EFFECTIVE RANDOMNESS FOR CONTINUOUS MEASURES

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Abstract. We investigate which infinite binary sequences (reals) are effectively random with respect to some continuous (i.e. non-atomic) probability measure. We prove that for every n, all but countably many reals are n-random for such a measure, where n indicates the arithmetical complexity of the Martin-Löf tests allowed. The proof is based on a Borel determinacy argument and presupposes the existence of infinitely many iterates of the power set of the natural numbers. In the second part of the paper we present a metamathematical analysis showing that this assumption is indeed necessary. More precisely, there exists a computable function $g(n)$ such that, for any n, the statement "All but countably many reals are $g(n)$-random with respect to a continuous probability measure" cannot be proved in ZFC_n^-. Here ZFC_n^- stands for Zermelo-Fraenkel set theory with the Axiom of Choice, where the Power Set Axiom is replaced by the existence of n-many iterates of the power set of the natural numbers. The proof of the latter fact rests on a very general obstruction to randomness, namely the presence of an internal definability structure.

1. Introduction

In [39], we investigated under what circumstances a real (i.e. an infinite binary sequence) is Martin-Löf random with respect to some, not necessarily uniform probability measure. We were able to show that if an infinite binary sequence (real) X is not computable (i.e. there is no algorithm to determine, uniformly in n, the n-th bit of X), then there exists a probability measure μ such that X is random for μ. On the other hand, it is not hard to see that if a real X is computable, then the only way that X is random with respect to μ is that a positive probability is assigned to $\{X\}$, i.e. X an *atom* of μ.

Hence having non-trivial random content (with respect to some measure) in the most general sense is equivalent to being non-computable. While this may

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be seen as an interesting characterization of computability, it also indicates that including arbitrary probability measures results in a classification that is too coarse. In order to obtain a more interesting structure of random reals, it seems reasonable to restrict the class of admissible probability measures.

In this paper, we investigate randomness with respect to *non-atomic*, or *continuous*, probability measures. The restriction to continuous measures seems reasonable, since atoms (which can always be retrieved effectively from a representation of a measure) can code additional information. In fact, in [39], atoms were used to "conceal" the rather low information content of a real (such as a K-trivial real), to make this real random for some probability measure.

It turns out that if restricted to the continuous case, a fascinating interplay between randomness and logical complexity emerges.

One can relativize the concept of a Martin-Löf test with respect to a parameter. This means that the test, which is essentially an effectively defined G_δ nullset, has to be effectively G_δ only in some parameter Z, thereby enlarging the family of admissible nullsets and, correspondingly, shrinking the set of random reals. If the parameter Z is an instance $\emptyset^{(n)}$ of the Turing jump, i.e. Z is real that can decide all Σ_n statements about arithmetic, we speak of n-randomness.

Our first main result concerns the size of the set of reals that are not n-random with respect to any continuous measure.

Theorem 1. *For any $n \in \omega$, all but countably many reals are n-random with respect to some continuous probability measure.*

The proof features a metamathematical argument. Let us denote by NCR_n the set of all reals that are not n-random with respect to any continuous probability measure. We show that for each n, NCR_n is contained in a countable model of a fragment of set theory. More precisely, this fragment is ZFC^-_n, where ZFC^-_n denotes the axioms of Zermelo-Fraenkel set theory with the Axiom of Choice, with the power set axiom replaced by a sentence that assures the existence of n iterates of the power set of the natural numbers.

One may wonder whether this metamathematical argument is really necessary to prove the countability of a set of reals, in particular, whether one needs the existence of infinitely many iterates of the power set of ω to prove Theorem 1, a result about sets of reals. It turns out that this is indeed the case. This is the subject of our second main result.

Theorem 2. *For any $k \in \omega$ there exists a number $n_k \in \omega$ such that the statement*

> *"There exist only countably many reals that are not n_k-random with respect to some continuous probability measure."*

is not provable in ZFC^-_k.

This metamathematical property of NCR is reminiscent of *Borel determinacy* [30]. Even before Martin proved that every Borel game is determined, Friedman [10] had shown that any proof of Borel determinacy had to use infinitely many iterates of the power set of ω. Borel determinacy is a main ingredient in our proof of Theorem 1. Theorem 2 establishes that this use is, in a certain sense, inevitable.

Theorem 2 is proved via a fine structure analysis of the countable models used to show NCR_n is countable. These models are certain levels L_β of Gödel's constructible hierarchy. In these L_β (or rather Jensen's version, the J-hierarchy) we exhibit cofinal sequences of non-random reals. These reals will be so-called *master codes* [3, 20], reals that code the L_β in a way that arithmetically reflects the strong stratification between the structures L_β. The main feature of this proof is a very general principle that manifests itself in various forms: an internal stratified definability structure (such as a jump hierarchy or the constructible universe) forms a strong obstruction to randomness.

The paper is organized as follows. In Section 2, we introduce effective randomness for arbitrary (continuous) probability measures. We also prove some fundamental facts on randomness we will use later. In particular, we will give various ways to obtain reals that are random for *some* continuous measure from standard Martin-Löf random reals (i.e. random with respect to Lebesgue measure). We also consider the definability strength of random reals. Section 3 features the proof that for any n, all but countably many reals are n-random with respect to some continuous measure (Theorem 1). Finally, Section 4 is devoted to the metamathematical analysis of Theorem 1. In particular, it contains a proof of Theorem 2.

We expect the reader to have basic knowledge in mathematical logic and computability theory. We tried to keep the paper as self-contained as possible, though some familiarity with forcing, the constructible universe, and the recursion theoretic hierarchies is certainly helpful. We assume no prior knowledge of fine structure theory and state or develop all the material we need.

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2. Randomness for Continuous Measures

In this section we introduce the basic notions of measure on the Cantor space 2^ω and define randomness for arbitrary probability measures. We then derive some facts about random reals needed later on in the text.

The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The topology generated by the *cylinder sets*

$$[\![\sigma]\!] = \{x : x\lceil_n = \sigma\},$$

where σ is a finite binary sequence, turns 2^ω into a compact Polish space. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use $\subseteq$ to denote the usual prefix partial ordering. This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in [\![\sigma]\!]$ if and only if $\sigma \subset x$. Finally, given $U \subseteq 2^{<\omega}$, we write $[\![U]\!]$ to denote the open set induced by U, i.e. $[\![U]\!] = \bigcup_{\sigma \in U} [\![\sigma]\!]$.

2.1. Turing functionals.

The notion of a Turing functional will be important in this paper, so we give a formal definition and explain in a precise way how functionals give rise to partial, continuous mappings from 2^ω to 2^ω.

A *Turing functional* Φ is a computably enumerable set of triples (m, k, σ) such that m is a natural number, k is either 0 or 1, and σ is a finite binary sequence. Further, for all m, for all k_1 and k_2, and for all compatible σ_1 and σ_2, if $(m, k_1, \sigma_1) \in \Phi$ and $(m, k_2, \sigma_2) \in \Phi$, then $k_1 = k_2$ and $\sigma_1 = \sigma_2$.

In the following, we will also assume that Turing functionals Φ are *use-monotone*, which means the following hold.

(1) For all (m_1, k_1, σ_1) and (m_2, k_2, σ_2) in Φ, if σ_1 is a proper initial segment of σ_2, then m_1 is less than m_2.
(2) For all m_1 and m_2, k_2 and σ_2, if $m_2 > m_1$ and $(m_2, k_2, \sigma_2) \in \Phi$, then there are k_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(m_1, k_1, \sigma_1) \in \Phi$.

We write $\Phi^\sigma(m) = k$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(m, k, \tau) \in \Phi$. In this case, we also write $\Phi^\sigma(m)\downarrow$, as opposed to $\Phi^\sigma(m)\uparrow$, indicating that for all k and all $\tau \subseteq \sigma$, $(m, k, \tau) \notin \Phi$.

If $X \in 2^\omega$, we write $\Phi^X(m) = k$ to indicate that there is an l such that $\Phi^{X\lceil_l}(m) = k$. This way, for given $X \in 2^\omega$, Φ^X defines a partial function from ω to $\{0, 1\}$ (identifying reals with sets of natural numbers). If this function is total, it defines a real Y, and in this case we write $\Phi(X) = Y$ and say that Y is Turing reducible to X via Φ, $Y \leq_{\mathrm{T}} X$.

By use-monotonicity, if $\Phi^\sigma(m)\downarrow$, then $\Phi^\sigma(n)\downarrow$ for all $n < m$. If we let $\overline{m}$ be maximal such that $\Phi^\sigma(\overline{m})\downarrow$, Φ^σ gives rise to a string τ of length $\overline{m} + 1$,

$$\tau = \Phi^\sigma(0) \ldots \Phi^\sigma(\overline{m}).$$

If $\Phi^\sigma(n)\uparrow$ for all n, we put $\tau = \varnothing$. On the other hand, if $\overline{m}$ does not exist, then Φ^σ gives rise to a real Y. We write $\Phi(\sigma) = \tau$ or $\Phi(\sigma) = Y$, respectively. This way a Turing functional induces a function from $2^{<\omega}$ to $2^{<\omega} \cup 2^\omega$ that is *monotone*, that is, $\sigma \subseteq \tau$ implies $\Phi(\sigma) \subseteq \Phi(\tau)$. Note that $\Phi(\sigma)$ is not necessarily a computable function, but we can effectively approximate it by prefixes. More precisely, there exists a computable mapping $(\sigma, s) \mapsto \Phi_s(\sigma)$ so that $\Phi_s(\sigma) \subseteq \Phi_{s+1}(\sigma)$, $\Phi_s(\sigma) \subseteq \Phi_s(\sigma^\frown i)$ $(i \in \{0, 1\})$, and $\lim_s \Phi_s(\sigma) = \Phi(\sigma)$.

If, for a real X, $\lim_n |\Phi(X\lceil_n)| = \infty$, then $\Phi(X) = Y$, where Y is the unique real that extends all $\Phi(X\lceil_n)$. In this way, Φ also induces a partial, continuous function from 2^ω to 2^ω. We will use the same symbol Φ for the Turing functional, the monotone function from $2^{<\omega}$ to $2^{<\omega}$, and the partial, continuous function from 2^ω to 2^ω. It will be clear from the context which Φ is meant. Φ is called *total* if $\Phi(X)$ is a real for all $X \in 2^\omega$. If Φ is total and $\Phi(X) = Y$, then Y is called *truth-table reducible* to X, $Y \leq_{\mathrm{tt}} X$.

Turing functionals can be relativized with respect to a parameter Z, by requiring that Φ is c.e. in Z. We call such functionals *Turing Z-functionals*. This way we can consider relativized Turing reductions. A real X is Turing reducible to a real Y relative to a real Z, written $X \leq_{\mathrm{T}(Z)} Y$, if there exists a Turing Z-functional Φ such that $\Phi(X) = Y$.

2.2. **Probability measures.** A *probability measure* on 2^ω is a *countably additive function* $\mu : \mathcal{F} \to [0, 1]$, where $\mathcal{F} \subseteq \mathcal{P}(2^\omega)$ is a σ-algebra and $\mu(2^\omega) = 1$. μ is called a *Borel probability measure* if $\mathcal{F}$ is the Borel σ-algebra over 2^ω. It is a basic result of measure theory that a probability measure is uniquely determined by the values it takes on an algebra $\mathcal{A} \subseteq \mathcal{F}$ that generates $\mathcal{F}$. It is not hard to see that the Borel sets are generated by the algebra of *clopen sets*, i.e. finite unions of basic open cylinders. Normalized, monotone, countably additive set functions on the algebra of clopen sets are induced by any function $\rho : 2^{<\omega} \to [0,1]$ satisfying

$$\rho(\varnothing) = 1 \qquad \text{and} \qquad \rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1) \tag{2.1}$$

for all finite sequences σ. Such a function is called an *additive premeasure*. Putting $\mu(\llbracket\sigma\rrbracket) = \rho(\sigma)$ yields an monotone, additive function on the clopen sets, which in turn uniquely extends to a Borel probability measure on 2^ω. In the following, we will deal exclusively with Borel probability measures, and hence we will identify such measures with the underlying function on cylinders satisfying (2.1). We sometimes write μE in place of $\mu(E)$ to avoid multiple parentheses and brackets. We will exclusively deal with Borel probability measures. In the following, we will simply write *measure* to denote a Borel probability measure on 2^ω.

The *Lebesgue measure* λ on 2^ω is obtained by distributing a unit mass uniformly along the paths of 2^ω, i.e. by setting $\lambda\llbracket\sigma\rrbracket = 2^{-|\sigma|}$. A *Dirac measure*, on the other hand, is defined by putting a unit mass on a single real, i.e. for $X \in 2^\omega$, let

$$\delta_X\llbracket\sigma\rrbracket = \begin{cases} 1 & \text{if } \sigma \subset X, \\ 0 & \text{otherwise.} \end{cases}$$

If, for a measure μ and $X \in 2^\omega$, $\mu(\{X\}) > 0$, then X is called an *atom* of μ. Obviously, X is an atom of δ_X. A measure that does not have any atoms is called *continuous*.

2.3. **Representation of measures and Martin-Löf randomness.** It was Martin-Löf's fundamental idea to define randomness by choosing a *countable family* of nullsets. For any non-trivial measure, the complement of the union of these sets will have positive measure, and any point in this set will be considered *random*.

Essentially, a Martin-Löf test is an effectively presented G_δ nullset, where the effectiveness is relative to underlying measure (and maybe some other parameter Z).

To incorporate measures into an effective test for randomness we represent them as reals. The following facts about the space of probability measures on 2^ω are well-known and can be found, for example, in [24] and [14].

Let $\mathcal{M}(2^\omega)$ be the set of all Borel probability measures on 2^ω, endowed with the weak-* topology. A compatible metric is given by

$$d_{\text{meas}}(\mu,\nu) = \sum_{n=1}^{\infty} 2^{-n} d_n(\mu,\nu),$$

where

$$d_n(\mu,\nu) = \frac{1}{2}\sum_{|\sigma|=n} |\mu[\![\sigma]\!] - \nu[\![\sigma]\!]|.$$

With this metric $\mathcal{M}(2^\omega)$ becomes a *compact Polish space*. A countable dense subset is given by the set of measures which assume positive, rational values on a finite number of rationals, i.e. $\mathcal{D}$ is the set of measures of the form

$$\nu_{\Delta,Q} = \sum_{\sigma\in\Delta} Q(\sigma)\delta_{\sigma^\frown 0^\omega},$$

where Δ is a finite set of finite strings (representing dyadic rational numbers) and $Q : \Delta \to [0,1]\cap\mathbb{Q}$ such that $\sum_{\sigma\in\Delta} Q(\sigma) = 1$.

Every compact Polish space is the continuous image of 2^ω, i.e. there exists a continuous surjection $\rho : 2^\omega \to \mathcal{M}(2^\omega)$. Due to the simple nature of the countable dense subset $\mathcal{D}$, ρ can be chosen effectively in the sense that for every $R \in 2^\omega$, $\rho^{-1}(\{\rho(R)\})$ is $\Pi^0_1(R)$. For details on the construction of ρ, see [4] ([38] has a similar development). If $\rho(R) = \mu$, then R is called a *representation* of μ. Essentially, the representations of a measure μ are encodings of Cauchy sequences of measures $\nu_{\Delta,Q}$ converging effectively to μ. If μ is given, R_μ will always denote a representation of μ.

Working with representations, we can apply computability theoretic notions to measures. We say a real X is *recursive in* μ if $X \leq_{\mathrm{T}} R_\mu$ for every representation R_μ of μ. On the other hand, we say a real computes a measure if its computes *some* representation of it. In particular, a measure is computable if it has a computable representation. For a measure being computable is equivalent to being able to compute its values on cylinders to any desired degree of precision.

Proposition 2.1. *A measure $\mu \in \mathcal{M}(2^\omega)$ is computable if and only if there exists a computable function $g_\mu : 2^{<\omega}\times\omega \to \mathbb{Q}$ such that for all $\sigma \in 2^{<\omega}$, $n \in \omega$,*

$$|g_\mu(\sigma,n) - \mu[\![\sigma]\!]| \leq 2^{-n}.$$

For a proof (in a more general setting), see [18]. Note, however, that a non-computable measure does not necessarily have a representation of least Turing degree [4].

A measure μ is *dyadic* if every measure of a cylinder is of the form $\mu[\![\sigma]\!] = m/2^n$ with m,n non-negative integers. For dyadic measures, it makes sense to speak of *exact computability*: A dyadic measure μ is *exactly computable* if the function $\sigma \mapsto \mu[\![\sigma]\!]$ is a computable mapping from $2^{<\omega}$ to

$\mathbb{Q}$. Note that for exactly computable measures, the relation $\mu[\![\sigma]\!] > \alpha$, α rational, is decidable, whereas in the general case for computable μ we only know it is Σ^0_1. Dyadic measures always have a representation of least Turing degree.

We can now give the definition of a general Martin-Löf test. The definition is a straightforward generalization of Martin-Löf n-tests and Martin-Löf n-randomness for Lebesgue measure (see, for instance, [34] or [8]).

Definition 2.2. Suppose μ is a probability measure on 2^ω, and R is a representation of μ. Suppose further that $Z \in 2^\omega$ and $n \geq 1$.

(1) An (R, Z, n)-*test* is a set $W \subseteq \omega \times 2^{<\omega}$ which is recursively enumerable in $(R \oplus Z)^{(n-1)}$, the $(n-1)$st Turing jump of $R \oplus Z$ such that

$$\sum_{\sigma \in W_n} \mu[\![\sigma]\!] \leq 2^{-n},$$

where $W_n = \{\sigma :\ (n, \sigma) \in W\}$

(2) A real X *passes* a test W if $X \notin \bigcap_n [\![W_n]\!]$.

(3) A real X is (R, Z, n)-*random* if it passes all (R, Z, n)-tests.

(4) A real X is *Martin-Löf n-random for μ relative to Z*, or simply (μ, Z, n)-*random* if there exists a representation R_μ such that X is (R_μ, Z, n)-random. In this case we say R_μ *witnesses* the μ-randomness of X.

If the underlying measure is Lebesgue measure λ, we often drop reference to the measure and simply say X is (Z, n)-*random.* We also drop the index 1 in case of $(\mu, Z, 1)$-randomness and simply speak of *μ-randomness relative to Z* or *μ-Z-randomness.* If $Z = \emptyset$, on the other hand, we speak of (μ, n)- or *μ-n-randomness.*

Levin [28] introduced the alternative concept of a *uniform test* for randomness, which is representation-independent (see also [12]). Day and Miller [4] have recently shown that randomness with respect to uniform tests, and randomness in the sense of Definition 2.2 (4) coincide.

Since there are only countably many Martin-Löf n-tests, it follows from countable additivity that the set of Martin-Löf (μ, Z, n)-random reals for any μ and any Z has μ-measure 1. Hence there always exist (μ, Z, n)-random reals for any measure μ, any real Z, and any $n \geq 1$.

However, μ, Z, and n put some immediate restrictions on the relative definability of any (μ, Z, n)-random real.

Proposition 2.3. *If X is (μ, Z, n)-random via representation R_μ, then X cannot be $\Delta^0_n(R_\mu \oplus Z)$.*

Proof. If X is $\Delta^0_n(R_\mu \oplus Z)$, then $X \leq_{\mathrm{T}} (R_\mu \oplus Z)^{(n-1)}$, and we can easily build a (μ, Z, n)-test covering X. □

It is also immediate from the definition of randomness that any atom of a measure is random with respect to it. This is a trivial way for a real to be random. The proposition below (a straightforward relativization of a result by Levin [45]) shows that atoms of a measure are also computationally trivial (relative to the measure).

Proposition 2.4. *If for a measure μ and a real X, $\mu\{X\} > 0$, then $X \leq_{\mathrm{T}} R_\mu$ for any representation R_μ of μ.*

Since we are interested in randomness for continuous measures, the case of atomic randomness is excluded a priori.

2.4. **Image measures and transformation of randomness.** One can obtain new measures from given measures by transforming them with respect to a sufficiently regular function. Let $f : 2^\omega \to 2^\omega$ be a Borel (measurable) function, i.e. for every Borel set A, $f^{-1}(A)$ is Borel, too. If μ is a measure on 2^ω and f is Borel, then the *image measure* μ_f is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$

It can be shown that every probability measure can be obtained from Lebesgue measure λ by means of a measurable transformation. For continuous measures, Oxtoby [35] proved that any continuous, positive measure on 2^ω can be transformed into Lebesgue measure on the set of irrationals in $[0, 1]$ via a homeomorphism. Here, a measure is *positive* if $\mu[\![\sigma]\!] > 0$ for $\sigma \in 2^{<\omega}$.

Levin [45], and independently Kautz [23] (see also [2]) proved an effective version of these results. If μ is computable, it can be obtained from Lebesgue measure via a Turing functional Φ that is defined on almost every real. If μ is, moreover, continuous and positive, then Φ has an inverse that transforms μ into λ.

A consequence of the Levin-Kautz Theorem is that every non-recursive real that is random with respect to a computable probability measure is Turing equivalent to a λ-random real. We will show now that for continuous measures, this can be strengthened to *truth-table equivalence*.

