n < \omega$. For p and q in $\mathcal{P}_\alpha$, $p \geqslant^n_F q$ if and only if $p \geqslant q$ and for all $\beta \in F$, $q \restriction \beta \Vdash p(\beta) \geqslant^n q(\beta)$.

Remark 15. Consider p, n and F as above. Just as the strings $T\langle 0\rangle, \dots, T\langle n\rangle$ determine a finite maximal antichain below T in the partial order for adding a single Laver real, n and F determine a finite maximal antichain below p in $\mathcal{P}_\alpha$, all the elements of which have the same support as p.

Lemma 16 provides the infrastructure for fusion arguments.

Lemma 16 (Laver (1976))**.** *Suppose that $1 \leqslant \alpha \leqslant \omega_2$, p_n, $n \in \omega$, are members of $\mathcal{P}_\alpha$ and F_n, $n \in \omega$, is an increasing chain of finite sets such that $\bigcup_{n\in\omega} F_n$ is equal to the union of the supports of the p_n, and for each $n \in \omega$, $p_n \geqslant^n_{F_n} p_{n+1}$. Then there is a $p_\omega \in \mathcal{P}_\alpha$ such that for all $n \in \omega$, $p_n \geqslant^n_{F_n} p_\omega$. Further, p_ω has support the union of the supports of the p_n's and is unique up to forced equivalence in $\mathcal{P}_\alpha$.*

Lemma 17 provides the means to decide formulas by thinning trees rather than extending their stems.

Lemma 17 (Laver (1976))**.** *Suppose that $1 \leqslant \alpha \leqslant \omega_2$, p is a condition in $\mathcal{P}_\alpha$, $F = \{\alpha_1 < \cdots < \alpha_i\}$ is a finite subset of α, and $n \in \omega$.*

1. *For φ a sentence in the forcing language, there is a q such that $p \geqslant^0_F q$ such that q decides φ, that is either $q \Vdash_\alpha \varphi$ or $q \Vdash_\alpha \neg\varphi$.*

2. *If $k < \omega$ and $p \Vdash \bigvee_{j<k} \varphi_j$, then there is an $I \subseteq \{0, \ldots, k\}$ such that I has cardinality less than or equal to $(n+i)^i$ and there is a q such that $p \geqslant^n_F q$ such that $q \Vdash \bigvee_{j \in I} \varphi_j$.*

3 Strong Dimension in Laver's Model

3.1 Reflecting Non-σ-finiteness

Lemma 18. *Suppose that $1 \leqslant \alpha \leqslant \omega_2$ and α has cofinality ω_1 in $\mathcal{M}$. For every $p \in \mathcal{P}_\alpha$, every finite F contained in α and every term t such that $p \Vdash_\alpha t \in 2^\omega$, there are q in $\mathcal{P}_\alpha$ with $p \geqslant^0_F q$ and $\beta < \alpha$ such that $q \Vdash_\alpha t \in \mathcal{M}[G_\beta]$.*

Proof. Let α, p and t be fixed as above. We proceed to find q by application of Lemma 16 in a fusion argument. We define a sequence $p = p_0 \geqslant^0_{F_0} p_1 \geqslant^1_{F_1} p_2 \ldots$ with associated finite sets $F = F_0 \subseteq F_1 \subseteq \ldots$. We arrange by standard bookkeeping, which we leave unspecified here, that $\bigcup\{F_n : n \in \omega\}$ is equal to the union of the supports of the p_n's. We determine p_{n+1} from p_n as follows. Let $\{p_{n,i} : i < k_n\}$ be the finite maximal antichain below p_n determined by p_n, n and F_n as in Remark 15, where k_n is the size of this antichain. By Lemma 17, for each $i < k_n$, let $q_{n,i}$ be a condition in $\mathcal{P}_\alpha$ such that $p_{n,i} \geqslant^0_{F_n} q_{n,i}$ and $q_{n,i}$ decides the value of t at argument n. Let p_{n+1} be the condition obtained by taking the disjunction of the $q_{n,i}$'s. Then, $p_n \geqslant^n_{F_n} p_{n+1}$ and for each n, the value of t at n is determined by the values of G_α on the ordinals in F_n. By Lemma 17, take p_ω to be the fusion of the p_n's. Thus, p_ω forces that the element of 2^ω denoted by t in $\mathcal{M}[G_\alpha]$ is an element of $\mathcal{M}[G_{\beta+1}]$, where β is the supremum of the support of of p_ω. Since p_ω has countable support and α has uncountable cofinality, $\beta < \alpha$, as required. □

By Lemma 18, if α has cofinality ω_1, $x \in 2^\omega$ and $x \in \mathcal{M}[G_\alpha]$, where G_α is $\mathcal{P}_\alpha$-generic over $\mathcal{M}$, then there is a $\beta < \alpha$ such that $x \in \mathcal{M}[G_\beta]$. This observation applies not only to elements of 2^ω but also to elements of

$\mathbb{R}^N$, Hausdorff systems and every other type of set that is determined by a conjunction of countably many Boolean properties.

