THE SLAMAN-WEHNER THEOREM IN HIGHER RECURSION THEORY

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1. Introduction

A central concern of computable model theory is the restriction that algebraic structure imposes on the information content of an object of study. One asks about a countable object, what information is coded intrinsically into this object, which cannot be avoided by passing to an isomorphic copy of the object? Given a countable structure $\mathcal{M}$, we define the degree spectrum of $\mathcal{M}$ to be

$$\mathrm{Spec}(\mathcal{M}) = \{X \in 2^\omega : \exists \mathcal{N} \cong \mathcal{M} \ \ (\mathcal{N} \leqslant_{\mathrm{T}} X)\},$$

where we identify $\mathcal{N}$ with its atomic diagram. In the language of mass problems, $\mathrm{Spec}(\mathcal{M})$ is the problem of computing a copy of $\mathcal{M}$. Since $\mathrm{Spec}(\mathcal{M})$ is degree-invariant, we often replace $\mathrm{Spec}(\mathcal{M})$ by the collection of Turing degrees of elements of $\mathrm{Spec}(\mathcal{M})$. One of the major aims of computable model theory is understanding which collections of Turing degrees can be the spectra of some countable structures. Much study has gone into this problem; we refer the reader to [3] for more information. We remark that Knight [6] has shown that unless $\mathcal{M}$ is trivial, if $X \in \mathrm{Spec}(\mathcal{M})$ then X is Turing equivalent to some copy of $\mathcal{M}$.

Richter [7] has shown that every cone can be a degree Spectrum; that is, for any Turing degree $\mathbf{d}$, there is a structure whose inherent information content is exactly $\mathbf{d}$. On the other hand, Slaman [8] and Wehner [9] have independently shown that the collection of all non-computable sets can also be the spectrum of a countable structure; in other words, there is a structure which captures non-computability.

Unlike the structures whose degree spectrum is a (nontrivial) cone, Slaman's and Wehner's structures are *almost computable*: the Lebesgue measure of their degree spectrum is 1. (Since the degree spectrum is $\Sigma^1_1(\mathcal{M})$, it is measurable; since it is invariant under Turing equivalence, it is either null or co-null). The notion of almost computable structures was defined and investigated by Kalimullin [4] and Csima and Kalimullin [1], who in particular show that there is an almost computable structure whose degree spectrum is not co-countable.

In this paper we give an upper bound on the possible complexity of almost computable structures:

Theorem 1.1. *If $\mathcal{M}$ is an almost computable structure, then there is some copy of $\mathcal{M}$ which is computable from Kleene's $\mathcal{O}$.*

Here Kleene's $\mathcal{O}$ is the standard complete Π^1_1 set of natural numbers. We remark that Nies and Kalimullin (see [5]) have recently announced an independent proof of Theorem 1.1; indeed, they announced that if $\mathcal{M}$ is almost computable, then every

Π^1_1-random set computes a copy of $\mathcal{M}$; Theorem 1.1 follows, because Kleene's $\mathcal{O}$ computes a Π^1_1-random set.

Corollary 1.2. *There are only countably many almost computable structures.*

We note that in Theorem 1.1, Kleene's $\mathcal{O}$ cannot be replaced by any hyperarithmetic set. This is because it follows from results in [2] that for any computable ordinal α, there is a countable structure $\mathcal{M}_\alpha$ whose degree spectrum consists of all sets X such that $\Delta^0_{\alpha+1}(X) \not\subseteq \Delta^0_{\alpha+1}$. This spectrum is co-countable, and so $\mathcal{M}_\alpha$ is almost computable. On the other hand, given a computable ordinal β, we let $\alpha = \beta \cdot \omega$, so $\alpha + \beta = \alpha$, and so $\Delta^0_{\alpha+1}(\emptyset^{(\beta)}) = \Delta^0_{\beta+\alpha+1} = \Delta^0_{\alpha+1}$ and so $\emptyset^{(\beta)} \notin \mathrm{Spec}(\mathcal{M}_\alpha)$.

Hence no hyperarithmetic set can serve as a bound for *all* almost complete sets. We do not know, though, whether every almost computable structure has a hyperarithmetic copy. In particular:

Question 1.3. *Is there a structure whose degree spectrum consists of all non-hyperarithmetic sets?*

Theorem 1.1 and Question 1.3 motivate the following question: does the Slaman-Wehner theorem hold if we replace Turing reducibility by higher reducibilities?