Proposition 2.5. *Let X be a real. For any $Z \in 2^\omega$ and any $n \geq 1$, the following are equivalent.*

(i) X is (μ, Z, n)-random for a continuous measure μ recursive in Z.
(ii) X is (ν, Z, n)-random for a continuous, positive, dyadic measure ν exactly computable in Z.
(iii) There exists a functional Φ recursive in Z so that Φ is an order-preserving homeomorphism of 2^ω, and $\Phi(X)$ is (λ, Z, n)-random.
(iv) X is truth-table equivalent relative to Z to a (λ, Z, n)-random real.

Here, the order on 2^ω is given by

$$X < Y \quad :\Leftrightarrow \quad X(N) < Y(N) \text{ where } N = \min\{n \colon X(n) \neq Y(n)\}.$$

Proof. We give a proof for $Z = \emptyset$ and $n = 1$. It is routine to check that the proof relativizes and generalizes to higher levels of randomness.

(i) $\Rightarrow$ (ii): Let X be μ-random, where μ is a continuous, computable measure. We construct a continuous, positive, dyadic, and exactly computable measure ν such that X is random with respect to ν, too. The construction is similar to Schnorr's rationalization of martingales [41]. We define ν by recursion on the full binary tree $2^{<\omega}$. Initially, let $\nu^*[\![\varnothing]\!] = 2$. Given $\nu^*[\![\sigma]\!]$, choose $\nu^*[\![\sigma^\frown 0]\!]$ and $\nu^*[\![\sigma^\frown 1]\!]$ as dyadic rationals such that

$$\mu[\![\sigma^\frown i]\!] < \nu^*[\![\sigma^\frown i]\!] \leq \mu[\![\sigma^\frown i]\!] + 2^{-(|\sigma|+1)}, \qquad i \in \{0, 1\},$$

and

$$\nu^*[\![\sigma^\frown 0]\!] + \nu^*[\![\sigma^\frown 1]\!] = \nu^*[\![\sigma]\!].$$

This is possible since the dyadic rationals are dense in $\mathbb{R}$ and can be done effectively in σ by Proposition 2.1. Finally, normalize by setting $\nu = \nu^*/2$. Then ν is an exactly computable measure. It is clear from the construction that for all σ, $\mu[\![\sigma]\!] < 2\nu[\![\sigma]\!]$. In particular, ν is positive. Finally, if (V_n) is a test for ν, by letting $W_n = V_{n+1}$ we obtain a μ-test that covers every real covered by (V_n). Hence if X is μ-random, then X is also ν-random.

(ii) $\Rightarrow$ (iii): Suppose ν is an exactly computable, continuous, positive, dyadic measure. Since ν is continuous and 2^ω is compact, for every m there exists l_m such that whenever $|\sigma| \geq l_m$, then $\nu[\![\sigma]\!] \leq 2^{-m}$. Without loss of generality, we can assume that $l_m < l_{m+1}$. It is easy to see that such l_m can be obtained recursively and uniformly in any representation of ν (see e.g. [39]).

We define inductively a mapping $\varphi : 2^{<\omega} \to 2^{<\omega}$ that will induce the desired homeomorphism. Let $\varphi(\varnothing) = \varnothing$. We will also define, for every $\tau \in 2^{<\omega}$, an auxiliary finite set $E_\tau \subseteq 2^{<\omega}$. It will hold that

(a) all strings in E_τ are of the same length, and this length depends only on the length of τ;
(b) $\tau_0 \subseteq \tau_1$ implies $E_{\tau_0} \supseteq E_{\tau_1}$;
(c) for all n, $\bigcup_{|\tau|=n} [\![E_\tau]\!] = 2^\omega$;
(d) for all τ, $0 < \nu[\![E_\tau]\!] \leq 2^{-|\tau|+2}$;

Put $E_\varnothing = \{\varnothing\}$. Suppose that for given τ, φ is defined on all strings in E_τ and prefixes thereof. Let

$$F_\tau = \{\sigma : |\sigma| = l_{2(|\tau|+1)} \ \&\ \sigma \text{ extends some string in } E_\tau\}.$$

Find the least (with respect to the usual lexicographic ordering) $\sigma \in F_\tau$ such that

$$\sum_{\substack{\eta \leq \sigma \\ \eta \in F_\tau}} \nu[\![\eta]\!] > \nu[\![E_\tau]\!]/2.$$

Let $E_{\tau^\frown 0} = \{\eta \in F_\tau : \eta < \sigma\}$ and put the remaining strings of F_τ into $E_{\tau^\frown 1}$. By choice of the length of strings in F_σ, both $E_{\tau^\frown 0}$ and $E_{\tau^\frown 1}$ are non-empty, and by construction (a), (b), and (c) are met. Moreover,

$$\nu[\![E_{\tau^\frown i}]\!] \leq \nu[\![E_\tau]\!]/2 + 2^{-2|\tau|} \qquad (i \in \{0,1\}),$$

which inductively yields the bound in (d).

Map all strings in $E_{\tau^\frown 0}$ to $\tau^\frown 0$, and all strings in $E_{\tau^\frown 1}$ to $\tau^\frown 1$. To make φ defined on all strings, map any string that extends some string in E_τ but is a true prefix of some string in F_τ to τ.

It is clear from the construction that φ induces a total, order preserving mapping $\Phi : 2^\omega \to 2^\omega$ by letting

$$\Phi(X) = \lim_n \varphi(X\lceil_n).$$

Φ is onto since for every σ, E_σ is not empty. We claim that Φ is also one-one. Suppose $\Phi(X) = \Phi(Y)$. This implies that for all n, $\varphi(X\lceil_n) = \varphi(Y\lceil_n)$, that is, for all n, $X\lceil_n$ and $Y\lceil_n$ belong to the same E_σ. Since ν is positive, the diameter of the E_σ goes to 0 along any path. Hence $X = Y$.

It remains to show that $\Phi(X)$ is Martin-Löf random. Suppose not, then there exists an λ-test (W_n) that covers $\Phi(X)$. Let

$$V_n = \bigcup_{\sigma \in W_{n+2}} E_\sigma.$$

Then (V_n) covers X. Furthermore, the (V_n) are uniformly enumerable since the mapping $\sigma \mapsto E_\sigma$ is computable by the construction of the E_σ. Finally,

$$\sum_{\tau \in V_n} \nu[\![\tau]\!] = \sum_{\sigma \in W_{n+2}} \nu[\![E_\sigma]\!] \leq \sum_{\sigma \in W_{n+2}} 2^{-|\sigma|+2} \leq 2^{-n}.$$

thus X is not ν-random, contradiction.

(iii) $\Rightarrow$ (iv): This is immediate.

(iv) $\Rightarrow$ (i): This follows from Theorem 5.7 in [39]

□

The result also suggests that if we are only interested in whether a real is random with respect to a continuous measure, representational issues do not really arise. We can restrict ourselves to dyadic measures, which have a minimal representation.

Remark 2.6. We will henceforth, unless explicitly noted, assume that *any measure is a dyadic measure*. We drop reference to the representation and write μ instead of R_μ.

2.5. **Making reals random.** While Proposition 2.5 gives a necessary and sufficient criterion, we need further techniques to show that a given real is random with respect to a continuous measure. In particular, we need to develop a method to "transfer" randomness via Turing reductions.

Demuth [5] showed that every non-recursive real truth-table below a Martin-Löf random real measure is Turing equivalent to a Martin-Löf-random real. The proof works by taking the image measure of λ given by the truth-table functional. This yields a computable measure μ. Since the functional would transform any μ-test covering Y into a λ-test covering X, it follows that Y is μ-random. Since Y is not recursive, it will not be an atom of the image measure, and hence Y is non-trivially random. Now apply the Levin-Kautz Theorem to μ and Y.

Demuth's result relativizes. First of all, we can replace Lebesgue measure λ by another measure μ. In case μ is not computable, however, we have to give information about the measure to the reduction, i.e. have Y tt-reducible to X relative to (a representation of) μ, say via a total functional Φ. This way we ensure that Y is μ_Φ-random relative to μ, where μ is a parameter of randomness, not a measure. To avoid Y being an atom of Φ_μ, we have to require that Y is not recursive in μ. Moreover, instead of looking at unrelativized 1-randomness with respect to some μ, we can use the same argument for n-randomness relative to a parameter Z, as long as we pass on the accordant information to the reduction and impose sufficient non-computability restrictions on Y. A relativized version of the Levin-Kautz Theorem then ensures that Y is (relatively) Turing equivalent to a relatively λ-random real.

The fully relativized result then reads as follows. Recall that we only consider dyadic measures and hence drop reference to a representation. Nevertheless, the results in this section are not dependent on the existence of a minimal representation and can be reformulated accordingly.

Proposition 2.7. *Suppose Y is (μ, Z, n)-random ($n \geq 1$) and X is truth-table reducible to Y relative to $(\mu \oplus Z)^{(k)}$ for some $k \leq n-1$ (i.e. $X \leq_{\mathrm{tt}((\mu\oplus Z)^{(k)})} Y$). Further suppose X is not recursive in $(\mu \oplus Z)^{(k)}$. Then X is Turing equivalent relative to $(\mu \oplus Z)^{(k)}$ to a $(\lambda, \mu \oplus Z, n)$-random real.*

We can combine this result with an observation on domination properties of functions computed by random reals of higher order. This allows for replacing, by the expense of two jumps, truth-table reducibility by Turing reducibility, which not only is often easier to handle, but also will later enable us to use the powerful tool of *Borel-Turing Determinacy*.

Kurtz [27] first observed that 2-random reals cannot compute fast growing functions.

Theorem 2.8 ([27])**.** *There exists a function $f \leq_{\mathrm{T}} \emptyset'$ such that for every 2-random X, if $g \leq_{\mathrm{T}} X$, then g is dominated by f, i.e. for all but finitely many $n \in \omega$, $f(n) \geq g(n)$.*

The proof is based on the following idea: Given a Turing functional Φ, $\emptyset'$ can decide, given rational q and $n \in \omega$, whether

$$\lambda \{Y \colon \Phi(Y) \text{ is defined for all } k \leq n\} > q.$$

For each n, let q_n be maximal of the form $i \cdot 2^{-n}$ so that the above holds, and let t_n be the use by which at least q_n-measure many computations have converged. Construct a function $f \leq_{\mathrm{T}} \emptyset'$ such that $f(n)$ dominates all function values $\Phi(X)$ computed with use t_n and within t_n steps. Then f dominates all functions computed by Φ with the exception of a measure zero set, which is captured by a $\emptyset'$-Martin-Löf-test, and hence cannot contain any 2-random real.

As above, this relativizes to other measures and parameters.

Proposition 2.9. *Given a measure μ and a real Z, there exists a function $f \leq_{\mathrm{T}} (\mu \oplus Z)'$ such that for every $(\mu, Z, 2)$-random X, if $g \leq_{\mathrm{T}(Z \oplus \mu)} X$, then g is dominated by f.*

Together with Proposition 2.7 this yields a sufficient criterion for continuous randomness.

Proposition 2.10. *Suppose $n \geq 3$ and Y is (μ, Z, n)-random. If $X \leq_{\mathrm{T}(\mu \oplus Z)} Y$ and X is not recursive in $(\mu \oplus Z)'$, then X is $(\nu, (\mu \oplus Z)'', n-2)$-random for some continuous measure $\nu \leq_{\mathrm{T}} (\mu \oplus Z)''$.*[1]

Proof. Suppose $X \leq_{\mathrm{T}(R_\mu \oplus Z)} Y$ via a Turing reduction Φ. By Proposition 2.9, the use of Φ on X is dominated by some function recursive in $(R_\mu \oplus Z)'$. We can modify Φ to $\widetilde{\Phi}$ such that $\widetilde{\Phi}$ is a truth-table reduction relative to $(R_\mu \oplus Z)'$ and $\widetilde{\Phi}(Y) = X$.

By Proposition 2.7, X is Turing equivalent relative to $(R_\mu \oplus Z)'$ to a $(\lambda, R_\mu \oplus Z, n)$-random real L. Any $(\lambda, R_\mu \oplus Z, n)$-random real is also $(\lambda, (R_\mu \oplus Z)', n-1)$-random, and so we can apply Proposition 2.9 to X and L to conclude that they are truth-table equivalent relative to $(R_\mu \oplus Z)''$. This in turn means that X is truth-table equivalent relative to $(R_\mu \oplus Z)''$ to a $(\lambda, (R_\mu \oplus Z)'', n-2)$-random real, which by Proposition 2.5 implies that X is $(\nu, (R_\mu \oplus Z)'', n-2)$-random for a continuous measure recursive in $(R_\mu \oplus Z)''$. ☐

We can save one jump if X is actually Turing *equivalent* to a λ-n-random real, since in that case it follows directly from 2.9 that X is in the same truth-table degree as a λ-$(n-1)$-random real (relative to a jump), and we can apply Proposition 2.5 directly.

Proposition 2.11. *Suppose $n \geq 2$ and Y is (μ, Z, n)-random for some measure μ. If $X \equiv_{\mathrm{T}(\mu \oplus Z)} Y$ and X is not recursive in $(\mu \oplus Z)'$, then X is $(\nu, (\mu \oplus Z)', n-1)$-random for some continuous measure $\nu \leq_{\mathrm{T}} (\mu \oplus Z)'$.*

2.6. **The definability strength of higher randomness.** By a theorem of Kučera [25] and Gács [11], every real is Turing-reducible to a 1-random one. This implies that 1-random reals can have arbitrary high definability power: every real is Δ^0_1 in some 1-random real. However, this situation is somewhat atypical. For $n > 1$, randomness behaves rather orthogonal to reals with high definability power, such as Turing jumps.

[1] meaning that the minimal representation of the dyadic continuous measure ν is computable from $(\mu \oplus Z)''$, see remarks at the end of Section 2.3

This manifests itself in the following two results. The first one is straightforward relativization of a result due to Kautz [23] (see also [39], where the basic computational properties of a measure relative to a representation are established).

Proposition 2.12. *Let μ be a continuous measure, and suppose X is μ-$(n+1)$-random, where $n \geq 1$. Then*

$$(X \oplus \mu)^{(n)} \equiv_{\mathrm{T}} X \oplus \mu^{(n)}$$

The other result generalizes a result of Downey, Nies, Weber, and Yu [7], who show that every weakly 2-random real forms a minimal pair with $0'$. It is of central importance to Section 4.

Lemma 2.13 (The "Stair Trainer Lemma")**.** *Suppose μ is a continuous measure and Z is μ-n-random, $n \geq 2$. If $Y \leq_{\mathrm{T}} \mu^{(n-1)}$ and $Y \leq_{\mathrm{T}} Z \oplus \mu$, then $Y \leq_{\mathrm{T}} \mu$.*

Proof. Suppose $Y \leq_{\mathrm{T}} Z \oplus \mu$ via a Turing functional Φ and $Y \leq_{\mathrm{T}} \mu^{(n-1)}$. Then Y is Δ^0_2 relative to $\mu^{(n-2)}$. Let $Y(n,s)$ be a $\mu^{(n-2)}$-recursive approximation of Y, i.e. $\lim_s Y(n,s) = Y(n)$. Given $i, s \in \omega$, put

$$U_{i,s} = \{X : \exists t > s\ \Phi_t(X \oplus \mu)(i) = Y(i,t)\},$$

$U_{i,s}$ is $\Sigma^0_1(\mu^{(n-2)})$ uniformly in i, s and hence $P = \bigcap_{i,s} U_{i,s}$ is $\Pi^0_2(\mu^{(n-2)})$. Note that P is the upper cone of Y under Φ. P cannot have μ-measure 0: If it had then by the monotone convergence theorem for measures, for the sequence of open sets (V_k) given by $V_k = \bigcup_{\langle i,s\rangle \leq k} U_{i,s}$, we have $\mu V_k \searrow 0$. Since each V_k is $\Sigma^0_1(\mu^{(n-2)})$, $\mu^{(n-1)}$ can decide whether $\mu V_k \leq 2^{-l}$ for given l. Hence, we could convert (V_k) into a (μ, n)-test. Since $\bigcap_k V_k = P$ and P contains Z, this would contradict the fact that Z is μ-n-random.

Hence pick r rational such that $\mu P > r > 0$, where r is rational. Define a tree T by letting

$$\sigma \in T \ :\Leftrightarrow\ \mu\{\tau : \Phi(\tau \oplus \mu) \supseteq \sigma\} > r,$$

and closing under initial segments. T is r.e. in μ and Y is an infinite path through T.

Since μ is a probability measure, T contains at most $\lceil 1/r \rceil$ strings at any level. Choose $\sigma = Y\lceil_n$ such that no $\tau \supseteq \sigma$ incompatible with Y is in T. To compute $Y\lceil_m$ from μ, it suffices to enumerate T above σ till a long enough extension shows up. $\square$

We will later need the following relativization of the previous lemma. The proof is similar.

Lemma 2.14. *Suppose μ is a continuous measure and Z is μ-$(k+n)$-random, $k \geq 0, n \geq 2$. If $Y \leq_{\mathrm{T}} \mu^{(k+n-1)}$ and $Y \leq_{\mathrm{T}} Z \oplus \mu^{(k)}$, then $Y \leq_{\mathrm{T}} \mu^{(k)}$.*

One interpretation of Lemma 2.13 is that random reals cannot "accelerate" definability. If a real is properly Σ^0_n (relative to a measure μ), then it cannot be Σ^0_k relative to $\mu \oplus X$ where $k < n$ and X is μ-$(n+1)$-random.

An application of the preceding result yields that Turing jumps cannot be n-random for a continuous measure, where $n \geq 2$. This fact will be important in Section 4.

Proposition 2.15 (The "Stair Trainer Technique")**.** *If $n \geq 2$, then for all $k \geq 0$, $\emptyset^{(k)}$ is not n-random with respect to a continuous measure.*

Proof. The case $k = 0$ is clear, so assume $k > 0$.

Suppose $\emptyset^{(k)}$ is μ-n-random for some μ. Then $\emptyset' \leq_{\mathrm{T}} \emptyset^{(k)}$ and also $\emptyset' \leq_{\mathrm{T}} \mu' \leq_{\mathrm{T}} \mu^{(n-1)}$. It follows from Lemma 2.13 that $\emptyset'$ is recursive in μ. Applying the same argument inductively to $\emptyset^{(i)}$, $i \leq k$, yields $\emptyset^{(i)} \leq_{\mathrm{T}} \mu$, in particular $\emptyset^{(k)} \leq_{\mathrm{T}} \mu$, which is impossible if $\emptyset^{(k)}$ is μ-n-random. □

The non-randomness property of the jumps extends to infinite jumps, too.

Proposition 2.16. *For $n \geq 3$, $\emptyset^{(\omega)}$ is not n-random with respect to a continuous measure.*

Proof. Assume for a contradiction that $0^{(\omega)}$ is μ-n-random for $n \geq 3$ and continuous μ. By the inductive argument of the previous proof, $0^{(k)} \leq_{\mathrm{T}} \mu$ for all $k \in \omega$. By a result of Enderton and Putnam [9], if X is a $\leq_{\mathrm{T}}$-upper bound for $\{0^{(k)} \colon k \in \omega\}$, then $0^{(\omega)} \leq_{\mathrm{T}} X''$. Therefore, $0^{(\omega)} \leq_{\mathrm{T}} \mu''$, but since $n \geq 3$ and $0^{(\omega)}$ is μ-n-random, this is impossible. □

These two results are prototypical for what will follow later. We will build on them to construct long sequences of reals that are not random with respect to any continuous measure.

3. The Countability Theorem

In this section we will prove Theorem 1, which we restate here for convenience.

Theorem 1. *Let $n \in \omega$. Then the set*

$$\mathrm{NCR}_n = \{X \in 2^\omega : X \textit{ is not } n\textit{-random for any continuous measure}\}$$

is countable.

The idea of the proof is as follows. In [39], we used the Kučera-Gács result [25, 11] that every real above $\emptyset'$ is in the same Turing degree as a random real. We relativized this result using the Posner-Robinson Theorem [37] to obtain that every non-recursive real is, relative to some real G, above G'.

We will follow a similar strategy here: First, we show that the set of continuously random reals contains an upper cone in the Turing degrees. The argument uses Martin's result [29] that every Turing invariant Borel set contains an upper cone with respect to Turing reducibility. The constructive nature of Martin's proofs yields that the base of the cone (a winning strategy in a Borel game) is contained in a level L_{β_n} of the constructible hierarchy that is countable. Finally, we generalize the Posner-Robinson Theorem to cones with base in L_{β_n}.

3.1. **The constructible hierarchy.** To make this paper more accessible for readers who are not experts in set theory, and to provide set-theorists with a quick overview of what part of set theory will be invoked in this section, we include a few basic facts about the constructible hierarchy. These will be supplemented over the course of the paper. We refer to the monographs by Devlin [6], Jech [19], and Moschovakis [33] for in-depth accounts.