Lemma 19. *Suppose that* $\mathcal{M}[G_{\omega_2}] \models$ *"*$A \subseteq \mathbb{R}^N$ *is non-*σ*-finite for* f*." There is an* $\alpha < \omega_2$ *such that* $A \cap \mathcal{M}[G_\alpha]$ *is an element of* $\mathcal{M}[G_\alpha]$ *and such that* $\mathcal{M}[G_\alpha] \models$ *"*$A \cap \mathcal{M}[G_\alpha]$ *is non-*σ*-finite for* f*."*

Proof. Since $\mathcal{P}_{\omega_2}$ has the ω_2-chain condition, there is a $\beta_0 < \omega_2$ and a countable collection of maximal antichains which decide the values of f on all rational arguments, and by continuity decide the values of f everywhere. Fix β_0 so that every condition from any of these antichains has support contained in β_0. Fix a canonical term t_f for f.

Let t_A be a term in the forcing language such that A is the set in $\mathcal{M}[G_{\omega_2}]$ denoted by t_A. Fix an initial condition p_0 which forces "t_A is not σ-finite for H^{t_f}."

We define a function $\beta^* : \omega_2 \to \omega_2$ as follows. Let $\beta \in [\beta_0, \omega_2)$. By (Laver, 1976, Lemma 10), $\mathcal{M}[G_\beta] \models 2^\omega = \aleph_1$. Fix a term for a map ψ in $\mathcal{M}[G_\beta]$ from ω_1 onto $(\mathbb{R}^N)^{\mathcal{M}[G_\beta]}$. Since $\mathcal{P}_{\omega_2}$ has the ω_2-chain condition and each condition has countable support, there is a $\beta^*(\beta) \in [\beta, \omega_2)$ such that, for each $\gamma < \omega_1$, there are two maximal antichains of conditions stronger than p_0 in $\mathcal{P}_{\omega_2}$ all of whose elements have support contained in $\beta^*(\beta)$ such that each element q of the first antichain decides whether $\psi(\gamma)$ belongs to A and each element r of the second antichain decides whether $\psi(\gamma)$ codes a countable collection of Hausdorff systems, all of which assign finite H^f-measure, and if r does force that $\psi(\gamma)$ codes such a system then there is a term t for a real in $\mathcal{M}[\beta^*(\beta)]$ such that r forces that t is an element of A and t is not covered by any of the Hausdorff systems coded by $\psi(\gamma)$. Antichains of the second type exist by the initial assumption that A is forced to be non-σ-finite for H^f in $\mathcal{M}[G_{\omega_2}]$.

Choose $\alpha \in [\beta_0, \omega_2)$ be of cofinality ω_1 so that for all $\beta \in [\beta_0, \alpha)$, $\beta^*(\beta)$ is less than α. By Lemma 18, for every element of $2^\omega \cap \mathcal{M}[G_\alpha]$ belongs to some $\mathcal{M}[G_\beta]$, where $\beta < \alpha$. Thus, for every element of $2^\omega \cap \mathcal{M}[G_\alpha]$, the maximal antichains that decide whether that real is an element of A belong to $\mathcal{M}[G_\alpha]$. Since the forcing relation is definable in $\mathcal{M}$. $A \cap \mathcal{M}[G_\alpha]$ is definable in $M[G_\alpha]$. Further, for every countable collection of Hausdorff systems in $\mathcal{M}[G_\alpha]$, all of which assign finite H^f measure, there is an element of $R^N \cap \mathcal{M}[G_\alpha]$ which is not covered by any of them. Hence, $A \cap \mathcal{M}[G_\alpha]$ is not σ-finite for H^f in $\mathcal{M}[G_\alpha]$. □

3.2 Exhibiting a σ-finite challenge

By the above, we have reduced to the situation of forcing over a model $\mathcal{M} \models ZFC + CH$ using the partial order $\mathcal{P}_{\omega_2}$, f and A are elements of $\mathcal{M}$, and $\mathcal{M} \models$ "A is not σ-finite for f." Since $\mathbb{R}^N$ is a countable union of totally bounded sets, there is a totally bounded set such that its intersection with A is not σ-finite for f. By replacing A with such an intersection, we may also assume that A is totally bounded. We let b denote an integer such that A is contained in the ball around the origin of radius b.