Question 1.4. *Is there a countable structure $\mathcal{M}$ such that the collection of sets $X \in 2^\omega$ such that some copy of $\mathcal{M}$ is hyperarithmetic in X is exactly the collection of non-hyperarithmetic sets? In other words, is there a countable structure whose spectrum in the hyperdegrees is exactly all nonzero hyperdegrees?*

If we go further up in the hierarchy of reducibilities given by higher recursion theory, we arrive at relative constructibility. To make the situation non-trivial, the standard assumption on the underlying set-theoretic universe is that ω_1 is inaccessible for reals. Under this assumption, we show that the Slaman-Wehner theorem fails for relative constructibility.

Theorem 1.5. *Suppose that ω_1 is inaccessible for reals. Then there is no countable structure $\mathcal{M}$ such that for all $X \in 2^\omega$, $L[X]$ contains a copy of $\mathcal{M}$ if and only if X is not constructible.*

2. Almost computable structures

In this section we prove Theorem 1.1: if $\mathcal{M}$ is a countable model such that $\lambda\,\mathrm{Spec}(\mathcal{M}) > 0$, then $\mathcal{M}$ has a copy computable from Kleene's $\mathcal{O}$ (here λ denotes Lebesgue measure on 2^ω). Fix such a structure $\mathcal{M}$.

Lemma 2.1. *There is a partial computable function $\Phi\colon 2^\omega \to 2^\omega$ such that*

$$\lambda\,\{X \in 2^\omega \,:\, \Phi(X) \cong \mathcal{M}\} > 1/2.$$

Proof. For a partial computable function Ψ, let

$$C_\Psi = \{X \in 2^\omega \,:\, \Psi(X) \cong \mathcal{M}\}\,.$$

there are only countably many partial computable functions, and the union of C_Ψ for all partial computable functions is

$$\{X \in 2^\omega \,:\, \exists \mathcal{N} \leqslant_T X \;\; (\mathcal{N} \cong \mathcal{M})\}\,,$$

which by assumption has measure 1. Hence there is a partial computable function Ψ such that $\lambda C_\Psi > 0$. By the Lebesgue density theorem, there is some finite string $\sigma \in 2^{<\omega}$ such that

$$\frac{\lambda\left(C_\Psi \cap [\sigma]\right)}{2^{-|\sigma|}} > 1/2.$$

We let $\Phi(X) = \Psi(\sigma X)$. ☐

We fix a partial computable function Φ given by Lemma 2.1, and let

$$C = C_\Phi = \{X \in 2^\omega \,:\, \Phi(X) \cong \mathcal{M}\}.$$

Let

$$A = \left\{(X,Y) \in (2^\omega)^2 \,:\, \Psi(X) \cong \Psi(Y)\right\}.$$

Then A is a Σ^1_1 class. For any $X \in 2^\omega$, let A_X be the section

$$A_X = \{Y \in 2^\omega \,:\, (X,Y) \in A\}.$$

Fix some rational number $q > 1/2$ such that $\lambda C \geqslant q$. Let

$$B = \{X \in 2^\omega \,:\, \lambda A_X \geqslant q\}.$$

Lemma 2.2. *B is a Σ^1_1 class.*

Proof. There is a computable function $\Psi\colon (2^\omega)^2 \to 2^\omega$ such that for all X and Y, $\Phi(X,Y)$ is a linear ordering of ω, and $(X,Y) \in A$ iff $\Phi(X,Y)$ is not well-founded. For $\alpha < \omega_1$, we let A_α be the set of pairs (X,Y) such that $\Phi(X,Y)$ is not embeddable into α.

By Spector's argument, for all X, $\lambda A_X = \lambda(A_{\omega_1^X})_X$, so $X \in B$ if and only if for all $\alpha < \omega_1^X$, $\lambda(A_\alpha)_X \geqslant q$. Now, we have a computable function Ξ which given X and a notation in $\mathcal{O}^X$ for an ordinal $\alpha < \omega_1^X$ gives a $\Delta^1_1(X)$-index for $\lambda(A_\alpha)_X$, since $(A_\alpha)_X$ is $\Delta^1_1(X)$, uniformly in X and α.