3.1.1. *The definition of L.* A set X is *first-order definable* in a set Y (from parameters) if there exists a first-order formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language of set theory (i.e. only using the binary relation symbol $\in$) such that for some $a_1, \ldots, a_n \in Y$,

$$X = \{y \in Y : (Y, \in) \models \varphi[y, a_1, \ldots, a_n]\}.$$

Here $(Y, \in)$ stands for the interpretation of Y as a structure of the language of set theory, i.e. Y is a set and $\in$ is interpreted as a binary relation over Y.

The constructible universe is built as a cumulative hierarchy of sets along the ordinals. In each successor step, instead of adding all subsets of the current set, only the *definable* ones are added. Formally, L is defined as follows. Given a set Y, let

$$\mathcal{P}_{\mathrm{DEF}}(Y) = \{X \subseteq Y : X \text{ is first order definable in } Y\ \},$$

where the underlying language is the language of set theory. Now put

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathcal{P}_{\mathrm{DEF}}(L_\alpha) \\ L_\beta &= \bigcup_{\alpha<\beta} L_\alpha \quad (\beta \text{ limit ordinal}) \end{aligned}$$

Finally, let

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha.$$

3.1.2. *Basic properties of L.* It is not hard to see that for $\alpha < \beta$, $L_\alpha \in L_\beta$. Furthermore, each L_α is a transitive set, and L is a transitive class. L is an *inner model* of ZF, that is, L is transitive, contains all ordinals, and satisfies the axioms of ZF. In fact, for every ordinal α, $\alpha \subseteq L_\alpha$ and $L_\alpha \cap \mathrm{Ord} = \alpha$.

The proof of the latter fact and the verification that L satisfies all the axioms in ZF relies on the *absoluteness* of Δ_0 formulas. Since we will make repeated use of absoluteness, we will briefly describe this concept.

Given a formula φ in the language of set theory and some class M, we can *relativize* φ to φ^M essentially by restricting all quantifiers occurring in φ to range over M, i.e. $\exists x\, \psi$ becomes $(\exists x \in M)\, \psi^M$, for example. A formula is Δ_0 if it contains no or only *bounded quantifiers* of the form $\forall x \in v$ or $\exists x \in v$, where x, v are set variables. Δ_0 formulas are *absolute* for all transitive classes M, i.e. if M is a transitive class, then for all $x_1, \dots, x_n \in M$.

$$\phi^M(x_1, \dots, x_n) \quad \text{if and only if} \quad \phi(x_1, \dots, x_n)$$

We will be concerned with reals and sets of reals and effective aspects of their definability. Therefore, we will also need absoluteness results for sets definable in *second order arithmetic*. Recall that the arithmetical sets correspond to the sets definable in second order arithmetic via a formula using only (natural) number quantifiers (Σ^0_n or Π^0_n), while the general definable sets (the analytical sets in the sense of Kleene) give effective counterparts of the projective hierarchy (Σ^1_n or Π^1_n).

The important *Shoenfield Absoluteness Theorem* [42] says that all Σ^1_2 relations are absolute for L (or, more generally, for all inner models M of $\mathsf{ZF}+\mathsf{DC}$). As a consequence, one obtains that all Σ^1_2 reals (i.e. sets of integers definable by means of a Σ^1_2 predicate) are in L. A relativized version of this statement holds as well: Every $\Sigma^1_2(X)$ relation is absolute for all inner models M of $\mathsf{ZF} + \mathsf{DC}$ such that $X \in M$.

We will often work not with full models of ZFC, but finite fragments thereof. Moreover, we will replace the power set axiom by sentences asserting that a certain number of iterates of the power set of ω exist. The Shoenfield Absoluteness Theorem actually holds between transitive models of some finite $T^* \subseteq \mathsf{ZF}$, as long as $\omega_1 \subseteq M$. The latter condition arises from a tree representation of Σ^1_2 sets, where the trees are defined over $\omega \times \omega_1$.

If ω_1 is not contained in M, absoluteness still holds for Π^1_1 relations, since for Π^1_1 sets tree representations (on which the Shoenfield Absoluteness Theorem is based) work over ω. One has to require though that the model M is well-founded, transitive and satisfies enough axioms of ZF that well-founded trees have a rank function. (This is due to Mostowski.)

Therefore, we have absoluteness also for certain countable levels L_β of the constructible hierarchy. These levels will satisfy a sufficiently large fragment of ZF, and this fragment will include the theory T^* needed to establish absoluteness.

We will need absoluteness to ensure that winning strategies of certain Borel games in a forcing extension $L_\beta[G]$ persist as winning strategies "in the real world".

3.2. **A cone of continuously random reals.** In this section we will prove the existence of an upper cone of random reals. We will use Martin's result on Borel-Turing determinacy [29].

Definition. A set $A \subseteq 2^\omega$ is *Turing invariant* if $X \in A$ and $Y \equiv_{\mathrm{T}} X$ implies that $Y \in A$.

In the following we denote the *upper Turing cone* of a real X by $\mathrm{cone}_{\mathrm{T}}(X)$, i.e.

$$\mathrm{cone}_{\mathrm{T}}(X) = \{Y \in 2^\omega : X \leq_{\mathrm{T}} Y\}.$$

Borel-Turing Determinacy ([29, 30])**.** *If $A \subseteq 2^\omega$ is a Turing invariant Borel set, then there exists a real Y such that either $\mathrm{cone}_{\mathrm{T}}(Y) \subseteq A$ or $\mathrm{cone}_{\mathrm{T}}(Y) \subseteq 2^\omega \setminus A$.*

Proposition 3.1. *For every $n \geq 1$, there exists a real X such that for all $Y \geq_{\mathrm{T}} X$, Y is n-random with respect to some continuous measure. The base of the cone is given by the encoding of a winning strategy in a Borel game.*

Proof. Suppose $X \equiv_{\mathrm{T}(Z)} R$ where R is $(n+1)$-random relative to Z. By Proposition 2.11 X is n-random with respect to some continuous measure.

Let $B \subseteq 2^\omega$ be the set

$$\{Y \in 2^\omega : \exists Z \, \exists R \, (Y \equiv_{\mathrm{T}} Z \oplus R \ \ \& \ \ R \text{ is } (Z, n+1)\text{-random})\}$$

Clearly, B is Turing invariant. To see that B is Borel, note that the set can be defined in the form

$$\exists e, d \ (e, d \text{ are indices of Turing functionals such that } \Phi_d(\Phi_e(Y)) = Y$$
$$\text{and one half of } \Phi_e(Y) \text{ is } (n+1)\text{-random relative to the other half}).$$

By Proposition 2.11, every real in B is n-random for a continuous measure (note that $Y \in B$ cannot be recursive in Z' since R is $(Z, n+1)$-random).

Furthermore, B is *cofinal* in the Turing degrees, i.e. if S is any real, then there exists an $Y \geq_{\mathrm{T}} S$ such that $Y \in B$, since we can always find a real R that is $(n+1)$-random relative to S and put $Y \equiv_{\mathrm{T}} S \oplus R$.

Now apply Borel-Turing Determinacy.

□

3.3. **An aside on effective descriptive set theory.** The exact complexity of a winning strategy in a Borel game is difficult to determine. Recently, Hachtman [15] was able to give a precise bound for Σ^0_4-games. In this subsection, we give a brief analysis of an even simpler case which has implications for randomness for continuous measures. Suppose $A \subseteq \omega^\omega$ is closed, hence it is represented by some tree T. The determinacy of closed games was proved by Gale and Stewart [13]. The idea is as follows. Suppose Player II does not have a winning strategy. Call a node σ in T *non-losing* for Player I if Player II does not have a winning strategy above σ. Consequently, $\varnothing$ is a non-losing node for I. But then there must also exist at least one non-losing node above $\varnothing$. Inductively, one can construct an infinite tree T^I of non-losing nodes. This tree has infinite paths, and the closedness of A implies that each such infinite path is in A. Hence the tree T^I describes a winning strategy for I.

In determining the winning strategy, one has to check whether a node is non-losing. This involves checking whether certain trees above some node have infinite paths, or equivalently, whether certain trees are well-founded under the inverse prefix lexicographical ordering.

If the set A is effectively closed, the trees to be checked are uniformly recursive. Checking a uniformly recursive sequence of computable trees for well-foundedness can be done recursively in Kleene's $\mathcal{O}$. Hence a tree of non-losing nodes, and thus a winning strategy for I, can be obtained recursively in $\mathcal{O}$.

To formalize this argument and generalize it to higher complexities, the theory of *inductive definability* provides a suitable framework. We refer to the monographs by Moschovakis [33] and Sacks [40] for details.

An analysis of the cofinal Borel set $B \subseteq 2^\omega$ defined in the proof of Proposition 3.1 reveals that for $n = 1$, its descriptive complexity is Σ^0_3. The condition $\Phi_d(\Phi_e(X)) = X$ is Π^0_2, while (relative) 2-randomness is Σ^0_3.

Using the fact that Π^0_1 games are determined by winning strategies recursive in Kleene's $\mathcal{O}$, Harrington and Kechris [16] showed that every Σ^0_3 set of reals cofinal in the Turing degrees contains reals of every Turing degree $\geq \mathcal{O}$. To pass from Σ^0_3 to Π^0_1, they assigned every Σ^0_3 set of reals B a Π^0_1 subset A of Baire Space that intersects the same Turing degrees as the Σ^0_3 set. More precisely, if B is of the form $\{X\colon \exists k\, \forall l\, \exists m\, R(k, l, X\lceil_m)\}$, where R is a recursive relation, one can put

$$A = \{\langle k, X, Y\rangle \colon \forall l\, (Y(l) = \min\{m\colon R(k, l, X\lceil_m)\})\}.$$

Since our set B is not only cofinal in the Turing degrees but also Turing invariant, we obtain that every real $\geq_{\mathrm{T}} \mathcal{O}$ is 1-random with respect to a continuous measure.

In [39, Theorem 5.9], we showed that this is indeed true of every non-hyperarithmetic real. The result relied on a theorem by Woodin [44], which establishes that outside Δ^1_1, the Posner-Robinson Theorem holds with truth-table reducibility, and we can thus apply Proposition 2.5.

In the next subsection, we will derive the continuous randomness of non-hyperarithmetic reals in a different way. Namely, based on the above observation it suffices now to show that if X is not hyperarithmetic, then for some G, $X \oplus G \geq_{\mathrm{T}} \mathcal{O}^G$, where $\mathcal{O}^G$ is the *hyperjump* of G, i.e. the complete Π^1_1 set relative to G.

This will be part of a more general forcing construction, which provides the second part of the proof of Theorem 1. The general idea, however, is the same as the one described for 1-randomness in the previous paragraph.

3.4. **From upper cone to co-countably many.** The determinacy argument of the previous subsection yields the existence of an upper cone of n-random reals. The base of the cone is given by a winning strategy for a certain Borel game. In the case $n = 1$, we saw that the complexity of the winning strategy is at most $\mathcal{O}$. However, for $n > 1$, the complexity of such a winning strategy is increasingly hard to determine.

Although Martin's proof of Borel Determinacy [30] is of a constructive nature, its set theoretic complexity is high, in the sense that it makes inductive use of the power set operation: The higher the level of a Borel set, the more iterates of the power set of ω one needs to construct a winning strategy, in form of trees whose nodes are trees whose nodes are trees etc.

Friedman [10] showed that this is in fact an intrinsic feature of Borel determinacy. As we will see in the next section, this supplements Theorem 1 with an interesting metamathematical twist.

Nevertheless, Martin's proof of Borel determinacy is constructive. Therefore, it is not hard to locate a winning strategy within the constructible hierarchy.

Definition 3.2. Given $n \in \omega$, ZFC_n^- denotes the axiom of ZFC, where the power set axiom is replaced by the sentence

"There exist n-many iterates of the power set of ω".

Hence, in ZFC_0^-, for instance, we have the existence of the set of all natural numbers (since the Axiom of Infinity holds and ω is absolute), and various other subsets of ω as given by applications of separation or replacement, but we lack the guaranteed existence of the set of all such subsets.

Models of ZFC_n^- will play an important role throughout this paper. In particular, we are interested in models inside the constructible universe.

Definition 3.3. Given $n \in \omega$, let β_n be the least ordinal such that

$$L_{\beta_n} \models \mathsf{ZFC}_n^-.$$

By the Gödel Condensation Lemma and the Löwenheim-Skolem Theorem, β_n, and hence L_{β_n}, is countable.

Proposition 3.4. *If $A \subseteq 2^\omega$ is Σ_n^0, then the Borel game $\mathcal{G}(A)$ with winning set A has a winning strategy S in L_{β_n}.*

The proof given by Martin [31] is *inductive*. The basic notion is an *unraveling of game*. Simply speaking, a tree T over some set B unravels $\mathcal{G}(A)$ if there exists a continuous mapping $\pi\colon [T] \to 2^\omega$ such that $\pi^{-1}(A)$ is clopen in $[T]$, and there is a continuous correspondence between strategies on T and strategies on $2^{<\omega}$.

Martin first shows that Π_1^0 games can be unraveled. The argument is completely constructive, hence can be carried out in L. The unraveling tree

T is given by the legal moves of some auxiliary game whose moves correspond to strategies in the original game on $2^{<\omega}$, that is, reals. To be able to collect all these legal moves requires the existence of the power set of ω.

The inductive step then shows how to unravel a given Σ^0_n set A. Suppose $A = \bigcup A_i$, where each A_i is Π^0_{n-1}. By induction hypothesis, each A_i can be unraveled by some T_i via some mapping π_i. Martin proves that the unravelings T_i can be combined into a single one, T_∞, that unravels each A_i via some π_∞. Since each of the sets $\pi_\infty^{-1}(A_i)$ is clopen, their union $\bigcup \pi_\infty^{-1}(A_i) = \pi_\infty^{-1}(A)$ is open, and can in turn be unraveled by some T. Again, the proof is constructive. The last step in the construction (unraveling $\pi_\infty^{-1}(A)$) passes to a tree of higher type – its nodes correspond to strategies over T_∞. Hence one more iterate of the power set of ω is introduced.

Martin's proof constructs a winning strategy in L_{β_n}, relative to L_{β_n}. By Shoenfield Absoluteness, this is also a winning strategy in V (see also Subsection 3.6).

In order to complete the proof of Theorem 1, it now suffices to prove a Posner-Robinson style theorem for reals not contained in L_{β_n} and use an absoluteness argument.

Theorem 3.5. *Suppose a real X is not in L_{β_n}, then there exists a model $L_{\beta_n}[G]$ of ZFC^-_n such that every real in $L_{\beta_n}[G]$ is Turing reducible to $X \oplus G$.*

3.5. **Kumabe-Slaman forcing.** This subsection is devoted to proving Theorem 3.5. We construct G by means of a notion of forcing due to Kumabe and Slaman. The forcing was an essential ingredient in the proof of the definability of the Turing jump by Shore and Slaman [43]. It allows for extending the Posner-Robinson Theorem beyond a simple instance of the Turing jump. It is based on the following partial order. The construction of the generic G follows [43] rather closely. We have to ensure, however, that forcing has the desired set theoretic properties.

Definition 3.6. Let $\mathbb{P}$ be the following partial order.

(1) The elements p of $\mathbb{P}$ are pairs $(\Phi_p, \vec{Z}_p)$ in which Φ_p is a finite, use-monotone Turing functional and $\vec{Z}_p$ is a finite collection of subsets of ω. As usual, we identify subsets of ω with elements of 2^ω.

(2) If p and q are elements of $\mathbb{P}$, then $p \geq q$ if and only if
   (a) (i) $\Phi_p \subseteq \Phi_q$ and
   (ii) for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,

(b) $\vec{Z}_p \subseteq \vec{Z}_q$,
(c) for every x, y, and $X \in \vec{Z}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

In short, a stronger condition than p can add computations to Φ_p, provided that they are longer than any computation in Φ_p and that they do not apply to any element of $\vec{Z}_p$.

If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter, then G gives naturally naturally rise to a functional $\Phi_G = \bigcup\{\Phi_p : p \in G\}$. To prove Theorem 3.5, we will construct a G that is $\mathbb{P}$-generic for L_{β_n} so that $L_{\beta_n}[G]$ is a model of ZF^-_n. Furthermore, we have to ensure that every element in $L_{\beta_n}[G]$ is computable from $G \oplus X$ and every statement in the forcing language about Φ_G is correctly decided by a condition in $\mathbb{P}$ that belongs to G. We will also show that it is possible to meet the relevant dense subsets of $\mathbb{P}$ and still arrange that $\Phi_G(X)$ codes a counting of all reals in $L_{\beta_n}[G]$.

We establish the standard facts about forcing over models of ZFC for $\mathbb{P}$. In our setting, $\mathbb{P}$ is not in the ground model since it is uncountable. We will therefore restrict $\mathbb{P}$ to L_{β_n}, i.e. work with $\mathbb{P}^{(n)} = \mathbb{P} \cap L_{\beta_n}$. If $n \geq 1$, then the presence of $\mathcal{P}(\omega)^{L_{\beta_n}}$ implies that $\mathbb{P}^{(n)} \in L_{\beta_n}$. For $n = 0$, this is not true and hence we have to treat this case separately.

The following two propositions are straightforward, following either the standard proofs (see for instance [26]), or the corresponding arguments in [43].

Proposition 3.7 (Definability of Forcing)**.** *For every n, the forcing relation*

$$p \Vdash_{\mathbb{P}, L_{\beta_n}} \varphi,$$

where φ is a sentence in the forcing language, is definable over L_{β_n}.

In particular, for each φ, the set

$$\{p \in \mathbb{P}^{(n)} : p \Vdash_{\mathbb{P}, L_{\beta_n}} \varphi\}$$

is in L_{β_n} (since comprehension holds in L_{β_n}). As usual, we drop the index $\mathbb{P}, L_{\beta_n}$, since the model and the partial order will be fixed.

Proposition 3.8 (Density)**.** *For any sentence φ and any n, the set*

$$\{p \in \mathbb{P}^{(n)} : p \Vdash \varphi \text{ or } p \Vdash \neg\varphi\}$$

is open dense in $\mathbb{P}^{(n)}$.

Since the L_{β_n} are not full models of ZFC, but satisfy only special instances of the power set axiom, we state the Generic Model Theorem adapted to our purposes.

Proposition 3.9 (Generic Model Theorem)**.** *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over L_{β_n}, then there exists a countable transitive set $L_{\beta_n}[G]$ such that*

(1) $M[G]$ is a model of ZFC_n^-,
(2) $M \subset M[G]$ and $G \in M[G]$.

For $n > 0$, the proof is the standard one for all axioms of $\mathsf{ZFC} - \mathsf{P}$ (see for instance [26], Theorem 4.2). Note that one does not need the power set axiom to verify these in $L_{\beta_n}[G]$.

In the case $n = 0$, we sketch how verify Replacement (Comprehension is similar). Suppose

$$p \Vdash f \text{ is a definable function with domain } \sigma.$$

One can find a set $\Gamma \in M$ of $\mathbb{P}$-names such that it is forced that every element of σ is named by a term in Γ. We argue then that there exists a definable map $F : \Gamma \to M$ such that for all $\gamma \in \Gamma$, $F(\gamma)$ is a subset of a set $A \times B$, where A is a maximal antichain extending p in $\mathbb{P}$ and B is a set of $\mathbb{P}$-names in M such that for all $(q, \beta) \in F(\gamma)$, $q \Vdash f(\gamma) = \beta$.

To see that such a mapping F exists, fix $\gamma \in \Gamma$. For each $\Phi_q \supseteq \Phi_p$ compatible with $\vec{X}_p$ find $(\Phi_q^*, \vec{X}^*) \leq (\Phi_q, \vec{X}_p)$ and β such that

$$(\Phi_q^*, \vec{X}^*) \Vdash f(\gamma) = \beta$$

and Φ_q^* is a minimal first part of such an extension.

Collect the results, i.e. the pairs of the form $((\Phi_q^*, \vec{X}^*), \beta)$ in a set $C \in M$ (which exists since replacement holds in M). We claim that the set of first components of C has a maximal antichain subset. For suppose that (Φ, Y) is incompatible with all $(\Phi_q^*, \vec{X}^*)$.

It is safe to assume that (Φ, Y) decides the value of $f(\gamma)$ (if not, replace it by an extension that does). Now, when $\Phi_q = \Phi$, we must have had $(\Phi_q^*, \vec{X}^*) = (\Phi_q, \vec{X}^*)$, i.e. $\Phi_q^* = \Phi_q = \Phi$ (since (Φ, Y) decides the value, it is minimal). But then $(\Phi_q^*, \vec{X}^*) \in C$ and (Φ, Y) are compatible, contradiction.

The standard way to ensure that cardinals are preserved in the forcing extension is to show that the partial order $\mathbb{P}$ has the *countable chain condition* (*c.c.c.*) relative to M. Recall that $\mathbb{P}$ has the countable chain condition if every antichain is countable. An antichain is a subset A of $\mathbb{P}$ whose elements are mutually *incompatible*, i.e. no two elements of A have a common extension. We will see next that the Kumabe-Slaman partial order has the c.c.c.