Next, since we are concerned only with the distinctions between null, σ-finite and non-σ-finite, we may replace H^f with a more tractable net measure using a restricted collection of basic open sets. Here, we follow (Rogers, 1998, Theorem 49, page 102). For integers $i_1, \ldots, i_n$ and a positive integer r, let $N_r(i_1, \ldots, i_n)$ be the set of points $(x_1, \ldots, x_n)$ such that for all j between 1 and n, $i_j/2^r \leqslant x_j \leqslant (i_j + 1)/2^r$. Let $\mathcal{N}$ be the set of such $N_r(i_1, \ldots, i_n)$. Define the pre-measure τ^f on $\mathcal{N}$ by $\tau(\varnothing) = 0$ and $\tau^f(N_r(i_1, \ldots, i_n) = f(\sqrt{n}/2^r) = f(\text{diameter}(N_r(i_1, \ldots, i_n))$. Define H^{τ^f} as in Definition 1 using the family of basic sets $\mathcal{N}$ with the pre-measure τ^f.

Theorem 20 ((Rogers, 1998, Theorem 49, page 102)). *Suppose that f is a gauge function and n is a positive integer. For any set $E \subseteq \mathbb{R}^N$,*

$$H^f(E) \leqslant H^{\tau^f}(E) \leqslant 3^n 2^{n(n+1)} H^f(E).$$

Thus, A is σ-finite for H^f if and only if it is σ-finite for H^{τ^f}.

Convention 21. For the remainder of this section, we will identify H^f with H^{τ^f}.

Remark 22. For any positive real numbers $a_1 < a_2$, there are only finitely many elements U of $\mathcal{N}$ such that $|U| \in [a_1, a_2]$ and U has nonempty intersection with the ball about the origin of radius b. Since A is taken to be a subset of this ball, all other elements of $\mathcal{N}$ in this diameter-range have empty intersection with A.

Definition 23. Suppose that f is a gauge function and G is an increasing function from ω to ω. For $r \in \omega$, let $k(r)$ be the maximal k such that $1/G(k) > \sqrt{n}/2^r$ if this number is defined and greater than 0 and let $k(r)$ equal 1 otherwise. Let $g_{f,G}$ be a gauge function such that for all r, $g(\sqrt{n}/r^r) = f(\sqrt{n}/r^r)/k(r)$.

In other words, for all elements U of $\mathcal{N}$, $g_{f,G}(|U|)$ is either equal to $f(|U|)$ or is equal to $f(|U|)/k$, where k is the greatest integer such that $1/G(k) > |U|$.

3.3 Consistency

For the duration of this section, fix $\mathcal{M}$ to be a countable model of *ZFC*+ *CH*. We force over $\mathcal{M}$ with the partial order $\mathcal{P}_{\omega_2}$ as defined in $\mathcal{M}$. Let G_{ω_2} be generic over $\mathcal{M}$ for this partial order.

Theorem 24. *If ZFC is consistent, then so is ZFC together with "For every N, every $A \subseteq \mathbb{R}^N$ and every gauge function f, if A is not σ-finite for H^f then there is a gauge function g such that $f \prec g$ and $H^g(A) > 0$. Consequently, A does not have strong dimension f.*

Proof. Suppose that f is a gauge function in $\mathcal{M}[G_{\omega_2}]$, A is a subset of $\mathbb{R}^N$ in $M[G_{\omega_2}]$, and $\mathcal{M}[G_{\omega_2}] \models$ "$A \subseteq \mathbb{R}^N$ is non-σ-finite for H^f." By Lemma 19, there is an α such that f and $A \cap \mathcal{M}[G_\alpha]$ are elements of $\mathcal{M}[G_\alpha]$ and such that $\mathcal{M}[G_\alpha] \models$ "$A \cap \mathcal{M}[G_\alpha]$ is non-σ-finite for f." By Theorem 14, $\mathcal{M}[G_{\omega_2}]$ is $\mathcal{P}_{\omega_2}$-generic over $\mathcal{M}[G_\alpha]$. By replacing $\mathcal{M}$ with $\mathcal{M}[G_\alpha]$, we may assume that both f and $A = A \cap \mathcal{M}[G_\alpha]$ belong to our ground model $\mathcal{M}$ and that b is given so that A is contained in the ball about the origin of radius b. We may also assume that the initial condition in $\mathcal{P}_1$ is such that all of its infinite paths are increasing functions from ω to ω. For $g = g_{f,G(0)}$ (as in Definition 23), we show that $\mathcal{M}[G_{\omega_2}] \models H^g(A) > 0$.