Hence

$$X \in B \Leftrightarrow \forall n \left(n \in \mathcal{O}^X \to q \leqslant \Xi(n,X)\right)$$

which is a Σ^1_1 definition of B. ☐

Lemma 2.3. *$B = C$.*

Proof. If $X \in C$, then for all $Y \in C$, $(X,Y) \in A$, so $C \subseteq A_X$, so $\lambda A_X \geqslant q$, so $X \in B$.

Suppose that $X \in B$. Since $\lambda A_X, \lambda C > 1/2$, the intersection $A_X \cap C$ is nonempty; let $Y \in A_X \cap C$. Then $\Phi(Y) \cong \mathcal{M}$ since $Y \in C$, and $\Phi(Y) \cong \Phi(X)$ since $Y \in A_X$. Hence $\Phi(X) \cong M$, so $X \in C$. ☐

Now Theorem 1.1 follows from ——'s basis theorem, that every nonempty Σ^1_1 class contains a set computable from Kleene's $\mathcal{O}$.

3. The constructible spectrum

We adopt the argument of the previous section to prove Theorem 1.5. We need to relativise the proof to a countable ordinal α, but by restricting to a co-null class, we may assume that α is countable in L.

Let $\mathcal{M}$ be a countable structure, and suppose that for any non-constructible $X \in 2^\omega$, $L[X]$ contains some copy of $\mathcal{M}$. We assume that ω_1 is inaccessible for reals: for all $X \in 2^\omega$, $\omega_1^{L[X]}$ is countable, and so $2^\omega \cap L[X]$ is countable.

Since L contains only countably many Borel codes, the collection of $X \in 2^\omega$ which are random over L is co-null. Since random real forcing does not collapse ω_1, for each X which is random over L, we have $\omega_1^{L[X]} = \omega_1^L$. Hence, the collection

$$N = \left\{ X \in 2^\omega \,:\, \omega_1^{L[X]} = \omega_1^L \right\}$$

is co-null.

For all $X \in 2^\omega$, fix a uniformly $\Delta_1^{L[X]}$ bijection j_X from $\omega_1^{L[X]}$ to $2^\omega \cap L[X]$.

Lemma 3.1. *For any $\alpha < \omega_1$, the relation $Y = j_X(\alpha)$ is Borel.*

Indeed, the relation is $\Delta_1^1(R)$ for any real code R for α.

Proof. Let $R \in 2^\omega$ be a code for α. Then $Y = j_X(\alpha)$ if and only if for some (for all) ω-models M of ZFC$^-$ which contains R, X and Y, $M \models$ "$Y = j_X(\alpha)$". □

For $X \in N \setminus L$, there is some $\alpha < \omega_1^L$ such that $j_X(\alpha)$ is isomorphic to $\mathcal{M}$. By Lemma 3.1, for any $\alpha < \omega_1^L$, the class

$$K_\alpha = \{X \in 2^\omega \,:\, j_X(\alpha) \cong \mathcal{M}\}$$

is $\mathbf{\Sigma_1^1}$, and so is measurable. Since ω_1^L is countable, and $N \setminus L$ is co-null, there is some $\alpha < \omega_1^L$ such that $\lambda K_\alpha > 0$. Again by Lebesgue density, there is some finite $\sigma \in 2^{<\omega}$ such that

$$C = \{X \in 2^\omega \,:\, \sigma X \in K_\alpha\}$$

has measure strictly greater than 1/2. For all $X \in 2^\omega$, let $\Phi(X) = j_{\sigma X}(\alpha)$.

Fix some $R \in L \cap 2^\omega$ which is a code for α. Let

$$A = \left\{ (X, Y) \in (2^\omega)^2 \,:\, \Phi(X) \cong \Phi(Y) \right\}.$$

Lemma 3.1 implies that A is $\Sigma_1^1(R)$. We again fix some rational $q > 1/2$ such that $\lambda C \geqslant q$, and let

$$B = \{X \in 2^\omega \,:\, \lambda A_X \geqslant q\}.$$

The argument of Lemma 2.2 shows that B is $\Sigma_1^1(R)$; and the argument of Lemma 2.3 shows that $B = C$. Hence there is a copy of $\mathcal{M}$ computable from $\mathcal{O}^R$; since $R \in L$, we have $\mathcal{O}^R \in L$, and so $\mathcal{M}$ has a constructible copy as required.

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