Proposition 3.10. *If M is a transitive model of $\mathsf{ZF} - \mathsf{P}$, then the partial order $\mathbb{P}$ satisfies the countable chain condition, relativized to M.*

Proof. Let $A \subseteq \mathbb{P}$ be an antichain. Note that if $(\Phi_p, \vec{Z}_p)$ and $(\Phi_q, \vec{Z}_q)$ are two elements of $\mathbb{P}$ with $\Phi_p = \Phi_q$, then they have a common extension $(\Phi_p, \vec{Z}_p \cup \vec{Z}_q)$. Hence all the Φ_p occurring in A must be pairwise different. There are at most countably many Φ_p. This argument relativizes to M. □

Next, we show that every dense set in L_{β_n} can be met via an extension adding no computations along X. This is crucial for the construction in [43].

Lemma 3.11. *Let $D \in L_{\beta_n}$ be dense, and suppose $X \notin L_{\beta_n}$. For any $p \in \mathbb{P}$, there exists a $q \leq p$ such that $q \in D$ and Φ_q does not add any new computation along X.*

Proof. Suppose $p = (\Phi_p, \vec{Z}_p)$ is in $\mathbb{P}$. We say a string τ is *essential* for (p, D) if, whenever $q < p$ and $q \in D$, there exists a triple $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that σ is compatible with τ. In other words, whenever one meets D above p, a computation along τ is added.

The following argumentation takes place in the ground model L_{β_n}, using absoluteness and the strong closure properties of L_{β_n}.

It is obvious that being essential is closed under taking initial segments. Therefore, the set

$$T(p, D) = \{\tau \colon \tau \text{ essential for } (p, D)\ \}$$

is a binary tree, and, due to the explicit definition of $T(p, D)$, it also holds that $T(p, D) \in L_{\beta_n}$.

Assume now for a contradiction that a q as postulated above does not exist. This means that for any $r \leq p$, either $r \notin D$ or Φ_r adds a computation along X. It follows that every initial segment $\tau \subset X$ is essential for (p, D). Thus $T(p, D)$ is an infinite binary tree. By a compactness argument relativized to L_{β_n}, there exists a real $Y \in 2^\omega \cap L_{\beta_n}$ such that Y is an infinite path through $T(p, D)$. Since $X \notin L_{\beta_n}$, $X \neq Y$.

Now consider the condition $q = (\Phi_p, \vec{Z}_p \cup \{Y\})$. Since $Y \in L_{\beta_n}$, this is a valid condition in $(\mathbb{P})^{L_{\beta_n}}$. As $\Phi_q = \Phi_p$, we trivially have $q \leq p$. Furthermore, if $r \leq q$, then $r \leq q = (\Phi_p, \vec{Z}_p \cup \{Y\}) \leq (\Phi_p, \vec{Z}_p) = p$. Since every initial segment of Y is essential for (p, D), every extension of p in D must add a computation along Y. But this means that there is no extension of r that is in D, contradicting the density of D. □

The actual construction of a generic real G now follows [43], except that instead of deciding a Π_n-complete predicate, we simply code a counting of all reals in $L_{\beta_n}[G]$.

3.6. **Completing the proof of Theorem 1.** We now put the pieces together to show that every real outside of L_{β_n} is n-random with respect to a continuous probability measure. Given $X \notin L_{\beta_n}$, choose G as in Theorem 3.5. Consider the game defined in the proof of Proposition 3.1. Its winning set is

$$B = \{X \in 2^\omega : \exists Z \, \exists R \, (X \equiv_{\mathrm{T}} Z \oplus R \;\;\&\;\; R \text{ is } (Z, n+1)\text{-random})\}$$

Using the Harrington-Kechris analysis described above, one can relate the game $\mathcal{G}(B)$ to a Σ^0_n-game in Baire space.

Now relativize to G and denote the relativized winning set by B^G. Since $G \in L_{\beta_n}[G]$ and $L_{\beta_n}[G]$ is a generic extension of L_{β_n}, by choice of β_n, $L_{\beta_n}[G]$ contains a winning strategy for the relativized game $\mathcal{G}(B^G)$ in $L_{\beta_n}[G]$. We simply have to mimic Martin's proof inside $L_{\beta_n}[G]$. This winning strategy is successful for all plays "inside L_{β_n}".

The property of being a winning strategy for a given Borel game is $\mathbf{\Pi}^1_1$. By the (relativized) Shoenfield Absoluteness Theorem, this means that a winning strategy in $L_{\beta_n}[G]$ is actually a winning strategy "in the real world", i.e. it wins on *all* plays, not just the ones in $L_{\beta_n}[G]$.

Hence, by Proposition 3.1, *every* real in the upper cone (relative to G) of the winning strategy is (μ, G, n)-random for some continuous measure μ. Since X is (relative to G) in every upper cone with base in $L_{\beta_n}[G]$, this applies to X. Finally, note that every (μ, G, n)-random real is (μ, n)-random.

It is possible "strip" any mentioning of randomness from the main result of this section and present it as a degree theoretic property of Turing-invariant Borel sets. To obtain a random real above X, we join it with a real that is X-random. Closure under Turing equivalence yields the set B used above. Note that we needed Turing-equivalence to an n-random real in order to apply the measure-existence results of Section 2. But the proof of Theorem 1 works for any Borel set with the accordant invariance and cofinality properties. This generalization was suggested by Carl Jockusch.

Proposition 3.12. *Suppose $B \subseteq 2^\omega \times 2^\omega$ is Borel and Turing invariant. Furthermore, suppose for all $X \in 2^\omega$ there exists $Y \geq_{\mathrm{T}} X$ such that $(X, X \oplus Y) \in B$. Then for all but countably many reals X there exists a real Y such that $(Y, Y \oplus X) \in B$.*

4. The Metamathematics of Randomness

In this section, we will show that the metamathematical argument used to prove the countability of NCR_n is in a certain sense indispensable. More precisely, we will prove the Theorem 2, which we reformulate here in a slightly stronger way.

Theorem 2. *There exists a computable function $G(n)$ such that for every $n \in \omega$,*

$$\mathsf{ZFC}^-_n \nvdash \text{“}\mathrm{NCR}_{G(n)} \textit{ is countable.”}$$

Before starting the proof, we outline its basic idea. For given n, we will exhibit a model $\mathcal{M}$ of ZFC^-_n in which $\mathrm{NCR}_{G(n)}$ is not countable. To this end, we find a sequence (Y_α) of reals that satisfies

(1) (Y_α) is cofinal in the Turing degrees of the model $\mathcal{M}$, hence not countable in $\mathcal{M}$,
(2) no Y_α is $G(n)$-random for a continuous measure in $\mathcal{M}$.

As we have seen in Propositions 2.15 and 2.16, instances of the jump are candidates for an increasing sequence of non-random reals. It makes therefore sense to look for a set-theoretic analogue of the jump hierarchy. This analogue is given by so-called *master codes* of the constructible hierarchy. Just as instances of the jump code levels of definable subsets of ω, master codes code levels of the constructible universe. The master codes for a higher level of L can be obtained from codes for lower levels by iterating definability. This will be crucial for our proof, since it allows for applying the “Stair Trainer”-argument of Propositions 2.15 and 2.16 in this setting.

In order to define this sequence, we have to present a few more facts on L. In particular, we are interested how at each step new sets are added to L. This is the heart of the *fine structure theory* of L due to Jensen [20]. We will give a brief review of the core concepts and results. Readers familiar with fine structure theory can skip ahead to Subsection 4.3.

Let us also remark that a lot of numerical details in the following sections are owed to the goal of obtaining a rather explicit bound for the non-randomness of master codes. Any reader content with the mere existence of a recursive function $G(n)$ as postulated above can skip over many details.

4.1. **Fine structure and Jensen's J-hierarchy.** Fine structure provides a level-by-level, quantifier-by-quantifier analysis of how new sets are generated in L. One problem with such an analysis for the L-hierarchy is the lack of sufficient closure properties of L_α (such as under pairing functions etc) when α is not a limit ordinal. Jensen therefore replaces the L-hierarchy by a slightly different hierarchy, the *J-hierarchy*, which has sufficiently nice closure properties at every level, not only at limit stages.

The sets J_α are obtained by closing under a scheme of so-called *rudimentary functions*. In contrast to L, $J_{\alpha+1}$ can contain sets of higher rank, not just *subsets* of J_α, e.g. (set-theoretic) pairs. The rudimentary functions are essentially a scheme of *primitive set recursion* [21].

- projection,
- set difference,
- pairing,
- composition (substitution),
- union, i.e. $f(x, y) = \bigcup_{z\in y} f(x, z)$.

$\mathrm{rud}(X)$ is the smallest set Y that contains $X \cup \{X\}$ and is closed under rudimentary functions (rud closed). The inclusion of $\{X\}$ when taking the rudimentary closure guarantees that new sets are introduced even if X is closed under rudimentary functions.

The J-hierarchy is introduced as a cumulative hierarchy induced by the rud-operation:

$$\begin{aligned} J_0 &= \emptyset \\ J_{\alpha+1} &= \mathrm{rud}(J_\alpha) \\ J_\lambda &= \bigcup_{\alpha<\lambda} J_\alpha \quad \text{ for } \lambda \text{ limit.} \end{aligned}$$

A fine analysis of the rudimentary functions reveals that the rud-operation can be completed by iterating some or all of *nine basic rudimentary functions*.

Proposition 4.1 ([20])**.** *Every rudimentary function is a combination of the following nine functions:*

$$\begin{aligned} F_0(x, y) &= \{x, y\}, \\ F_1(x, y) &= x \setminus y, \\ F_2(x, y) &= x \times y, \\ F_3(x, y) &= \{(u, z, v) \colon z \in x \ \wedge \ (u, v) \in y\}, \\ F_4(x, y) &= \{(u, v, z) \colon z \in x \ \wedge \ (u, v) \in y\}, \\ F_5(x, y) &= \bigcup x, \\ F_6(x, y) &= \mathrm{dom}(x), \\ F_7(x, y) &= \in \cap \, (x \times x), \\ F_8(x, y) &= \{\{x(z)\} \colon z \in y\}. \end{aligned}$$

The S-operator is defined as taking a *one-step application* of any of the basic functions,

$$S(X) = [X \cup \{X\}] \cup \left[\bigcup_{i=0}^{8} F_i[X \cup \{X\}] \right]. \tag{4.1}$$

We have

$$\mathrm{rud}(X) = \bigcup_{n \in \omega} S^{(n)}(X).$$

The S-hierarchy is defined as the cumulative hierarchy induced by the S-operator and refines the J-hierarchy.

$$\begin{aligned} S_0 &= \emptyset, \\ S_{\alpha+1} &= S(S_\alpha), \\ S_\lambda &= \bigcup_{\alpha<\lambda} S_\alpha \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

We obviously have

$$J_\alpha = \bigcup_{\beta < \omega\alpha} S_\beta = S_{\omega\alpha}.$$

We list a few basic properties of the sets J_α. For details and proofs, see [20] or [6].

- Each J_α is transitive and *amenable*, i.e. it is a model of a sufficiently large fragment of set theory (more precisely, it is a model of KP-set theory without Σ_0-collection).
- The hierarchy is *cumulative*, i.e. $\alpha \leq \beta$ implies $J_\alpha \subseteq J_\beta$.
- $\mathrm{rank}(J_{\alpha+1}) = \mathrm{rank}(J_\alpha) + \omega$. Each successor step adds ω new ordinals. $J_\alpha \cap \mathrm{Ord} = \omega\alpha$, in particular, $J_1 = V_\omega$ and $J_1 \cap \mathrm{Ord} = \omega$.
- $(J_\alpha)_{\alpha \in \mathrm{Ord}}$ and $(L_\alpha)_{\alpha \in \mathrm{Ord}}$ generate the same universe: $L = \bigcup_\alpha J_\alpha$. Moreover, $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$, and $J_\alpha = L_\alpha$ if and only if $\omega\alpha = \alpha$. Finally, $J_{\alpha+1} \cap \mathcal{P}(J_\alpha) = \mathcal{P}_{\mathrm{DEF}}(J_\alpha)$, that is, $J_{\alpha+1}$ contains precisely those subsets of J_α that are first order definable over J_α.
- The Σ_n-satisfaction relation over J_α, $\models^{\Sigma_n}_{J_\alpha}$, is Σ_n-definable over J_α, uniformly in α.
- The mapping $\beta \mapsto J_\beta$ $(\beta < \alpha)$ is Σ_1-definable over any J_α.
- There is a Π_2 formula $\varphi_{\mathsf{V=J}}$ such that for any transitive set M,

$$M \models \varphi_{\mathsf{V=J}} \;\Leftrightarrow\; \exists \alpha\, M = J_\alpha.$$

The J-hierarchy shares all important metamathematical features with the L-hierarchy. We cite the two most important facts. The L-versions of the two propositions together constitute the core of Gödel's proof that GCH and AC hold in L.

Proposition 4.2. *The sequence $(J_\beta \colon \beta < \alpha)$ is uniformly Σ_1-definable over J_α. Furthermore, there exists a Σ_1-definable well-ordering $<_J$ of L and for any $\alpha > 1$, the restriction of $<_J$ to J_α is uniformly Σ_1-definable over J_α.*

Proposition 4.3 (The Condensation Lemma for J)**.** *For any α, if $X \preceq_{\Sigma_1} J_\alpha$, then there is an ordinal β and an isomorphism π between X and J_β. Both β and π are uniquely determined.*

For proofs of these results for the J-hierarchy again refer to [20] or [6].

4.2. **Projecta and master codes.** The definable well-ordering $<_J$ together with the definability of the satisfaction relation can be used to show that each J_α has **definable Skolem functions**, essentially by selecting the $<_J$-least witness that satisfies an existential formula. The definable Skolem functions can in turn be used to define a canonical counting of J_α.

Proposition 4.4 ([20])**.** *For each α, there exists a $\Sigma_1(J_\alpha)$-definable surjection from $\omega\alpha$ onto J_α.*

While a simple cardinality argument yields that $|J_\alpha| = |\omega\alpha|$, Jensen's result shows that an α-counting of J_α already exists in $J_{\alpha+1}$. The counting is obtained by taking (essentially) the *Skolem hull* of $\omega\alpha$ under the canonical Σ_1-Skolem function. The resulting set X is a Σ_1-elementary substructure of J_α, hence by the Condensation Lemma isomorphic to some J_β. The isomorphism taking X to J_β is the identity on all ordinals below $\omega\alpha$, and one can show that this in turn implies that the isomorphism must the identity on X, i.e. $X = J_\beta$.

Boolos and Putnam [3] first observed that if a new real is defined in $L_{\alpha+1}$, i.e. if

$$\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset,$$

then the strong absoluteness properties of L can be used to get a *definable* ω-counting of L_α (instead of just an α-counting as above). Because, if a new subset of ω is constructed in $L_{\alpha+1} \setminus L_\alpha$, one can take the Skolem hull of ω instead of $\omega\alpha$. The resulting $X \cong L_\beta$ is still equal to L_α, since the definition of the new real can be "mimicked" in the elementary substructure L_β. If $\beta < \alpha$, then this would contradict the fact that $Z \notin L_\alpha$.

Proposition 4.5 ([3])**.** *If $\mathcal{P}(\omega) \cap (L_{\alpha+1} \setminus L_\alpha) \neq \emptyset$, then there exists a surjection $f : \omega \to L_\alpha$ in $L_{\alpha+1}$.*

Of course, at some stages no new reals constructed. [3] showed that the first such stage is precisely the ordinal β_0, i.e. the least ordinal β such that $L_\beta \models \mathsf{ZF}^-$. By Gödel's work, on the other hand, we know that no new real is constructed after stage ω_1^L.

Jensen [20] vastly extended these ideas into the framework of *projecta* and *master codes*, which form the core concepts of *fine structure theory*.

Definition 4.6. For $n, \alpha > 0$, the Σ_n*-projectum* ρ_α^n is equal to the least $\gamma \leq \alpha$ such that $\mathcal{P}(\omega\gamma) \cap (\Sigma_n(J_\alpha) \setminus J_\alpha) \neq \emptyset$.

We put $\rho_\alpha^0 = \alpha$. Hence $1 \leq \rho_\alpha^n \leq \alpha$ for all n. As ρ_α^n is non-increasing in n, we can also define

$$\rho_\alpha = \min_n \rho_\alpha^n \quad \text{and} \quad n_\alpha = \min\{k \colon \rho_\alpha^k = \rho_\alpha\}.$$

Jensen proved that the projectum ρ_α^n is equal to the least $\delta \leq \alpha$ such that there exists a function f that is $\Sigma_n(J_\alpha)$-definable over J_α such that $f(D) = J_\alpha$ for some $D \subseteq \omega\delta$, establishing the analogy with the Boolos-Putnam result. From this it follows that if $\rho_\alpha^n < \alpha$, it must be a cardinal in J_α, for all n.

Jensen gave another characterization of the projectum, which in fact he used as his original definition in [20]. Suppose $\langle M, \in\rangle$ is a set-theoretic structure. We can extend this structure by adding an additional relation $A \subset M$. If we do this, we would like the structure to satisfy some basic set theoretic closure properties. For instance, we would like our universe to satisfy the comprehension axiom with respect to the new relation, that is, whenever we pick an $x \in M$, the collection of elements in x that satisfy R should be in M. Such structures are called *amenable*.

Definition 4.7. Given $A \subseteq M$, the structure $\langle M, A\rangle$ is called *amenable*, if M is an amenable set and

$$\forall x \in M \; [x \cap A \in M].$$

Jensen showed that

$$\begin{aligned} \rho_\alpha^n = {} & \text{the largest ordinal } \gamma \leq \alpha \text{ such that} \\ & \langle J_\gamma, A\rangle \text{ is amenable for any } A \subseteq J_\gamma \text{ that is in } \Sigma_n(J_\alpha). \end{aligned}$$

This means the projectum ρ_α^n identifies the "stable" core of J_α with respect to Σ_n definability over J_α.

Being amenable with rud-closed domain can also be characterized via relative rud-closedness. This will be important later.

Definition 4.8 ([20])**.** A function f is *A-rud* if it can be obtained as a combination of the basis functions $F_1, \ldots, F_8$ and the function

$$F_A(x, y) = x \cap A.$$

A structure $\langle M, A\rangle$ is *rud closed* if $f[M^n] \subseteq M$ for all A-rud functions f.

Proposition 4.9 ([20])**.** *A structure $\langle M, A\rangle$, $A \subseteq M$, is rud closed if and only if M is rud closed and $\langle M, A\rangle$ is amenable.*

The existence of a definable surjection between (a subset of) $\omega\rho_\alpha^n$ and $\Sigma_n(J_\alpha)$ allows for coding $\Sigma_n(J_\alpha)$ into its projectum. One way this can be implemented is via so-called *master codes*.

Definition 4.10. A Σ_n *master code* for J_α is a set $A \subseteq J_{\rho_\alpha^n}$ that is $\Sigma_n(J_\alpha)$, such that for any $m \geq 1$,

$$\Sigma_{n+m}(J_\alpha) \cap \mathcal{P}(J_{\rho_\alpha^n}) = \Sigma_m(\langle J_{\rho_\alpha^n}, A\rangle).$$

A Σ_n master code does two things:

(1) It "accelerates" definitions of new subsets of $J_{\rho_\alpha^n}$ by n quantifiers.
(2) It replaces parameters from J_α in the definition of these new sets by parameters from $J_{\rho_\alpha^n}$ (and the use of A as an "oracle").

The existence of master codes follows rather easily from the existence of a $\Sigma_n(J_\alpha)$-mapping from $\omega\rho_\alpha^n$ onto J_α. However, for $n > 1$, this mapping is *not uniform*. Jensen exhibited a uniform, canonical way to define master codes, by iterating Σ_1-definability.

Put

$$A_\alpha^0 = \emptyset, \quad p_\alpha^0 = \emptyset.$$

Assuming that A_α^n is a Σ_n code, it is not hard to see that every set $x \in J_{\rho_\alpha^n}$ is Σ_1-definable over $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ with parameters from $J_{\rho_\alpha^{n+1}}$ and one parameter from $J_{\rho_\alpha^n}$ (used to define a surjection from $\omega\rho_\alpha^{n+1}$ onto $J_{\rho_\alpha^n}$). Hence we can put

$$p_\alpha^{n+1} = \text{the } <_J\text{-least } p \in J_{\rho_\alpha^n} \text{ such that every } u \in J_{\rho_\alpha^n} \text{ is } \Sigma_1 \text{ definable over } \langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \text{ with parameters from } J_{\rho_\alpha^{n+1}} \cup \{p\}\ .$$

The p^n_α are called the *standard parameters*.

Using p^{n+1}_α, we can code the Σ_1 elementary diagram of the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ into a set A^{n+1}_α:

$$A^{n+1}_\alpha := \{(i,x)\colon i\in\omega \ \wedge\ x \in J_{\rho^{n+1}_\alpha} \ \wedge\ \langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \varphi^{(2)}_i(x, p^{n+1}_\alpha)\},$$

where $(\varphi^{(k)}_i)$ is a standard Gödel numbering of all Σ_1 formulas with k free variables. It is not hard to verify that A^{n+1}_α is a Σ_{n+1} master code for J_α. Furthermore, the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is amenable for each $\alpha > 1$, $n \geq 0$. We will call the structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ the *standard* Σ_n J*-structure* for J_α.