For the sake of a contradiction, suppose that O is a term, $p \in \mathcal{P}_{\omega_2}$ is stronger than p_0, and $p \Vdash$ "O is a Hausdorff system for H^g-size 1 covering A." Let O_i denote a term for the ith component of the term O. We many assume that p forces that every element of every O_i has nonempty intersection with the ball about the origin of radius b. Note,

$$p \Vdash \left(\begin{array}{l} O_i \text{ is a cover of } A;\ U \in O_i \text{ implies } |U| < 1/(i+1); \text{ and} \\ \sum_{U \in O_i} g(|U|) \leqslant 1. \end{array} \right)$$

By the definition of g from f and G, for each $k \in \omega$, p forces that the sum of $\{f(|U|) : U \in O_i \text{ and diameter}(U) \in [1/G(k+1), 1/G(k))\}$, is less than or equal to k.

We define a stronger condition q such that $p(0) \geqslant^0 q(0)$ and $p \geqslant q$. We also define a function C which maps the elements of $q(0)$ to finite subsets of $\mathcal{N}$ so that the following conditions hold.

1. For each $\sigma \in q(0)$, if σ is contained or equal to the stem of $q(0)$ or σ is an immediate extension of the stem of $q(0)$ then $C(\sigma) = \varnothing$.

2. For every $\sigma * i * j \in q(0)$, the elements of $C(\sigma * i * j)$ have diameters in the interval $[1/j, 1/i)$.

3. For every $n \in \omega$, there is a c such that for all $\sigma \in q(0)$, if σ has length n then $\sum_{U \in C(\sigma)} f(|U|)$ is less than c.
4. For $C^\infty(G(0)) = \bigcup_{n\in\omega} C(G(0)\restriction n)$, $q \Vdash$"$C^\infty(G(0))$ is a cover of A."
5. For every $\sigma \in q(0)$ and for every $U \in \mathcal{N}$, either there are finitely many i such that $\sigma * i \in q$ and $U \in C(\sigma * i)$ or there are finitely many i such that $\sigma * i \in q(0)$ and $U \notin C(\sigma * i)$.

We construct q by a fusion argument similar to the one that appears in (Laver, 1976, Lemma 14). The elements of $q(0)$ are simultaneously specified by recursion on their length. The elements of the other coordinates are specified by fusion according to the pattern of Lemma 16. Let $p_0 = p_1 = p$ and let a be the length of $p_1(0)\langle 0\rangle$, the stem of the first coordinate of p_1. Define $C(\sigma) = \varnothing$ for all $\sigma \in p_1(0)$ such that the length of σ is less than or equal to $a + 1$.

We define $p_0, p_1, p_2, \ldots$ and $F_0, F_1, F_2, \ldots$ so that $\bigcup_{j\in\omega} F_j$ is equal to $\bigcup_{j\in\omega}$ support $p_j \setminus \{0\}$ and so that all the nodes in $p_j(0)$ of length less than or equal $a + j$ belong to $p_{j+1}(0)$. The F-sequence is defined by standard bookkeeping, with the proviso that $0 \in F_0$. Let $|F_i|$ denote the cardinality of F_i.