Definition 4.11. We denote the standard J-structure over J_α at the 'ultimate' projectum n_α by

$$\langle J_{\rho_\alpha}, A_\alpha\rangle := \langle J_{\rho^{n_\alpha}_\alpha}, A^{n_\alpha}_\alpha\rangle.$$

One consequence of the A^n_α being master codes is that we can obtain the sequence of projecta of an ordinal by iterating taking Σ_1-projecta relative to $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$. Given an amenable structure $\langle J_\alpha, A\rangle$, the Σ_n*-projectum* $\rho^n_{\alpha,A}$ of $\langle J_\alpha, A\rangle$ is defined to be the largest ordinal $\rho \leq \alpha$ such that $\langle J_\rho, B\rangle$ is amenable for any $B \subseteq J_\rho$ that is in $\Sigma_n(\langle J_\alpha, A\rangle)$.

Proposition 4.12 ([20])**.** *For* $\alpha > 1, n \geq 0$,

$$\rho^{n+1}_\alpha = \rho^1_{\rho^n_\alpha, A^n_\alpha}.$$

In particular, the standard Σ_{n+1} J-structure for $J_\alpha = \langle J_\alpha, \varnothing\rangle$ is the standard Σ_1 J-structure for $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$.

4.3. **ω-Copies of J-structures.** We later want to apply the recursion theoretic techniques of Section 2 to countable J-structures. We therefore have to code them as subsets of ω. If the projectum ρ^N_α is equal to 1, all "information" about the J-structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ is contained in the master code A^n_α, which is simply a real, and hence lends itself directly to recursion theoretic analysis. Starting with the work by Booloas and Putnam [3], this has been studied in a number of papers (e.g. [22], [17]).

The problem in our setting is that we need to uniformly work our way through arithmetic copies of J-structures even when the projectum is greater than 1. For this purpose we have to code two objects, the sets J_α (which keep track of the basic set theoretic relations) and the standard codes over each J_α, which keep track of the definable objects quantifier by quantifier. This adds a certain complexity to our coding mechanism. We tried to stay as close as possible to Jensen's original framework.

Definition 4.13. Let $X \subseteq \omega$. The *relational structure* induced by X is $\langle F_X, E_X\rangle$, where

$$xE_Xy \Leftrightarrow \langle x,y\rangle \in X$$

and

$$F_X = \mathrm{Field}(E_X) = \{x\colon \exists y xE_Xy \text{ or } yE_Xx \text{ for some } y\}.$$

The idea is that a number x represents a code for the set whose codes are the numbers y with yE_Xx,

$$\mathrm{Set}_X(x) = \{y\colon yE_Xx\}.$$

The relational structure $\langle F_X, E_X\rangle$ is *extensional* if

$$\forall x,y \in F_X\ [(\forall z\ zE_Xx \Leftrightarrow zE_Xy)\ \Rightarrow x = y],$$

that is

$$\forall x,y \in F_X\ (x \neq y \Rightarrow \mathrm{Set}_X(x) \neq \mathrm{Set}_X(y)).$$

Mostowski's Collapsing Theorem states that if $\langle F_X, E_X\rangle$ is extensional and well-founded, it is isomorphic to a unique structure $(M, \in)$, where M is a transitive set. In this sense we can speak of a countable set theoretic structure *coded* by X. If $\varphi(v_1,\dots,v_n)$ is a formula in the language of the set theory, we can interpret it over $\langle F_X, E_X\rangle$ and write

$$X \models \varphi[a_1,\dots,a_n]$$

for $\langle F_X, E_X\rangle \models \varphi[a_1,\dots,a_n]$ with $a_i \in F_X$.

J-structures have an additional set A, and we capture this on the coding side via pairs $\langle X, M\rangle$, where $M \subseteq F_X$. Semantically, A and M are seen as interpreting a predicate added to the language. This way we can consider the satisfaction relation $\langle X, M\rangle \models \varphi$, where φ is a set-theoretic formula with an additional unary predicate.

We are particularly interested in relational structures that code countable standard J-structures. The following is a generalization of the definition due to Boolos and Putnam [3]

Definition 4.14. An *ω-copy* of a countable set-theoretic structure $\langle S, A\rangle$, $A \subseteq S$, is a pair $\langle X, M\rangle$ of subsets of ω such that the structure coded by X is extensional and there exists a surjection $\pi : S \to \mathrm{Field}(E_X)$ such that

$$\forall x,y \in S\,[\,x \in y \iff \pi(x)E_X\pi(y)\,], \tag{4.2}$$

and

$$M = \{\pi(x)\colon x \in A\}. \tag{4.3}$$

The definition thus means an ω-copy $\langle X, M\rangle$ of $\langle S, A\rangle$ is isomorphic to $\langle S, A\rangle$ when seen as structures over the language of set theory.

If $A = \varnothing$, then necessarily $M = \varnothing$, and in this case we say X is an ω-copy of S.

The standard Σ_n J-structure $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ codes the Σ_n-definable sets over J_α in a set-theoretically "efficient" way, by locating counterparts for each set inside the Σ_n-projectum.

We will now show that ω-copies of J-structures allow us to decode this information effectively, provided the coding of natural numbers and pairs is implemented in an effective way.

If $\rho_\alpha^n = 1$, we have $J_{\rho_\alpha^n} = L_\omega = V_\omega$, i.e. the *hereditarily finite sets*. In this case, an ω-copy is easily obtained using a bijection between ω and V_ω. We fix the following bijection (given by Ackermann [1]): Put $\pi_\omega(0) = \varnothing$ and define recursively,

$$\pi_\omega(2^{a_0} + 2^{a_1} + \cdots + 2^{a_n}) = \{\pi_\omega(a_0), \ldots, \pi_\omega(a_n)\}$$

for pairwise distinct a_i.

Now we can let $\langle x, y\rangle \in X$ if and only if $\pi_\omega(x) \in \pi_\omega(y)$ and $x \in M$ if and only if $\pi_\omega(x) \in A_\alpha^{n+1}$. Then $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ via π_ω^{-1}. Let us call this the *canonical copy* of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, which only exists for $\rho_\alpha^{n+1} = 1$. We also have in this case (assuming $\alpha > 1$) that the isomorphism π_ω is an element of J_α. In general, while an ω-copy always exists (as long as $J_{\rho_\alpha^n}$ is countable), the isomorphism π between the J-structure and its ω-copy may be very complicated (in particular not an element of J_α).

However, we will show that from a canonical copy we can extract ω-copies of all $\langle J_{\rho_\alpha^i}, A_\alpha^i\rangle$, $i \le n$, in an effective and uniform way.

By choice of p_α^{n+1}, for every $u \in J_{\rho_\alpha^n}$, there exists a Σ_1-formula $\psi(v_0, v_1, v_2)$ and $x \in J_{\rho_\alpha^{n+1}}$ such that u is the only solution over $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ to $\exists v_0\, \psi(v_0, x, p_\alpha^{n+1})$.

Definition 4.15. A pair (i, x), $i \in \omega, x \in J_{\rho_\alpha^{n+1}}$ is an *n-code* if there exists a $u \in J_{\rho_\alpha^n}$ such that u is the unique solution to

$$\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \models \varphi_i^{(3)}(v_0, x, p_\alpha^{n+1}).$$

(Recall that $(\varphi_i^{(k)})$ is a standard Gödel numbering of all Σ_1 formulas with k free variables.)

We can check the property of being an n-code using Σ_1 formulas: (i, x) is an n-code if and only if

$$\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle \models \exists v_0\, \psi_i(v_0, x, p^{n+1}_\alpha) \tag{4.4}$$

and

$$\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle \not\models \exists v_0, v_1\, (\psi_i(v_0, x, p^{n+1}_\alpha) \wedge \psi_i(v_1, x, p^{n+1}_\alpha) \wedge v_0 \neq v_1). \tag{4.5}$$

This means a standard code has the information necessary to sort out n-codes among its elements. We would like this to hold also for ω-copies. If $\langle X, M \rangle$ is an ω-copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha \rangle$ via π, and $(i, x) \in J_{\rho^{n+1}_\alpha}$ is an n-code, then we call $\pi((i, x))$ a π-n-code.

For a canonical copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha \rangle$, it is decidable whether some number is a π-n-code, due to the effective way we can translate between finite sets and their codes. For arbitrary ω-copies, this may not be true as there may be no effective way to "decode" the π-code of a pair. Being able to decode the π-code of a pair hinges on knowledge of the following two functions

(i) the mapping $n_X : n \mapsto \pi(n)$ $(n \in \omega)$,
(ii) the mapping $h_X : (\pi(x), \pi(y)) \mapsto \pi((x, y))$.

To facilitate notation, we will sometimes let $i_X, j_X \dots$ denote $n_X(i), n_X(j), \dots$.

If $\alpha > 1$, then $\omega \in J_\alpha$, and an ω-copy of any such J_α must contain a witness for ω. In this case, we can recover n_X recursively in X'.

Proposition 4.16. *If X is an ω-copy of J_α, $\alpha > 1$, then n_X is computable in X'.*

Proof. We approximate $n_X(i)$ from below. Let $z = \pi(\omega)$. At stage 0, put $n_{X,0}(i) = 0$ for all i. At stage s, we test whether $s E_X z$. If yes, we can determine how s relates to the previous elements of z discovered, that is, we can compute the finite linear order of the elements of z seen so far, say $n_0 E_X \dots E_X n_k$. We put $n_{X,s}(i) = n_i$ for $i \leq k$. □

A similar argument shows that, when $\rho^n_\alpha = 1$, the jump of *any* ω-copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle$ computes the canonical copy.

Proposition 4.17. *If $\langle X, M \rangle$ is an ω-copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle$, and $\rho^n_\alpha = 1$, then $\langle X, M \rangle'$ uniformly computes the canonical copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle$.*

Proof. Suppose $\langle X, M \rangle$ is an ω-copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha \rangle$ via π. We need to show that the mapping $\pi_\omega \circ \pi^{-1}$ is recursive in $\langle X, M \rangle'$.

Given $0_{\langle X^n_\alpha, M^n_\alpha\rangle}$ (the representative of $\varnothing$ in $\langle X, M\rangle$), we can approximate $\pi_\omega \circ \pi^{-1}$ from below by computing at stage s the current "snapshot" of the finite E_X-diagram of the numbers up to s. □

We can also recover the function h_X arithmetically in X.

Proposition 4.18. *If X is an ω-copy of J_α, then the function h_X is computable in $X^{(2)}$.*

Proof. We have

$$h_X(\pi(x), \pi(y)) = b \;\Leftrightarrow\; \exists c, d \left[\forall z\, (zE_X c \;\Leftrightarrow z = \pi(x))\right.$$
$$\left.\wedge\; \forall z\, (zE_X d \Leftrightarrow z = \pi(x) \;\vee\; z = \pi(y)) \;\wedge\; \forall z\, (zE_X b \;\Leftrightarrow\; z = c \;\vee\; z = d)\right]$$

□

Definition 4.19. Suppose $\langle X, M\rangle$ is an ω-copy via π of a rud closed structure $\langle J, A\rangle$. We say $\langle X, M\rangle$ is *effective* if the functions n_X and h_X are recursive in $X \oplus M$.

Lemma 4.20. *The canonical copy of $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$, if it exists, is effective.*

Proof. The mapping π_ω^{-1} satisfies conditions (i) and (ii) required in Definition 4.19 naturally. □

The main property of effective copies is that we can effectively "decompose" codes for pairs.

Lemma 4.21. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π, then for every $u, v \in J_{\rho^{n+1}_\alpha}$, the mapping*

$$\pi((u, v)) \mapsto (\pi(u), \pi(v))$$

is recursive in $X \oplus M$. If $i \in \omega$, then the mapping

$$\pi((i, u)) \mapsto (i, \pi(u))$$

is recursive in $X \oplus M$.

Proof. The first mapping can be computed by inverting h_X (which must be one-one), the second mapping by additionally inverting n_X. □

Lemma 4.22. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π, then it is decidable in $X \oplus M$ whether a number $y \in \omega$ is a π-n-code.*

Proof. Suppose $y \in M$ (if not, it cannot be a π-n-code). Then $y = \pi((i,x))$ for some $(i,x) \in A^{n+1}_\alpha$. Since the copy is effective, we have $\pi((i,x)) = h_X(\pi(i),\pi(x))$, and by Lemma 4.21 we can find i and $\pi(x)$ recursively in $X \oplus M$.

Recall that $(\varphi^{(2)}_i)$ is a standard Gödel numbering of the Σ_1 formulas with two free variables. There exist recursive functions g_1, g_2 such that $\varphi^{(2)}_{g_1(i)}$ and $\varphi^{(2)}_{g_2(i)}$ are Σ_1 formulas equivalent (over $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$) to the formulas in (4.4) and (4.5), respectively. Then (i,x) is a n-code if and only if $(g_1(i),x) \in A^{n+1}_\alpha$ and $(g_2(i),x) \notin A^{n+1}_\alpha$.

By the effectiveness of the ω-copy, the latter two conditions are equivalent to

$$h_X(\pi(g_1(i)),\pi(x)) \in M \text{ and } h_X(\pi(g_2(i)),\pi(x)) \notin M,$$

which is recursive in $X \oplus M$. □

Two n-codes (i_0,x_0) and (i_1,x_1) *represent the same set* $u \in J_{\rho^n_\alpha}$ if u is the unique solution to $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \varphi^{(3)}_{i_0}(v_0,x_0,p^{n+1}_\alpha)$ and $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \varphi^{(3)}_{i_1}(v_0,x_1,p^{n+1}_\alpha)$. A similar property can be defined for π-n-codes.

Lemma 4.23. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho^{n+1}_\alpha}, A^{n+1}_\alpha\rangle$ via π, then it is decidable in $X \oplus M$ whether two numbers are π-n-codes of the same set.*

Proof. (i_0,x_0) and (i_1,x_1) represent different sets if and only if

$$\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle \models \exists v_0,v_1,v_2\ \big[\varphi^{(3)}_{i_0}(v_0,x_0,p^{n+1}_\alpha)\ \wedge\ \varphi^{(3)}_{i_1}(v_1,x_1,p^{n+1}_\alpha) \wedge \\ (v_2 \in v_0 \wedge v_2 \notin v_1) \vee (v_2 \notin v_0 \wedge v_2 \in v_1)\big].$$

Let $g_3(i_0,i_1)$ be a Gödel number for the Σ_1 formula

$$\psi(x,p^{n+1}_\alpha) \equiv \exists v_0,v_1,v_2\ \big[\varphi^{(3)}_{i_0}(v_0,(x)_0,p^{n+1}_\alpha)\ \wedge\ \varphi^{(3)}_{i_1}(v_1,(x)_1,p^{n+1}_\alpha) \wedge \\ (v_2 \in v_0 \wedge v_2 \notin v_1) \vee (v_2 \notin v_0 \wedge v_2 \in v_1)\big].$$

Then (i_0,x_0) and (i_1,x_1) represent different sets if and only if

$$(g_3(i_0,i_1),(x_0,x_1)) \in A^{n+1}_\alpha,$$

which in turn holds if and only if

$$h_X(\pi(g_3(i_0,i_1)),h_X(\pi(x_0),\pi(x_1))) \in M.$$

Since g_3 is computable, it follows from Lemma 4.21 that it is decidable in $X \oplus M$ whether two numbers are π-n-codes and whether hey represent the same set. □

Proposition 4.24. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ via π, n_X, h_X, it computes an ω-copy $\langle Y, N\rangle$ of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$. Furthermore, the computation is uniform, and h_Y and n_Y can be computed uniformly from $X \oplus M \oplus h_X \oplus n_X$.*

Proof. By Lemmas 4.22 and 4.23, the set

$$U = \{y \in \omega \colon y \text{ is the } <_\omega\text{-least } \pi\text{-}n\text{-code for some set } u \in J_{\rho_\alpha^n}\}$$

is recursive in $X \oplus M$.

Let σ be the mapping

$$\sigma : u \in J_{\rho_\alpha^n} \mapsto \text{ the unique } \pi\text{-}n\text{-code of } u \text{ in } U\ ,$$

and put

$$Y = \{\langle \sigma(x), \sigma(y)\rangle \colon x, y \in J_{\rho_\alpha^n}\}, \quad N = \{\sigma(x) \colon x \in A_\alpha^n\}.$$

Then $\langle Y, N\rangle$ is clearly an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$. To show that it is recursive in $X \oplus M$, we note that for $u, w \in J_{\rho_\alpha^n}$, if (i, x) is an n-code for u and (j, y) is an n-code for w,

(4.6)
$$u \in w \;\Leftrightarrow\; \langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \models \exists v_0, v_1\, (\varphi_i(v_0, x, p_\alpha^{n+1}) \wedge \varphi_j(v_1, y, p_\alpha^{n+1}) \wedge v_0 \in v_1).$$

Moreover,

$$u \in A_\alpha^n \;\Leftrightarrow\; \langle J_{\rho_\alpha^n}, A_\alpha^n\rangle \models \exists v_0\, (\varphi_i(v_0, x, p_\alpha^{n+1}) \wedge v_0 \in A_\alpha^n). \tag{4.7}$$

There are recursive functions g_4, g_5 that output Gödel numbers for Σ_1 formulas equivalent to the ones in (4.6) and (4.7), respectively. Given two numbers $a, b \in U$, we can use Lemma 4.21 to find (i, a_0) and (j, b_0) such that $a = h_X(\pi(i), \pi(a_0))$, $b = h_X(\pi(j), \pi(b_0))$. Then

$$a E_Y b \;\Leftrightarrow\; h_X(\pi(g_4(i, j)), h_X(a_0, b_0)) \in M.$$

Likewise,

$$a \in N \;\Leftrightarrow\; h_X(\pi(g_5(i)), \pi(a_0)) \in M.$$

To see that the functions h_Y and n_Y are uniformly recursive in $X \oplus M \oplus h_x \oplus n_X$ note that we can

(i) given $i \in \omega$, effectively compute the Gödel number of a Σ_1 formula that is satisfied by u if and only if u *is* the natural number i,

(ii) given n-codes (i, x), (j, y) for elements u, w in $J_{\rho_\alpha^n}$, compute a Gödel number for a Σ_1 formula whose only solution is (u, w).

□

By iterating the procedure described above, we obtain the following.

Corollary 4.25. *If $\langle X, M\rangle$ is an effective copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, then it computes ω-copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle, \ldots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0\rangle = \langle J_\alpha, \varnothing\rangle = J_\alpha.$$

If a copy is not effective, we can use Proposition 4.18 to decode the predecessor J-structures.

Corollary 4.26. *If $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$, then $(X \oplus M)^{(2)}$ computes ω-copies of*

$$\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle, \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle, \ldots, \text{ and } \langle J_{\rho_\alpha^0}, A_\alpha^0\rangle = \langle J_\alpha, \varnothing\rangle = J_\alpha.$$

Once we have an ω-copy of J_α, we can use it to compute ω-copies of all J-structures "below" it. We introduce the following notation.

Definition 4.27. Given a structure $\langle F_X, E_X\rangle$ induced by $X \subseteq \omega$ and $z \in \omega$, we define the *segment* $\langle F_X, E_X\rangle\lceil_z$ given by z, as

$$F_X\lceil_z = \{x \in F_X : x E_X z\} \text{ and } E_X\lceil_z = E_X\lceil_{F_X\lceil_z}.$$

In particular, $\langle F_X, E_X\rangle\lceil_z = \varnothing$ if $z \notin F_X$.

If there is no danger of confusing it with the usual initial segment notation for reals, we will abbreviate $\langle F_X, E_X\rangle\lceil_z$ by $X\lceil_z$.

Proposition 4.28. *If X is an ω-copy of J_α, then X computes an ω-copy of $\langle J_{\rho_\beta^n}, A_\beta^n\rangle$, for all $n \in \omega$, $\beta < \alpha$.*

Proof. Both $J_{\rho_\beta^n}$ and A_β^n are elements of J_α. Let π be the isomorphism between J_α and X, and let $x_\beta, a_\beta^n \in F_X$ be such that

$$x_\beta = \pi(J_{\rho_\beta^n}), \quad a_\beta^n = \pi(A_\beta^n).$$

Then $\langle X\lceil_{x_\beta}, \mathrm{Set}_X(a_\beta^n)\rangle$ is an ω-copy of $\langle J_{\rho_\beta^n}, A_\beta^n\rangle$, clearly recursive in X. □

A similar argument yields an analogous fact for the S-operator.

Proposition 4.29. *If X is an ω-copy of J_α, then X computes an ω-copy of $S^{(n)}(J_\beta)$, for all $n \in \omega$, $\beta < \alpha$.*

4.4. **Defining ω-copies.** In the previous section we saw how to effectively extract information from ω-copies of J-structures. Next, we describe how ω-copies of new J-structures can be defined from ω-copies of given J-structures.