Suppose that p_j has been defined and that C has been defined on all nodes in $p_j(0)$ of length less than or equal to $a + j$. Consider, τ in $p_j(0)$ of length $a + j + 1$. Let $p_{j,\tau}$ be the extension of p_j obtained by restricting $p_j(0)$ to $p_j(0)_\tau$, the tree of elements of $p_j(0)$ which are compatible with τ, and leaving the other coordinates of p_j fixed. Let $i = \tau(a + 1)$, the number that appears in τ immediately after the stem of $p_0(0)$. Let J be the finite collection of subsets S of $\mathcal{N}$ such that, first, every U in S has nonempty intersection with the ball about the origin of radius b, second, every U in S has diameter in $(1/\tau(a + j), 1/\tau(a + j - 1)]$ and, third, $\sum_{U\in S} f(|U|)$ is less than or equal to $a + j - 1$. By the remarks above, $p_{j,\tau}$ forces that the set of elements of O_i with diameter in $(1/\tau(a+j), 1/\tau(a+j-1)]$ is a set in J. By Lemma 17, take let r^τ and I^τ be such that $p_{j,\tau} \geqslant^{j+1}_{F_{j+1}} r^\tau$, I^τ is a collection of less than or equal to $(j + 1 + |F_{j+1}|)^{|F_{j+1}|}$ many sets, such that r^τ forces that the set of elements of O_i with diameter in $(1/\tau(a + j), 1/\tau(a + j - 1)]$ belongs to I^τ. Define $C(\tau)$ to be the union of I^τ. Note that $\sum_{U\in C(\tau)} f(|U|)$ is less than or equal to $(j + 1 + |F_{j+1}|)^{|F_{j+1}|}(a + j - 1)$, which is a constant that depends only on the length of τ. Let p_{j+1} be the disjunction of $\{r^\tau : \ \tau \in p_j(0) \text{ and } |\tau| = a + j + 1\}$.

Define p_∞ so that $p_\infty(0) = \bigcap_{j\in\omega} p_j(0)$ and $p_\infty\restriction[1, \omega_2)$ is equal to the fusion of $\{p_j \upharpoonright [1, \omega_2 : j < \omega\}$, which is forced to exist as a condition by $p_\infty(0)$, according to Lemma 16. By construction, clauses (1-4) are satisfied by p_∞.

Finally, we extend p_∞ to q to also satisfy clause (5) by thinning $p_\infty(0)$. Define $\tau \in T \subseteq p_\infty(0)$ by recursion on the length of τ. For step 0, $p_\infty(0)\langle 0\rangle$, the stem of T, and all of its initial segments belong to T. For the recursion step, fix an enumeration $U_0, U_1, \ldots$ of $\mathcal{N}$ in ordertype ω. Suppose that $\tau \in T$ at the end of step s and has maximal length among such. We decide which immediate extensions of τ will be in T by an nested recursion. In this recursion, we also define an auxiliary sequence of sets $X_0, X_1, \ldots$.

For step 0 of the nested recursion, let i_0 be least such that $\tau * i_0 \in p_\infty(0)$. If the set of $n > i_0$ such that $\tau * n \in p_\infty(0)$ and $U_0 \in C(\tau * n)$ is infinite, let X_0 be this set. Otherwise, let X_0 be the set of n such that $i_0 < n$, $\tau * n \in p_\infty(0)$ and $U_0 \notin C(\tau * n)$. For the recursion step, suppose that $i_0 < i_1 < \cdots < i_j$ and $X_0 \supset X_1 \supset \cdots \supset X_j$ are defined. Let i_{j+1} be the least element of X_j. If the set of $n > i_{j+1}$ such that $n \in X_j$ and $U_{j+1} \in C(\tau * j)$ is infinite, then let X_{j+1} be this set. Otherwise, let X_{j+1} be the set of $n \in X_j$ such that $n > i_{j+1}$ and $U_{j+1} \notin C(\tau * n)$. Let q be the condition for which $q(0) = T$ and $q \upharpoonright [1, \omega_2) = p_\infty \upharpoonright [1, \omega_2)$. Since it extends p_∞, q satisfies clauses (1-4). By the construction of T, q also satisfies clause (5). This completes the definition of q and C, with the required properties.

For $\sigma \in q(0)$, we define

$$C^+(\sigma) = \left\{ U : \begin{array}{l} \text{For all but finitely many } i, \text{ if } \sigma * i \in q(0) \text{ then} \\ \text{then } U \in C(\sigma * i) \end{array} \right\}.$$

For $\sigma \in q(0)$ and $x \in \mathcal{M}$, say that σ *2-covers* x if $C(\sigma) \cup C^+(\sigma)$ covers $\{x\}$.

In preparation for the construction to follow, consider $\sigma \in q(0)$ and $n \in \omega$. Define $E(\sigma, n)$ to be the set of x such that σ does not 2-cover x and for all $i > n$, if $\sigma * i \in q(0)$ then $\sigma * i$ 2-covers x. We now show that $E(\sigma, n)$ has finite measure with respect to H^f.