The J-hierarchy has two types of operations that we need to capture: Defining new sets using the S-operator, and taking projecta and defining standard codes. We will analyze both operations from an arithmetic perspective.

An arithmetic analogue of the S-operator. We introduce an arithmetic version of the S-operator by naming the witnesses $b_0, \dots, b_8$ for the basic functions $F_0, \dots, F_8$ in a systematic fashion.

Let $X \subseteq \omega$. We use $b = (x, y)$ as a shortcut for the $\Sigma^0_2(X)$-formula

$$\begin{aligned} (4.8) \quad \varphi_P(x, y, b) \equiv \exists c, d \, \big[& \forall z \, (z E_X c \Leftrightarrow z = x) \\ & \wedge \forall z \, (z E_X d \Leftrightarrow z = x \vee z = y) \wedge \forall z \, (z E_X b \Leftrightarrow z = c \vee z = d) \big] \end{aligned}$$

postulating that b is a code (in X) of the ordered pair of the sets coded by x and y.

Lemma 4.30. *There exists a Π^0_5-definable function $\overline{S}(X) = Y$ such that, if X is an ω-copy of a countable set U, $\overline{S}(X)$ is an ω-copy of $S(U)$.*

Proof. We build the arithmetic structure $Y = \overline{S}(X)$ in the following stages.

(1) Move X into a reserved column of ω:

$$\langle x, y \rangle \in X \mapsto \langle 2^x, 2^y \rangle \in Y.$$

(2) Add 3 as an element representing $\{F_X\}$:

$$\langle 2^x, 3 \rangle \in Y \text{ for all } x \in F_X.$$

Let us call this structure X^+. This will remain fixed through the rest of the construction while we add new relations to Y.

(3) For all $x, y \in F_{X^+}$, add (if necessary) witnesses for all $b_0(x, y), \dots, b_8(x, y)$ required (see also Definition 4.37). The following is a straightforward construction of $\overline{S}$ preserving extensionality, though it is arguably not the most elegant way to do it.

Given $x, y \in F_{X^+}$, for $i = 0, \dots, 8$:

$i = 0$**:** Check whether there exists $b_0 = b_0(x, y) \in F_Y$ such that

$$\forall z \, (z E_Y b_0 \;\Leftrightarrow\; (z = x \vee z = y)).$$

If so, do nothing. If not, add $\langle x, 5^{\langle x,y \rangle + 1} \rangle, \langle y, 5^{\langle x,y \rangle + 1} \rangle$ to Y.

$i = 1$: Check whether there exists $b_1 = b_1(x, y) \in F_Y$ such that

$$\forall z\,(zE_Y b_1 \iff (zE_Y x \wedge \neg zE_Y y)).$$

If so, do nothing. If not, add $\langle a, 7^{\langle x,y\rangle+1}\rangle$ to Y for all a such that $aE_{X^+}x$ and $\neg aE_{X^+}y$.

$i = 2$: Check whether there exists $b_2 = b_2(x, y) \in F_Y$ such that

$$\forall z \in F_Y\,(zE_Y b_2 \iff \exists c, d \in\ (z = (c, d) \wedge cE_{X^+}x \wedge dE_{X^+}y)).$$

If so, do nothing. If not, first check which ordered pairs already exist in Y: For each $aE_{X^+}x, bE_{X^+}y$, check whether

$$\exists c_0 \in F_Y\, \forall z(zE_Y c_0 \iff z = a),$$
$$\exists c_1 \in F_Y\, \forall z(zE_Y c_0 \iff z = a \vee z = b).$$

If c_0 or c_1 (or both) do not exist, add

$$\langle a, 11^{a+1}\rangle, \langle a, 13^{\langle a,b\rangle+1}\rangle, \langle b, 13^{\langle a,b\rangle+1}\rangle,$$

respectively, to Y. Then check whether

$$\exists d \in F_Y\, \forall z(zE_Y d \iff z = c_0 \vee z = c_1),$$

where c_0, c_1 are the witnesses for $\{a\}$ and $\{a, b\}$, respectively (either already existing or added in the previous step). If it does not exist, add

$$\langle c_0, 17^{\langle c_0,c_1\rangle+1}\rangle, \langle c_1, 17^{\langle c_0,c_1\rangle+1}\rangle$$

to Y. Either way, let $p(a, b)$ be the representative in F_Y for the ordered pair (a, b).

Finally, for all $aE_{X^+}x, bE_{X^+}y$, add

$$\langle p(a, b), 19^{\langle x,y\rangle+1}\rangle$$

to Y.

$i = 3$: Check whether there exists $b_3 = b_3(x, y) \in F_Y$ such that

$$\forall z \in F_Y(zE_X b_3 \iff \exists c, d, r\, \exists q, s \in F_{X^+}(z = (q, c)\wedge c = (r, s)\wedge d = (q, s)\wedge rE_{X^+}x\wedge dE_{X^+}y)).$$

(Here the pair $d = (q, s)$ has to exist completely in X^+, i.e. the witnesses in (4.8) have to exist in F_{X^+}, too.)

If so, do nothing. If not, first check which ordered pairs already exist in Y: For each $aE_{X^+}x, (q, s)E_{X^+}y$, check whether

$$\exists c_0 \in F_Y\, \forall z(zE_Y c_0 \iff z = a),$$
$$\exists c_1 \in F_Y\, \forall z(zE_Y c_0 \iff z = a \vee z = s).$$

If c_0 or c_1 (or both) do not exist, add

$$\langle a, 23^{a+1}\rangle, \langle a, 29^{\langle a,s\rangle+1}\rangle, \langle b, 29^{\langle a,s\rangle+1}\rangle,$$

respectively, to Y.
Next check whether

$$\exists d \in F_Y \, \forall z (z E_Y d \iff z = c_0 \vee z = c_1),$$

where c_0, c_1 are the witnesses for $\{a\}$ and $\{a, s\}$, respectively (either already existing or added in the previous step). If it does not exist, add

$$\langle c_0, 31^{\langle c_0,c_1\rangle+1}\rangle, \langle c_1, 31^{\langle c_0,c_1\rangle+1}\rangle$$

to Y. As before, let $p(a, s)$ be the representative in F_Y for the ordered pair (a, b). In a similar way, we check for, and possibly add, representatives for $\{q\}$ (using 37^{q+1} as a code), and for $(q, p(a, s))$ (using $41^{\langle q+p(a,s)\rangle+1}$ as a code).
Assume now we have a representative $t(q, a, s)$ for every triple (q, a, s) such that $aE_{X^+}x, (q, s)E_{X^+}y$. For each such a, q, s, add

$$\langle t(q, a, s), 43^{\langle x,y\rangle+1}\rangle$$

to Y.

$i = 4$: Completely analogous to $i = 3$.

$i = 5$: Check whether there exists $b_5 = b_5(x, y) \in F_Y$ such that

$$\forall z \, (z E_Y b_5 \iff \exists c \, (c E_{X^+} x \wedge z E_{X^+} c))$$

If not, add

$$\langle z, 47^{\langle x,y\rangle+1}\rangle$$

for all z for which the right hand side in the preceding formula is satisfied.

$i = 6$: Check whether there exists $b_6 = b_6(x, y) \in F_Y$ such that

$$\forall z \, (z E_Y b_6 \iff \exists c, d \, (d = (z, c) \wedge d E_{X^+} x)).$$

(Again, the pair $d = (z, c)$ must exist fully in X^+.) If not, add

$$\langle z, 53^{\langle x,y\rangle+1}\rangle$$

for all z for which the right hand side in the preceding formula is satisfied.

$i = 7$**:** Check whether there exists $b_7 = b_7(x,y) \in F_Y$ such that

$$\forall z\,(zE_Y b_7 \;\Leftrightarrow\; \exists c,d\,(z=(c,d) \,\wedge\, cE_{X^+}d \,\wedge\, cE_{X^+}x \,\wedge\, dE_{X^+}x)).$$

(Here, we consider *all* pairs (c,d) satisfying $cE_{X^+}d \wedge cE_{X^+}x \wedge dE_{X^+}x$.) If not, we first add all necessary pairs (c,d), similar to $i = 2,3,4$, using 59^{c+1}, $61^{\langle c,d\rangle+1}$, and $67^{\langle 59^{c+1},61^{\langle c,d\rangle+1}\rangle+1}$ as possible witnesses.

Assuming that all pairs (c,d) with $cE_{X^+}d \wedge cE_{X^+}x \wedge dE_{X^+}x$ have a representative $p(c,d)$, add

$$\langle p(c,d), 71^{\langle x,y\rangle+1}\rangle$$

to Y for all such c,d.

$i = 8$**:** Check whether there exists $b_8 = b_8(x,y) \in F_Y$ such that

$$\forall z\,(zE_Y b_8 \;\Leftrightarrow\; \exists a,b,c\,(a=(b,c) \wedge aE_{X^+}x \wedge bE_{X^+}y \wedge \forall d\,(dE_Y z \Leftrightarrow d=c))).$$

(Here we consider only pairs $a=(b,c)$ that are fully present in X^+.) If not, for all a,b,c such that $a=(b,c) \wedge aE_{X^+}x \wedge bE_{X^+}y$, we first check whether there exists a representative for $\{c\}$ exists in F_Y, possibly adding $\langle c, 73^{c+1}\rangle$ to Y. Assuming $s(c)$ is a representative for $\{c\}$, add

$$\langle s(c), 79^{\langle x,y\rangle+1}\rangle.$$

All steps in the above construction can be carried out recursively in a sufficiently powerful oracle. Inspecting the logical complexity of the questions asked during the construction, we see that an $X^{(5)}$-oracle suffices.

Then to decide whether a number a is in Y, we test whether it is of the right form (i.e. whether it is the power of a prime used during the construction), and if so, which operation $F_i(x,y)$ it is potentially witnessing. We can run the construction of Y up to that point and then see whether a new witness needs to be added for $F_i(x,y)$ (or one of its auxiliary sets), telling us whether a is in Y or not. □

We can subject the $\overline{S}$-operator to an analysis similar to that of the jump operator by Enderton and Putnam [9].

Lemma 4.31. *If $A \in 2^\omega$ is a Π^0_5-singleton, so is $\overline{S}(A)$.*

Proof. If A is the unique solution to $P(X)$, then $\overline{S}(A)$ is the unique solution to

$$P(X_{[2]}) \text{ and } X = \overline{X_{[2]}},$$

where $X_{[2]} = \{a \colon 2^a \in X\}$. $\square$

Using a universal Π^0_5-predicate, the uniformity of the above proof implies that there exists a Π^0_5-predicate $Q(n,X,Y)$ such that

$$Q(n,X,Y) \;\Leftrightarrow\; Y = \overline{S}^{(n)}(X).$$

Lemma 4.32. *If Z is such that $Z \geq_{\mathrm{T}} \overline{S}^{(n)}(X)$ for all n, then*

$$Z^{(5)} \geq_{\mathrm{T}} \bigoplus_n \overline{S}^{(n)}(X).$$

Proof. Define the predicate $\overline{Q}(n,e)$ as

$$\overline{Q}(n,e) \;:\Leftrightarrow\; \Phi^Z_e \text{ is total and } Q(n,X,\Phi^Z_e).$$

$\overline{Q}(n,e)$ is Π^0_5 and hence recursive in $Z^{(5)}$. To decide whether $a \in \overline{S}^{(n)}(X)$, find, recursively in $Z^{(5)}$, the least e such that $\overline{Q}(n,e)$ and compute $\Phi^Z_e(a)$. $\square$

Corollary 4.33. *If X is an ω-copy of J_α and $Z \geq_{\mathrm{T}} \overline{S}^{(n)}(X)$ for all n, then $Z^{(5)}$ computes an ω-copy of $J_{\alpha+1}$.*

Proof. We can use $\bigoplus_n \overline{S}^{(n)}(X)$ to define a copy of $J_{\alpha+1}$ by 'stacking' the elements of $\overline{S}^{(n+1)}(X)$ coded with base 3 and higher at the next 'available' prime column. Essentially this means that instead of moving $\overline{S}^{(n)}(X)$ into the column given by powers of 2, we leave it unchanged and add new elements for $\overline{S}^{(n+1)}(X)$ starting at the smallest available prime column. $\square$

4.4.1. *An arithmetic version of the standard code.* To define an arithmetic copy the Σ_n-standard code for J_α, we can simply interpret the set theoretic definitions as formulas of arithmetic. More precisely, suppose R is a definable predicate over a J-structure $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$, and $\langle X, M\rangle$ is an ω-copy via π. Since the structure $\langle F_X, E_X, M\rangle$ is isomorphic to $\langle J_{\rho^n_\alpha}, \in, A^n_\alpha\rangle$, we can use the same formula that defines R over $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ and obtain a definition of $\pi[R]$ arithmetic in $\langle X, M\rangle$. The problem, however is that a bounded quantifier in set theory will not necessarily correspond to a bounded quantifier in arithmetic. This means the transfer of complexities between the Lévy-hierarchy and the arithmetical hierarchy may not result in uniform bounds.

However, we will use only a fixed, finite number of set-theoretic definitions. Most importantly, we use the uniform definability of the satisfaction relation $\models$ over transitive, rud closed structures.

Proposition 4.34 (Jensen [20])**.** *For $n \geq 1$, the satisfaction relation $\models^{\Sigma_n}_{\langle M,A\rangle}$ is uniformly $\Sigma_n(\langle M,A\rangle)$ for transitive, rud closed structures $\langle M,A\rangle$).*

Definition 4.35. For each n, let $d_{\models}^{(n)}$ be the complexity of the formula defining $\models^{\Sigma_n}$ when counting *all* quantifier changes, including bounded ones.

We can now arithmetically define an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ over an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$.

Lemma 4.36. *Suppose $\langle X, M\rangle$ is an ω-copy of $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ (witnessed by π). Then there exists an ω-copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ $\Sigma^0_{d_{\models}^{(1)}}$-definable in $\langle X, M\rangle$.*

Proof. Since $\rho_\alpha^{n+1} \leq \rho_\alpha^n$, Proposition 4.28 implies that $\langle X, M\rangle$ computes an ω-copy of $J_{\rho_\alpha^{n+1}}$.

A_α^{n+1} is Σ_1 over $\langle X, M\rangle$. Hence its ω-version is $\Sigma^0_{d_{\models}^{(1)}}$ in $\langle X, M\rangle$. □

We will denote the copy of $\langle J_{\rho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ thus obtained by $\langle X^+, M^+\rangle$.

4.5. **Recognizing J-structures.** Our goal is to show that the sequence of canonical copies of J-structures with projectum $= 1$ in L_{β_n} cannot be $G(n)$-random with respect to a continuous measure. We will assume for a contradiction that such a copy, say $\langle X, M\rangle$, *is* random for (a representation of) a continuous measure μ. The basic idea to derive a contradiction is to look at the initial segment of ω-copies computable in (some fixed jump of) μ. Since $\langle X, M\rangle$ is μ-random, it cannot be among those. But we can "reach" $\langle X, M\rangle$ from the ω-copies of J_α's computable in μ by iterating arithmetic operations and taking uniform limits, which is incompatible with randomness by the Stair Trainer Lemma.

The problem is that we cannot arithmetically define the set of ω-copies of structures J_α. We can, however, define a set of "pseudocopies", subsets of ω that behave in most respects like actual ω-copies, but that may code structures that are not well-founded.

By comparing the structures coded by these pseudocopies, we can also linearly order the latter (up to isomorphism), depending on whether a coded structure embeds into another. With the help of the supposedly μ-random $\langle X, M\rangle$, we can then single out those pseudocopies that are actual ω-copies of some J_α.

To define pseudocopies, we specify a number of properties which will ensure that, if the copy is well-founded, it codes (i.e. is isomorphic to) some countable J_α, $\alpha > 1$.

(1) The relation E_X is non-empty and extensional.

(2) X is rud-closed.
(3) The structure coded by X satisfies $\varphi_{V=J}$.
(4) X contains (a copy of) ω as a element.

To formalize these properties in arithmetic, we can take, as before, any formula in the language of set theory and interpret it over the structure $\langle F_X, E_X\rangle$. This way we can define relations over F_X intended to represent the corresponding set-theoretic relation. Since we need an explicit bound on the complexity of the formulas involved, we write out rather formal definitions without bounded quantifiers.

Items (1) and (2) will guarantee that the representation is faithful if E_X is well-founded. Extensionality can be formalized by a $\Pi^0_2(X)$ formula:

$$\forall x, y(\forall z(zE_X x \leftrightarrow zE_X y) \ \rightarrow \ x = y).$$

To define rud-closedness we just have to require that all functions $F_0, \dots, F_8$ have witnesses.

Definition 4.37. A structure $\langle X, M\rangle$ is *rud closed* if

(4.9)

$$\begin{aligned}
&\forall x, y \in F_X \ \exists b_0, \dots, b_9 \in F_X \\
&\qquad \forall z\,(zE_X b_0 \ \Leftrightarrow \ (z = x \ \vee \ z = y)) \ \wedge \\
&\qquad \forall z\,(zE_X b_1 \ \Leftrightarrow \ (zE_X x \ \wedge \ \neg zE_X y)) \ \wedge \\
&\qquad \forall z\,(zE_X b_2 \ \Leftrightarrow \ \exists c, d\,(z = (c,d) \ \wedge \ cE_X x \ \wedge \ dE_X y)) \ \wedge \\
&\qquad \forall z\,(zE_X b_3 \ \Leftrightarrow \ \exists c, d, q, r, s\,(z = (q,c) \ \wedge \ c = (r,s) \ \wedge \ d = (q,s) \ \wedge \ rE_X x \ \wedge \ dE_X y)) \ \wedge \\
&\qquad \forall z\,(zE_X b_4 \ \Leftrightarrow \ \exists c, d, q, r, s\,(z = (q,c) \ \wedge \ c = (r,s) \ \wedge \ d = (q,r) \ \wedge \ sE_X x \ \wedge \ dE_X y)) \ \wedge \\
&\qquad \forall z\,(zE_X b_5 \ \Leftrightarrow \ \exists c\,(cE_X x \ \wedge \ zE_X c)) \ \wedge \\
&\qquad \forall z\,(zE_X b_6 \ \Leftrightarrow \ \exists c, d\,(d = (z,c) \ \wedge \ dE_X x)) \ \wedge \\
&\qquad \forall z\,(zE_X b_7 \ \Leftrightarrow \ \exists c, d\,(z = (c,d) \ \wedge \ cE_X d \ \wedge \ cE_X x \ \wedge \ dE_X x)) \ \wedge \\
&\qquad \forall z\,(zE_X b_8 \ \Leftrightarrow \ \exists a, b, c\,(a = (b,c) \ \wedge \ aE_X x \ \wedge \ bE_X y \ \wedge \forall d\,(dE_X z \ \Leftrightarrow \ d = c))) \ \wedge \\
&\qquad \forall z\,(zE_X b_9 \ \Leftrightarrow \ zE_X x \ \wedge \ z \in M)
\end{aligned}$$

A quantifier count yields that being rud closed is a $\Pi^0_5(X \oplus M)$ property, while for any individual triple (x, y, b) and given $0 \leq i \leq 8$, the property of b being a witness for $b_i(x, y)$ in the formula above is $\Pi^0_3(X)$.

Note that if $M = \varnothing$, the existence of b_9 is trivial.

By Mostowski's Collapsing Theorem, if X satisfies (1) and E_X is well-founded, then $\langle F_X, E_X\rangle$ is isomorphic to a transitive set structure $\langle S, \in\rangle$, and by (2) S will be rud-closed.

To formalize (3), let $\Pi^0_{c_J}$ be the arithmetic complexity of the formula $\varphi_{\mathsf{V}=\mathsf{J}}$. Hence we can formalize $M \models \varphi_{\mathsf{V}=\mathsf{J}}$ by a $\Sigma^0_{d_{\models}^{(c_J+1)}}$-predicate.

Finally, for (4), we can define ω using the usual Σ_0 set theoretic formula (the least infinite ordinal). In transitive, rud-closed structures, $\varphi_\omega(x)$ holds if and only if $x = \omega$. Let c_ω be one larger than the complexity of this formula, counting *all* quantifier changes, including bounded ones. Interpreting φ_ω over $\langle F_X, E_X\rangle$, we obtain a $\Sigma^0_{c_\omega}(X)$ property. We require a pseudocopy $\langle F_X, E_X\rangle$ to satisfy

$$\exists x\ \varphi_\omega(x).$$

This x will be unique and define ω with respect to $\langle F_X, E_X\rangle$. Let us denote this unique number by ω_X.

Given ω_X, we can also recover the mapping $i \mapsto n_X(i)$ as in Proposition 4.16. As the definition of ω_X is uniform, we obtain that $i \mapsto n_X(i)$ is uniformly $\Sigma^0_{c_\omega+2}(X)$.

Let c_{PC} be one larger than the largest complexity occurring in the formulas defining properties (1)-(4). Conjoining the preceding formulas, we obtain the following.

Proposition 4.38. *For given $n \geq 0$, there exists a $\Sigma^0_{c_{\mathrm{PC}}}$-formula $\varphi_{\mathrm{PC}}(X)$ of second order arithmetic such that if $\varphi_{\mathrm{PC}}(X)$ holds for a real X, then X defines an extensional relational structure $\langle F_X, E_X\rangle$. Moreover, if $\langle F_X, E_X\rangle$ is well-founded, then it is an ω-copy of a countable J_β, $\beta > 1$.*

4.6. **Comparing pseudocopies.** If two pseudocopies X and Y define well-founded structures, they are ω-copies of sets J_α and J_β, respectively. Since $\alpha < \beta$ implies $J_\alpha \in J_\beta$, it follows that one structure must embed into the other as an initial segment.

We want to find an arithmetic formula that compares two pseudocopies in this respect. The problem is that the isomorphism relation between countable structures is generally not arithmetic. In our case, however, we can make use of the special set-theoretic structure present in the pseudocopies, by comparing the subsets of the cardinals present.

The complexity of the arithmetic operations involved in these comparisons will depend on the largest cardinal present in a pseudocopy.

Let us introduce the following notation. Recall that β_N denotes the least ordinal such that $L_{\beta_N} \models \mathsf{ZF}^-_N$. For any ordinal α, let

$$P_\alpha = \max\{n \colon \mathcal{P}^{(n)}(\omega) \text{ exists in } J_\alpha\}, \tag{4.10}$$

if this maximum exists. It can be shown that for all $\alpha < \beta_N$, $P_\alpha \leq N$ [3], and hence P_α is defined and uniformly bounded for all $\alpha < \beta_N$. This will be useful later when we need an a priori bound on the complexity of the comparison operations to be outlined in the following.

Using the predicate φ_ω, we can formalize the (non-) existence of power sets of ω for pseudocopies. The relation $y = \mathcal{P}(x)$ is Π_1. Let θ be a Σ_0 formula such that

$$y = \mathcal{P}(x) \quad \Leftrightarrow \quad \forall u\, \theta(u, x, y).$$

Then

$$y = \mathcal{P}(\omega) \quad \Leftrightarrow \quad \forall u \forall x\, (\varphi_\omega(x) \rightarrow \theta(u, x, y)),$$

which is Π_1. Denote the preceding formula by $\varphi^{(1)}_{\mathcal{P}}(y)$. It follows that

$$y = \mathcal{P}(\mathcal{P}(\omega)) \quad \Leftrightarrow \quad \forall u \forall x\, (\varphi^{(1)}_{\mathcal{P}}(x) \rightarrow \theta(u, x, y)),$$

which is Π_2. Continuing inductively, we obtain formulas $\varphi^{(n)}_{\mathcal{P}}(y)$ $(n \geq 1)$ defining the predicate

$$y = \mathcal{P}^{(n)}(\omega)$$

by means of a Π_n formula.

As before, let c_P be the full complexity of the formula for $y = \mathcal{P}(\omega)$, counting all quantifier changes.

Definition 4.39. A pseudocopy X is an *n-pseudocopy* if it satisfies the $\Pi^0_{c_P+n+2}(X)$ predicate

$$\exists y (y = \mathcal{P}^{(n)}(\omega)) \wedge \forall z (z \neq \mathcal{P}^{(n+1)}(\omega)).$$

Using the power sets of ω in each pseudocopy, we can check whether two pseudocopies have the same reals, sets of reals, etc.

First, we can check whether every real in X has an analogue in Y:

$$\forall u\, (u \subseteq \omega_X \rightarrow \exists v (v \subseteq \omega_Y \wedge \forall i (n_X(i) E_X u \leftrightarrow n_Y(i) E_Y v))).$$

By extensionality, such a v, if it exists, is unique. We can therefore define the mapping $f^{X,Y}_0(u) = v$ which maps the representation of a real in $\langle F_X, E_X\rangle$ to its representation in $\langle F_Y, E_Y\rangle$. Expanding all defined symbols, the above formula is $\Pi^0_{c_P+5}(X,Y)$. We similarly check whether every real in Y has an analogue in X:

$$\forall u\,(u \subseteq \omega_Y \rightarrow \exists v(v \subseteq \omega_X \wedge \forall i(n_Y(i) E_Y u \leftrightarrow n_X(i) E_X v))).$$

This gives rise to the function $f_0^{Y,X}$. Let $\varphi^{(0)}_{\text{comp}}(X,Y)$ be the conjunction of the two formulas above. For two pseudocopies X,Y, $\langle F_X, E_X\rangle$ and $\langle F_Y, E_Y\rangle$ have the same reals if and only if $\varphi^{(0)}_{\text{comp}}(X,Y)$ holds.

If $\varphi^{(0)}_{\text{comp}}(X,Y)$ holds, we can use the functions $f_0^{X,Y}$ to check whether X and Y induce the same sets of reals:

$$\begin{aligned}\varphi^{(1)}_{\text{comp}}(X,Y) \equiv\ & \varphi^{(0)}_{\text{comp}}(X,Y) \wedge \\ & \forall u\,(u \subseteq \mathcal{P}(\omega_X) \rightarrow \exists v(v \subseteq \mathcal{P}(\omega_Y) \wedge \forall z(z E_X u \leftrightarrow f_0^{X,Y}(z) E_Y v))) \\ & \wedge\ \forall u\,(u \subseteq \mathcal{P}(\omega_Y) \rightarrow \exists v(v \subseteq \mathcal{P}(\omega_X) \wedge \forall z(z E_Y u \leftrightarrow f_0^{Y,X}(z) E_X v)))\end{aligned}$$

Expanding defined terms yields a $\Pi^0_{c_P+10}(X,Y)$ property (presumably a rather crude upper bound). As before, the formula gives rise to a function $f_1^{X,Y}(u) = v$, mapping X's representation of a set of reals to Y's representation.

We can continue this comparison through the iterates of the power set of ω, using the formulas $\varphi^{(n)}_{\mathcal{P}}$ and the inductively defined functions $f_n^{X,Y}$. This will yield $\Pi^0_{c_P+2n+8}$ formulas $\varphi^{(n)}_{\text{comp}}(X,Y)$ with the following property:

> If X and Y are well-founded pseudocopies in which $\mathcal{P}^{(n)}(\omega)$ exists, then $\varphi^{(n)}_{\text{comp}}(X,Y)$ holds if and only if X and Y have (representations of) the same subsets of $\mathcal{P}^{(i)}(\omega)$, for all $0 \leq i \leq n$.

Given two n-pseudocopies, the above formulas allow for an arithmetic definition of isomorphic pseudocopies.

Proposition 4.40. *For given n and for any two n-pseudocopies X,Y that code well-founded structures $\langle F_X, E_X\rangle$ and $\langle F_Y, E_Y\rangle$, respectively, if $\varphi^{(n)}_{\text{comp}}(X,Y)$, then X and Y code the same J_α.*

Proof. Assume for a contradiction X and Y are not isomorphic. Since they are well-founded pseudocopies, there must exist countable α, β such that $\langle F_X, E_X\rangle \cong (J_\alpha, \in)$ and $\langle F_Y, E_Y\rangle \cong (J_\beta, \in)$. Without loss of generality, $\alpha < \beta$. Since $\langle F_X, E_X\rangle$ and $\langle F_Y, E_Y\rangle$ code the same subsets of $\mathcal{P}^{(n)}(\omega)$ no new subset of $\mathcal{P}^{(n)}(\omega)$ is constructed between α and β. But this implies $\mathcal{P}^{(n+1)}(\omega)$ exists at $\alpha+1$, which is an immediate contradiction if $\beta = \alpha + 1$.

If $\beta > \alpha + 1$, since $\mathcal{P}^{(n+1)}(\omega)$ does not exist in J_β, a new subset of $\mathcal{P}^{(n)}(\omega)$ must be constructed between $\alpha + 1$ and β, contradiction. $\square$

Corollary 4.41. *For given n, there exists a $\Pi^0_{c_P+2n+8}(X,Y)$-definable function $G : \omega \to \omega$ such that if X and Y are well-founded n-pseudocopies and $\varphi^{(n)}_{\text{comp}}(X,Y)$, then G maps F_X bijectively onto F_Y, and for all $x, y \in F_X$,*

$$xE_X y \iff G(x) E_Y G(y).$$

Let us write $X \cong Y$ if, for some n, X and Y are both n-pseudocopies and $\varphi^{(n)}_{\text{comp}}(X,Y)$. By the previous corollary, the isomorphism between the structures can be computed recursively in $(X \oplus Y)^{(c_P+2n+8)}$.

If $\mathcal{P}^{(n)}(\omega)$ exists for all n in two pseudocopies, the relation $\cong$ is not arithmetically definable, since the existence of an isomorphism between n-pseudocopies is expressed by a different formula for each n.

However, since we will be working inside models L_{β_N}, we can assume there is an *a priori* bound on the largest cardinal in each pseudocopy. For fixed $N \in \omega$, let

$$\mathcal{PC}_N = \{X : X \text{ is an } n\text{-pseudocopy for some } n \leq N\}.$$

This is a $\Sigma^0_{c^N_{\text{PC}}}$ set of reals, where

$$c^N_{\text{PC}} = \max\{c_{\text{PC}}, c_P + N + 2\} + 1.$$

Restricted to $\mathcal{PC}_N$, the $\cong$-relation is definable as a $\Sigma^0_{c_{\cong}}$-predicate, where

$$c^N_{\cong} = \max\{c^N_{\text{PC}}, c_P + 2N + 8\} + 1,$$

namely

$$\begin{aligned} X \cong_N Y :\Leftrightarrow\ & X, Y \text{ are 0-pseudocopies and } \varphi^{(0)}_{\text{comp}}(X,Y) \lor \\ & X, Y \text{ are 1-pseudocopies and } \varphi^{(1)}_{\text{comp}}(X,Y) \lor \\ & \quad \vdots \\ & X, Y \text{ are } N\text{-pseudocopies and } \varphi^{(N)}_{\text{comp}}(X,Y) \end{aligned}$$

Working inside $\mathcal{PC}_N$, we can also use φ_{comp} to arithmetically define a pre-order $\prec$ on pseudocopies. The idea is that $X \prec Y$ if one embeds its structure into the other. For this purpose we have to identify the "internal" J-hierarchy of a pseudocopy.

In any J_β, the sequence of J_α $(\alpha < \beta)$ is uniformly Σ_1-definable. Let the accordant arithmetic complexity (counting all quantifiers) be bounded by $\Sigma^0_{c_J}$.

For $z \in F_X$, let us write $J(X,z)$ if $X\lceil_z$ codes a J_α. In this case, we call $X\lceil_z$ a *J-segment*. Furthermore, we denote by

$$J_X = \{z\colon J(X,z)\}$$

the J-structure inside X (the *internal J-structure of* X) and, given $z \in J_X$, write J_z for $X\lceil_z$. Finally, we write J_α^X to denote X's internal copy of J_α (if it exists).

If X is well-founded, J_X must be linearly ordered by E_X. So if it is not (a $\Sigma^0_1(X)$ property), we can exclude X as ill-founded right away. From now on suppose J_X is always linearly ordered.

Definition 4.42. We define

$$X \prec_N Y \quad :\Leftrightarrow X, Y \in \mathcal{PC}_N \ \wedge \ \exists z\, (J(Y,z) \in \mathcal{PC}_N \ \wedge \ X \cong_N J_z)).$$

This is a $\Sigma^0_{c^N_\prec}$-property, with $c^N_\prec = \max\{c^N_\cong, c_J\} + 2$. We let $X \preceq_N Y$ if $X \prec_N Y$ or $X \cong_N Y$. $\preceq_N$ is reflexive and transitive. Hence $\preceq_N$ defines a partial order on $\mathcal{PC}_N$.

If both X and Y are well-founded pseudocopies in $\mathcal{PC}_N$, we have either $X \prec_N Y$ or $X \cong_N Y$ or $Y \prec_N X$, that is, "true" pseudocopies (i.e. those that code a J_α) are linearly ordered by $\preceq_N$ (up to isomorphism). Hence comparability can only fail if (at least) one of the pseudocopies is not well-founded.

We now show that if comparability fails, we can arithmetically expose an ill-foundedness in one of the pseudocopies. We assume in the following that X, Y are pseudocopies in $\mathcal{PC}_N$. Since we will be working inside L_{β_N}, we also assume that any J-segment in either of the pseudocopies is an element of $\mathcal{PC}_N$.

So suppose X, Y are $\preceq_N$-incomparable. We can use the $\cong_N$-relation to see if two J-segments of X and Y align: Consider the predicate

$$J(X,x) \ \wedge \ J(Y,y) \ \wedge \ J_x \cong_N J_y.$$

It yields a partial function from F_X to F_Y, $\Sigma^0_{c^N_\prec}(X \oplus Y)$-definable. Since both structures contain a copy of ω, the function is not empty. Denote the domain of this function by D_X and the range by R_Y. Both D_X and R_Y are linearly ordered.

We consider the set of ordinals in a pseudocopy:

$$\mathrm{Ord}_X = \{z\colon \ z \text{ is an ordinal in } X\},$$

$$\mathrm{Ord}_Y = \{z\colon \ z \text{ is an ordinal in } Y\}.$$

Here *ordinal in* X means of course that $z \in F_X$ and satisfies the $\Pi^0_1(X)$ property (with no bounded quantifiers)

$$\{w \colon w E_X z\} \text{ is transitive and linearly ordered by } E_X.$$

(Similarly for Ord_Y.) $\mathrm{Ord}_X, \mathrm{Ord}_Y$ are closed downward under E_X, (E_Y, respectively), and linearly ordered by E_X (E_Y). If one structure is ill-founded, it must exhibit an instance of ill-foundedness among its ordinals.

Let

$$\mathrm{Ord}(D_X) = \mathrm{Ord}_X \cap \{J_z \colon z \in D_X\}, \quad \mathrm{Ord}(R_Y) = \mathrm{Ord}_Y \cap \{J_z \colon z \in D_Y\}.$$

Since X and Y are incomparable, there must be at least one instance of ill-foundedness either at $\mathrm{Ord}(D_X)$ or $\mathrm{Ord}(R_Y)$.

If $\mathrm{Ord}(D_X)$ is cofinal in Ord_X, then $X = \bigcup_{z \in \mathrm{Ord}(D_X)} J_z$. Likewise $\mathrm{Ord}(R_Y)$ being cofinal in Ord_Y implies $Y = \bigcup_{z \in \mathrm{Ord}(R_Y)} J_z$. But this would imply $X \cong_N Y$. Hence at least one of $\mathrm{Ord}(D_X), \mathrm{Ord}(R_Y)$ has to be bounded in $\mathrm{Ord}_X, \mathrm{Ord}_Y$, respectively.

Case 1: $\mathrm{Ord}(D_X)$ *is cofinal in* Ord_X*,* $\mathrm{Ord}(R_Y)$ *is bounded in* Ord_Y*.*

Since $X \not\preceq_N J_y$ for any y, Y must omit $\bigcup_{z \in R_Y} J_z$ and hence is ill-founded. The case when $\mathrm{Ord}(D_X)$ is bounded and $\mathrm{Ord}(R_Y)$ is cofinal is analogous.

Case 2: *Both* $\mathrm{Ord}(D_X), \mathrm{Ord}(R_Y)$ *are bounded in* $\mathrm{Ord}_X, \mathrm{Ord}_Y$*, respectively.*

A well-founded structure will have an ordinal present witnessing that the cut defined by $\mathrm{Ord}(D_X)$ (or $\mathrm{Ord}(R_Y)$, respectively) is principal. So we can discard any of the two structures for which this is not the case. That is, if

$$\neg \exists b \in \mathrm{Ord}_X \ (a \in \mathrm{Ord}(D_X) \leftrightarrow a E_X b)$$

(and similarly for $\mathrm{Ord}(R_Y)$).

If the cut *is* principal in both structures, then D_X and R_Y have a E-maximal element, and since the partial mapping cannot be extended past D_X, R_Y, one of X, Y must exhibit it is ill-founded by adding more sets than the other when passing to the next higher J-instance. But this can be detected arithmetically.

We thus obtain the desired linearization of $\preceq_N$. However, its definition involves quantification over *all* pseudocopies in $\mathcal{PC}_N$, and is therefore, if unrestricted, not arithmetic. However, in the proof of the Second Main Theorem, we will only use a set of pseudocopies that are computable in a given real. We therefore define the following restriction of $\mathcal{PC}_N$.

Definition 4.43. Given a real Z, let $\mathcal{PC}_N(Z)$ be the set of pseudocopies computable in Z.

$\mathcal{PC}_N(Z)$ is $\Sigma^0_{c^N_{\mathrm{PC}}}$ in Z.

Lemma 4.44. *For every natural number N and every real Z, there exists a $\Sigma^0_{c^N_{\prec}+6}(Z)$ set of reals $\mathcal{PC}^*_N(Z) \subseteq \mathcal{PC}_N(Z)$ with the following properties:*

(1) For every $X \in \mathcal{PC}^*_N(Z)$, each J-segment of X is in $\mathcal{PC}^*_N(Z)$,
(2) $\preceq_N$ is a total preorder on $\mathcal{PC}^*_N(Z)$, i.e. $\mathcal{PC}^*_N(Z)/\cong_N$ is linearly ordered.

Proof. Property (1) can be ensured via

$$X \in \mathcal{PC}_N(Z) \;\wedge\; \forall z\,(J(X,z) \;\rightarrow\; J_z \in \mathcal{PC}_N)$$

For (2), the linearization is given by the following predicate. For all e such that $\Phi^Z_e \in \mathcal{PC}_N(Z)$, one of the following holds (let $Y = \Phi^Z_e$):

(i) $X \preceq_N Y$ or $Y \prec_N X$,
(ii) $\mathrm{Ord}(D_X)$ is cofinal in Ord_X and $\mathrm{Ord}(R_Y)$ is bounded in Ord_Y,
(iii) $\mathrm{Ord}(D_X)$ is bounded in Ord_X, $\mathrm{Ord}(R_Y)$ is bounded in Ord_Y, and either
  (a) the cut defined by $\mathrm{Ord}(D_X)$ is principal while the cut defined by $\mathrm{Ord}(R_Y)$ is not, or
  (b) both cuts are principal, and if $\bar{x}, \bar{y}$ are the maximal E-elements of D_X and R_Y, respectively, and $J^+_{\bar{x}}$ and $J^+_{\bar{y}}$ are the successors of $J_{\bar{x}}$ and $J_{\bar{y}}$ in the J-hierarchy of X and Y, respectively, then $\cong_N$ fails between $J^+_{\bar{x}}$ and $J^+_{\bar{y}}$ because $J^+_{\bar{y}}$ has sets that cannot be matched with counterparts in $J^+_{\bar{x}}$.

The complexity of this property can be bounded by inspecting the involved notions. D_X and R_Y are at most $\Sigma^0_{c^N_{\prec}+1}(X \oplus Y)$, and a "brute force" estimate using this bound will give an upper bound of $\Sigma^0_{c^N_{\prec}+6}$ for $\mathcal{PC}^*_N$.

The argument above also establishes that $\mathcal{PC}^*_N/\cong_N$ is indeed linearly ordered by $\prec_N$. □

Comparing S-operations. We will also need to recognize arithmetic structures that are the result of an application of the S-operator to an arithmetic structure.

Lemma 4.45. *There exists a Π^0_{12} formula $\varphi_S(X,Y)$ such that $\varphi_S(X,Y)$ holds if and only if X, Y are extensional arithmetic structures, Y extends X*

(i.e. $X = Y\lceil_z$ for some $z \in F_Y$) and, if X is an ω-copy of a countable set U, then Y is an ω-copy of $S(U)$.

Proof. The idea is to match each element of Y not in X^+ with the ones in $\overline{S}(X)$. We can assume that $F_X \subseteq \{2^n : n \geq 1\}$. The formula $\varphi_{\mathcal{S}}$ says the following:

(1) There exists a y_0 representing $\{X\}$.
(2) For every element u in F_Y there exists *exactly* one element v of $F_{\overline{S}(X)}$ such that
   - for the elements z (in the sense of E_Y) of v from $F_{X\cup\{y_0\}}$, zE_Yv if and only if $zE_{\overline{S}(X)}v$,
   - if zE_Yv but z is not from $F_{X\cup\{y_0\}}$, then z has been added as an auxiliary set for witnesses for F_2, F_3, F_4, F_7, or F_8. This means that z must be either a pair or a triple formed with elements of $F_{X\cup\{y_0\}}$ (or sets used to build the tuple), and we can extend the comparison between elements of u and v accordingly to account for this.
(3) Similarly, for every element v in $F_{\overline{S}(X)}$ there exists *exactly* one element u of F_Y such that the elements of v and u align in the manner defined in the previous item.

The quantifier bound can obtained by inspection of the subformulas involved. (Recall that $\overline{S}(X)$ is Π^0_5-definable over X, Lemma 4.30.) □

The formula $\varphi_{\overline{S}}$ can be generalized in a straightforward way to a formula $\phi_{\mathcal{S}}(n, X, Y)$ recognizing whether Y is an ω-copy of $S^{(n)}(U)$, whenever X is an ω-copy of U. The complexity of $\phi_{\mathcal{S}}(n, X, Y)$ is also bounded by Π^0_{12}.