First, note that since $E(\sigma, n)$ excludes elements that are 2-covered by σ, $E(\sigma, n)$ has no element which is covered by $C^+(\sigma)$. For $i > n$ such that $\sigma * i \in q(0)$, let $E_i(\sigma, n)$ be the set covered by $C(\sigma * i) \backslash C^+(\sigma) \bigcup C^+(\sigma * i)$. $E_i(\sigma, n)$ is the set of x such that x is 2-covered by $\sigma * i$, when sets in $C^+(\sigma)$ are excluded, so $E(\sigma, n) \subseteq E_i(\sigma, n)$.

By application of property (3) of C, let c be such that for all $\tau \in q(0)$ such that the length of τ is two more than the length of σ

$$\sum \left\{ f(|U|) : \exists \tau_0 (\tau_0 \subseteq \tau \text{ and } U \in C(\tau_0)) \right\} \leqslant c.$$

The contribution to $E_i(\sigma, n)$ from $C(\sigma * i)\backslash C^+(\sigma)$ is covered by $C(\sigma * i)$ and so

$$\sum\Big\{f(|U|) : U \in C(\sigma * i)\backslash C^+(\sigma)\Big\} \leqslant c.$$

The contribution to $E_i(\sigma, n)$ from $C^+(\sigma * i)$ is also bounded by c, since every finite partial f-sum of diameters taken from $C^+(\sigma * i)$ is contained in the f-sum of diameters taken from some $C(\sigma * i * j)$ and these f-sums are uniformly bounded by c. Thus,

$$\sum\Big\{f|U| : U \in C(\sigma * i)\backslash C^+(\sigma) \bigcup C^+(\sigma * i)\Big\} < 2c$$

Next, note that for every $U \in \mathcal{N}$, there are only finitely many i such that $U \in C(\sigma * i)\backslash C^+(\sigma)$. This is because every $U \in \mathcal{N}$ is either in finitely many $C(\sigma * i)$ or in cofinitely many of them, and removing $C^+(\sigma)$ excludes every U which is in the cofinite case. Further, any element U of $C^+(\sigma * i)$ is an element of $C(\sigma * i * j)$ for cofinitely in j many of the $C(\sigma * i * j)$ such that $\sigma * i * j \in q(0)$. In particular, any U in $C^+(\sigma * i)$ has $|U| < 1/i$. Consequently, for every $\delta > 0$ there is an i such that for all $j > i$, if $U \in C(\sigma * j)\backslash C^+(\sigma) \bigcup C^+(\sigma * j)$ then $\delta > |U|$.

Thus, the collection

$$\Big\{C(\sigma * i)\backslash C^+(\sigma) \ \cup \ C^+(\sigma * i) : i > n \text{ and } \sigma * i \in q(0)\Big\}$$

contains a Hausdorff system for H^f-size $2c$ which covers $E(\sigma, n)$, and therefore $E(\sigma, n)$ has finite measure with respect to H^f.

Let E be the union of the sets $E(\sigma, n)$, for σ in $q(0)$ and $n \in \omega$. E is an element of $\mathcal{M}$. In $\mathcal{M}$, E is σ-finite for H^f.

We work in $\mathcal{M}$. Since A is not σ-finite for H^f, let a be an element of A which is not in E. We build an extension T_a of $q(0)$ such that for all $\tau \in T_a$, $a \notin C(\tau)$. We begin by setting the stem of T_a to be $q(0)\langle 0\rangle$, the stem of $q(0)$. By definition of C, for all i with $q(0)\langle 0\rangle * i \in q(0)$, $C(q(0)\langle 0\rangle * i) = \varnothing$, so the set of x such that $q(0)\langle 0\rangle$ 2-covers x is empty. Now, assume that we have reached step n of our recursion and no τ in T_a from an earlier step 2 covers a. Since a is not an element of E, for every τ of maximal length which was added to T_a there are infinitely many i such that $\tau * i$ is in $q(0)$ and does not 2-cover a. We conclude step $n + 1$ of the construction by adding each such $\tau * i$ to T_a. Finally, T_a is the tree obtained in the limit. Let r be the condition with $r(0) = T_a$ and for $\beta \in [1, \omega_2)$, $r(\beta) = q(\beta)$. Then, $q \geqslant r$ and r forces that $C(G(0))$ is not a cover of A, which is a contradiction to property (4) of C. □

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