Furthermore, we can define an operator that maps any copy of $S^{(n)}(X)$ to our canonically defined copy $\overline{S}^{(n)}(X)$.

Lemma 4.46. *There exists a Π^0_{12}-definable functional $\Psi_S(n, X, Y)$ such that if X is an ω-copy of a countable set U and Y is an ω-copy of $S^{(n)}(U)$ extending X, then $\Psi_S(n, X, Y) = \overline{S}^{(n)}(X)$.*

Proof. We can use φ_S to identify the levels $S^{(i)}(X)$ over X, for each $i \leq n$. Then, given $v \in F_Y$, we can construct possible "histories" how v was added to F_Y. Such a history is a finite sequence indicating which composition of functions $F_0, \ldots, F_8$ (or auxiliary functions, see Lemma 4.30) v witnesses. The histories can be "lexicographically" ordered in the sense of Lemma 4.30, and the sequence leading up to v will be mapped to witnesses in $\overline{S}^{(n)}$ corresponding to the leftmost such sequence.

The complexity of this mapping is bounded by Π^0_{12}, which is the complexity of reconstructing the levels $S^{(i)}(X)$. Identifying the leftmost history for a given v is Π^0_9 (a crude upper bound). □

Finally, we can combine the previous lemma with the transfer function of Corollary 4.41 to transform a copy of $S^{(n)}(J_\alpha)$ into a copy of $\overline{S}^{(n)}(Y)$, when Y is an ω-copy of J_α.

Corollary 4.47. *Suppose X is an ω-copy of some J_α with $P_\alpha = k$. Suppose further that Z is an ω-copy of $S^{(n)}(J_\alpha)$, for some $n \in \omega$. Then $\overline{S}^{(n)}(X)$ is recursive in $(X \oplus Z)^{(c_P+2k+21)}$.*

Proof. As Z computes an ω-copy of J_α, Corollary 4.41 implies that $(Z \oplus X)^{(c_P+2k+8)}$ computes the isomorphism between the two ω-copies of J_α. Now apply Lemma 4.46. □

4.7. **Effective copies are not random for continuous measures.** We now want to use the framework of ω-copies to show that for any $\alpha < \beta_N$, the canonical copy of a standard J-structure $\langle J_{\rho^k_\alpha}, A^k_\alpha \rangle$ cannot be K-random for a continuous measure, with K sufficiently large. For the rest of this section, consider N fixed. To simplify notation, let $c = c^N_\prec + 21$, which is greater than the complexity of all arithmetic definitions (N-pseudocopies, comparison of pseudocopies and S-operators, linearization) introduced in the previous sections. Let $G(N) = 6^{N+2}c$.

Theorem 4.48. *Suppose $N \geq 0$, $\alpha < \beta_N$, and for some $k > 0$, $\rho^k_\alpha = 1$. Then the canonical copy of the standard J-structure $\langle J_{\rho^k_\alpha}, A^k_\alpha \rangle$ is not $G(N)$-random with respect to any continuous measure.*

The argument rests mostly on various applications of the Stair Trainer Lemma (which establishes that random reals cannot accelerate definability) in the context of ω-copies of J-structures, as introduced in the previous sections.

For the rest of this section, assume for a contradiction that the canonical copy $R = \langle X, M \rangle$ of some standard J-structure $\langle J_{\rho^m_\alpha}, A^m_\alpha \rangle$, where $\rho^m_\alpha = 1$, is $G(N)$-random with respect to a continuous measure μ.

To obtain a contradiction similar to the proofs of Propositions 2.15 and 2.16, we inductively define a hierarchy of (pseudo)copies arithmetic in μ.

Recall the total preorder $(\mathcal{PC}^*_N(Z), \prec_k)$ (Lemma 4.44). Let us first pass to a true linear order on ω, defined by

$$\mathrm{PC}^*_N(Z) = \{e \in \omega \colon \Phi^Z_e \text{ is total and } \Phi^Z_e \in \mathcal{PC}^*_N(Z) \\ \text{and for all } d < e,\ \Phi^Z_d \neq \Phi^Z_e\},$$

By passing to Z-indices, $\prec_N$ induces a linear ordering on $\mathrm{PC}^*_N(Z)$. We will denote it by $\prec_N$, too. The arithmetic complexity of $\mathrm{PC}^*_N(Z)$ is bounded by $\Sigma^0_{c^N_\prec+6}(Z)$.

We first note that every structure in a well-founded initial segment must correspond to a well-founded pseudocopy.

Lemma 4.49. *Let I be a well-founded initial segment of $\mathrm{PC}^*_k(Z)$. Then, for every $e \in I$, Φ^Z_e is a well-founded pseudocopy.*

Proof. Suppose Φ^Z_e, $e \in I$, is an ill-founded pseudocopy. Then it has an ill-founded sequence of ordinals, and hence also an ill-founded internal J-sequence. But if I is an initial segment, the entire internal J-structure must be present in I (via some Z-index), contradicting the fact that I is well-founded. □

We argue that, for every k, μ can arithmetically recognize the longest well-founded initial segment of $\prec_k$. This will follow from the following lemma. Similar to the Stair Trainer Lemma, it says that random reals are not very helpful in recognizing well-founded initial segments.

Lemma 4.50. *Let $j \geq 0$. Suppose μ is a continuous measure and $\prec$ is a linear order on a subset of ω such that the relation $\prec$ and the field of $\prec$ are both recursive in $\mu^{(j)}$. Suppose further X is $(j+5)$-random relative to μ, and $I \subseteq \omega$ is the longest well-founded initial segment of $\prec$. If I is recursive in $(X \oplus \mu)^{(j)}$, then I is recursive in $\mu^{(j+4)}$.*

Proof. Suppose $I \leq_{\mathrm{T}} (X \oplus \mu)^{(j)}$, X is $(j+5)$-random relative to μ, but $I \not\leq_{\mathrm{T}} \mu^{(j+4)}$. By Proposition 2.10, there is a continuous measure $\mu_I \leq_{\mathrm{T}} \mu^{(j+2)}$ such that I is $(\mu_I, \mu'', j+3)$-random.

For given $a \in \mathrm{Field}(\prec)$, let $\mathcal{I}(a)$ be the set of all reals $Z \subseteq \omega$ such that Z is an initial segment of $\prec$, and all elements of Z are bounded by a. $\mathcal{I}(a)$ is a $\Pi^0_1(\mu^{(j)})$ class. Let T_a be a tree recursive in $\mu^{(j)}$ such that $[T_a] = \mathcal{I}(a)$. Given $n \in \omega$, let $T\lceil_n = \{\sigma \in T \colon |\sigma| = n\}$ be the n-th level of T. We have $\mathcal{I}(a) = \bigcap_n [\![T\lceil_n]\!]$.

Now, if $a \in I$, then $\mathcal{I}(a)$ is countable (since in this case each initial segment Z is an initial segment of the well-founded part of $\prec$ and there are at most countably many such initial segments). Since μ_I is continuous, it follows that $\mathcal{I}(a)$ has μ_I-measure zero.

If, on the other hand, $a \notin I$, then $I \in \mathcal{I}(a)$. Since I is $(\mu_I, \mu'', j+3)$-random and $\mathcal{I}(a)$ is $\Pi^0_1(\mu^{(j)})$, $\mathcal{I}(a)$ does not have μ_I-measure zero. Otherwise we could recursively in $\mu^{(j+2)}$, compute a sequence (l_n) such that $\mu_I[\![T_a\lceil_{l_n}]\!] \leq 2^{-n}$. This would be a $(\mu_I, \mu, j+1)$-test that covers I, but I is $(\mu_I, \mu'', j+3)$-random.

We obtain the following characterization of I.

$$a \in I \quad \Leftrightarrow \quad \forall n\, \exists l\, (\mu_I[\![T_a\lceil_l]\!] \leq 2^{-n})$$

Since $\mu_I \leq \mu^{(j+2)}$, the property on the right hand side is $\Pi^0_2(\mu^{(j+2)})$, hence I is recursive in $\mu^{(j+4)}$. But this contradicts the fact that I is $(\mu_I, \mu'', j+3)$-random. $\square$

We now define, arithmetically in μ, sets PC_k of (indices of) pseudocopies that represent true J-structures. Let us put $c_0 = c$.

Let I_0 be the longest well-founded initial segment of $\mathrm{PC}^*_N(\mu)$. The order $\prec_N$ and its field are recursive in $\mu^{(c)}$ (Lemma 4.44). As R is sufficiently random, the length of I_0 is at most α.

Since R is a 'true' copy, we can use it to test for any pseudocopy M given by $\mathrm{PC}^*_N(\mu)$ whether it embeds into J_α. This can be done recursively in $(\mu \oplus R)^{(c)}$, so in particular I_0 is recursive in $(\mu \oplus R)^{(c)}$. Since R is at least $(c+5)$-random for μ, Lemma 4.50 implies that I is recursive in $\mu^{(c+4)}$.

Now let PC_0 be the set of all numbers $e \in I_0$ such that the structure represented by Φ^μ_e is locally countable. This means that for every $x \in M$ there exists a surjection $f : \omega \to x$ in M. Put $\gamma_0 = 1$.

Now suppose for $k < N$, we have defined $\mathrm{PC}_0, \dots, \mathrm{PC}_k$, $\gamma_0, \dots, \gamma_k$, $c_0, \dots, c_k$.

Let

$$\gamma_{k+1} = \sup\{\beta \colon J_\beta \text{ is represented by some } e \in \mathrm{PC}_k\}.$$

Let $c_{k+1} = 6c_k$. Note that c_{k+1} is greater than the complexity of comparing two ω-copies of J-structures in which ω_{k+1} exists and is the largest cardinal (see Corollary 4.41).

Now let I_{k+1} be the longest well-founded initial segment of $\mathrm{PC}^*_N(\mu^{(c_{k+1})})$. As above, I_{k+1} is recursive in $(\mu \oplus R)^{(c+c_{k+1})}$, and since R is $6^{N+2}c$-random for μ, Lemma 4.50 implies that I is recursive in $\mu^{(c+c_{k+1}+4)}$. Let PC_{k+1} be the set of all $e \in I_{k+1}$ such that $\Phi_e^{\mu^{(c_{k+1})}}$ is locally γ_{k+1}-countable, i.e. for every element x in the structure there exists a surjection $f : \gamma_{k+1} \to x$ in the structure.

Finally, let

$$\gamma = \sup\{\beta \colon J_\beta \text{ is represented by some } e \in \mathrm{PC}_N\}.$$

We will use a Stair Trainer argument to derive a contradiction at one of the PC_k, by showing that one of them contains an additional ordinal.

We first show that, arithmetically in μ, we can produce an ω-copy of J_{γ_k}, for all k.

Lemma 4.51. *For all $k \le N$, there exists an ω-copy of J_{γ_k} recursive in $\mu^{(6c_k)}$. Moreover, there exists an ω-copy of J_γ recursive in $\mu^{(36c_N)}$.*

Proof. We give the proof for J_{γ_k}. The proof for J_γ is similar. We may assume γ_k is greater than every ordinal represented by PC_k. Otherwise, the assertion is trivial.

Case 1: γ_k is a successor ordinal, $\gamma_k = \delta + 1$.

By assumption, there exists an ω-copy Y of J_δ recursive in $\mu^{(c_k)}$. By Proposition 4.29, the real R computes ω-copies of $S^{(n)}(J_\delta)$, for all n.

The following lemma lets us transfer these copies to the "μ-side".

Lemma 4.52. *Suppose μ is a continuous measure and Y is an ω-copy of J_δ, recursive in $\mu^{(m)}$. Suppose further that R is $(m + c_P + 2P_\delta + 27)$-random with respect to μ and computes an ω-copy of $S^{(n)}(J_\alpha)$, for all n. Then, for all n, $\overline{S}^{(n)}(Y)$ is recursive in $\mu^{(m+c_P+2P_\delta+21)}$.*

The lemma follows from Proposition 2.12, Lemma 2.14, Lemma 4.30, and Corollary 4.47.

Since R is $6^{N+2}c$-random and $P_\delta \le N$, we obtain that for all n, $\overline{S}^{(n)}(Y)$ is recursive in $\mu^{(c_k+c_P+N+21)}$. By Corollary 4.33, $\mu^{(c_k+c_P+N+26)}$ computes an ω-copy of $J_{\delta+1} = J_{\gamma_k}$. Note that by definition of c and c_k, $c_k+c_P+N+26 \le 6c_k$.

Case 2: γ_1 is a limit ordinal.

We need to construct a copy of J_{γ_k} as a limit of ω-copies of the structures leading up to it.

Let E be the subset of PC_k given by

$$e \in E \;\Leftrightarrow\; \forall x < e \, (x \in I_k \Rightarrow x \prec_N e),$$

that is, E is the set of indices where a new J-structure occurs. Let us write

$$E = \{e_0 < e_1 < e_2 < \dots\}.$$

Note that the set E is $\Sigma^0_{c_k}(\mu)$.

We now build an ω-copy of J_{γ_k}. Let $Y_0 = \Phi_{e_0}^{\mu^{(c_k)}}$. Initialize by putting

$$U_0 = \{\langle 2^{x+1}, 2^{y+1}\rangle : \langle x, y\rangle \in Y_0\}.$$

Suppose now we have defined $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l$ with the property that

$$F_{U_i} \subseteq \bigcup_{k=0}^{i} \{p_i^m : m \in \omega\}.$$

Let $Y_{l+1} = \Phi_{e_{l+1}}^{\mu^{(c_k)}}$. Let z be such that $Y_{l+1}\lceil_z$ is isomorphic to U_l, and let f_{l+1} be the induced isomorphism between $Y_{l+1}\lceil_z$ and U_l. Put $U_{l+1} = U_l$ and, for any x for which $xE_{Y_{l+1}}z$ does *not* hold, add

$$\langle f_{l+1}(y), p_{l+1}^{x+1}\rangle \text{ to } U_{l+1} \text{ for all } yE_{Y_{l+1}}x.$$

Putting

$$U = \bigcup_{l\in\omega} U_l$$

yields an ω-copy of J_{γ_k} recursive in $\mu^{(c_k+8)}$, and by definition of c_k, $c_k + 8 \leq 6c_k$.

This completes the proof of Lemma 4.51. □

Let us denote the ω-copy of J_{γ_k} obtained by Lemma 4.51 by M_k. Furthermore, let G_k denote the isomorphism between M_k and the ω-copy of J_{γ_k} canonically computed from R. Note that G_k is recursive in $(M_k \oplus R)^{(c)}$.

Next, we work along the internal definability hierarchy over J_{γ_k} to obtain, again arithmetically in μ at a fixed level, ω-copies of the standard J-structures over J_{γ_k}.

Lemma 4.53. *Suppose Y is an ω-copy of a J-structure J_δ. Further suppose μ is a continuous measure and Y is recursive in $\mu^{(m)}$. Finally, suppose that R is $(m + c_P + 2P_\delta + d_{\models}^{(1)} + 9)$-random with respect to μ and computes an ω-copies of all $\langle J_{\rho_\delta^n}, A_\delta^n\rangle$, $n \geq 0$. Then, for every n, there exists an ω-copy $\langle X_\delta^n, M_\delta^n\rangle$ of $\langle J_{\rho_\delta^n}, A_\delta^n\rangle$ recursive in $\mu^{(m+c_P+2P_\delta+9)}$.*

Proof. This follows inductively using Proposition 2.12, Lemma 2.14, Lemma 4.36, and Corollary 4.41. □

Applying this lemma to the ω-copy of J_{γ_k} recursive in $\mu^{(6c_k)}$ obtained from Lemma 4.51, we get that, for *all* n, there exists an ω-copy of $\langle J_{\rho^n_{\gamma_k}}, A^n_{\gamma_k}\rangle$ recursive in $\mu^{(9c_k)}$. For J_γ we get ω-copies of $\langle J_{\rho_\gamma^n}, A_\gamma^n\rangle$ recursive in $\mu^{(39c_k)}$.

Now we successively look at the projection properties of $J_{\gamma_1}, \ldots, J_{\gamma_N}, J_\gamma$.

Suppose J_{γ_1} projects to ω. R computes a copy of $\langle J_{\rho_{\gamma_1}}, A_{\gamma_1}\rangle$ and $\mu^{(9c_0)}$ computes another copy. Since $J_{\rho_{\gamma_1}} = J_1 = V_\omega$, and $A_{\gamma_1} \subseteq J_1$, we just need to transfer A_{γ_1} to μ's copy of V_ω. $\mu^{(9c_0)}$ can recover the canonical coding of V_ω from its copy within $c_\omega + 2$ jumps. Hence, the canonical copy of $\langle J_{\rho_{\gamma_1}}, A_{\gamma_1}\rangle$ is recursive in $\mu^{(9c_0+c_\omega+2)}$. On the other hand, it is recursive in R. As R is $6^{N+2}c$-random for μ, Lemma 2.13 and the definition of c_0 imply that the canonical copy is recursive in μ.

As J_{γ_1} projects to ω, J_{γ_1+1} is locally countable and also projects to ω. We can apply the above argument one more time to obtain the canonical copy of J_{γ_1+1} recursively in μ. But this implies that $\gamma_1 + 1 \in \mathrm{PC}_0$, a contradiction.

Therefore, J_{γ_1} cannot project to ω.

Now suppose for $i = 1, \dots, k < N$, we have shown that J_{γ_i} cannot project to γ_{i-1}, and assume $J_{\gamma_{k+1}}$ projects to γ_k. R computes a copy of $\langle J_{\rho_{\gamma_{k+1}}}, A_{\gamma_{k+1}}\rangle$. By the argument above, $\mu^{(9c_{k+1})}$ computes another ω-copy of this structure. By Corollary 4.26 and Proposition 4.28, $\mu^{(9c_{k+1}+2)}$ computes a copy of J_{γ_k}. Using at most c additional jumps (recall c is greater than the comparison complexity between structures with at most N cardinals), $\mu^{(9c_{k+1}+2)}$ computes the isomorphism between this version and M_k, the copy of J_{γ_k} obtained via Lemma 4.51. Recall that M_k is recursive in $\mu^{(6c_k)}$. Hence there exists an ω-copy $\langle M_k, A^M_k\rangle$ of $\langle J_{\rho_{\gamma_{k+1}}}, A_{\gamma_{k+1}}\rangle$ recursively in $\mu^{(9c_{k+1}+2+c)}$. On the other hand, this copy can be obtained by applying G_k to the copy of $\langle J_{\rho_{\gamma_{k+1}}}, A_{\gamma_{k+1}}\rangle$ which can be obtained recursively in R. It will send R's version of $A_{\gamma_{k+1}}$ (which by assumption is a subset of J_{γ_k}) to M_k's version. Therefore, $\langle M_k, A^M_k\rangle$ is also recursive in $\mu^{(c_k+c)} \oplus R$. Since R is sufficiently random, $\mu^{(c_k+c)} \oplus R$ is Turing-equivalent to $\mu^{(c_k+c)} \oplus R$.

By the Stair Trainer Lemma 2.14, it follows that $\langle M_k, A^M_k\rangle$ is recursive in $\mu^{(c_k+c)}$. As $c_k + c \leq c_{k+1}$, an argument as in the case $k = 1$ now yields a contradiction.

We have established that no J_{γ_k}, $k = 0, \dots, N$ can project (using the choice of c_k). But this means each γ_k is actually an ω_k. Since ω_{N+1} does not exist in L_{β_N}, we obtain that J_γ must project to one of $\gamma_0, \dots, \gamma_N$, and applying the above argument one more time yields a final contradiction.

This completes the proof of Theorem 4.48.

4.8. **Finishing the proof of Theorem 2.** We restate Theorem 2. Let $G(n)$ be the recursive function defined at the beginning of Subsection 4.7.

Theorem 2. *For every $n \in \omega$,*

$$\mathsf{ZFC}_n^- \nvdash \text{“}\mathrm{NCR}_{G(n)} \textit{ is countable.”}$$

Proof. For any $n \geq 0$, the set $\mathcal{X}$ of canonical copies of standard J-structures $\langle J_{\rho_\alpha^k}, A_\alpha^k \rangle$ with $\rho_\alpha^k = 1$ is not countable in L_{β_n}. For suppose $f : \omega \twoheadrightarrow \mathcal{X}$ were a counting of $\mathcal{X}$ such that $f \in L_{\beta_n}$. We may assume f is given as a real. By the closure properties of L_{β_n}, $f' \in L_{\beta_n}$. Let $\gamma < \beta_n$ be the least ordinal such that $f \in J_{\gamma+1} \setminus J_\gamma$, and let m be such that f' is $\Sigma_m(J_\gamma)$. It follows that $\rho_\gamma^m = 1$. f' is computable in the canonical copy $\langle X_\alpha^m, M_\alpha^m \rangle$ of $\langle J_{\rho_\gamma^m}, A_\gamma^m \rangle$. It follows that $\langle X, M \rangle$ is not in the range of f. Since by Theorem 4.48, the set of canonical copies of standard J-structures is a subset of $\mathrm{NCR}_{G(n)}$. Therefore, $\mathrm{NCR}_{G(n)}$ is not countable in L_{β_n}. □

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