THE INDUCTIVE STRENGTH OF RAMSEY'S THEOREM FOR PAIRS

C. T. CHONG, THEODORE A. SLAMAN, AND YUE YANG

Abstract. We address the question, "Which number theoretic statements can be proven by computational means and applications of Ramsey's Theorem for Pairs?" We show that, over the base theory RCA_0, Ramsey's Theorem for Pairs does not imply Σ^0_2-induction.

1. Introduction

Ramsey's Theorem is the assertion that for any natural number k and any function F on the size-k of subsets of the natural numbers into a finite range, there is an infinite set H such that F is constant on the size-k subsets of H. We say that such an H is F-homogenous. If we think of F as a coloring of size-k sets, then H is monochromatic. Ramsey's Theorem is the initial point for a rich and widely applicable theory, in both finite and infinite combinatorics.

In addition to its mathematical importance, Ramsey's Theorem has an important metamathematical position. Proofs of it and of its extensions are naturally found by invoking large-scale infrastructure. For example, if we start with a nonprincipal ultrafilter on the natural numbers, the proof of Ramsey's Theorem follows in an intuitive way. Conversely, any proof of Ramsey's Theorem necessarily involves systems of complicated sets. In [7], Jockusch showed that there is a computable partition F_3 of triples such that any infinite F_3-homogeneous set computes the Halting Problem. Jockusch also showed that there is a computable partition F_2 of pairs which has no infinite homogeneous set which is computable relative to the Halting Problem. Thus, there is a considerable amount of information in every infinite F_3-homogeneous set and there is no simply-defined infinite F_2-homogeneous set. Jockusch's Theorem is also the initial point of an rich metamathematical investigation, which is what we pursue here.

We are particularly interested in calibrating the strength of Ramsey's Theorem with respect to its arithmetical consequences. In other words, we are interested in the question, "Which number theoretic statements can be proven by computational means and applications of Ramsey's Theorem for Pairs?" This question is part of the broader study of familiar infinitary methods, such as compactness, category, measure, and infinitary combinatorics, with respect to their applicability to questions about the finite. In most cases, even when an infinitary principle adds

---

2010 *Mathematics Subject Classification.* Primary 03B30, 03F35, 03D80; Secondary 05D10.

Key words and phrases. Reverse mathematics, RCA_0, Σ^0_2-bounding, Ramsey's Theorem for Pairs, Stable Ramsey's Theorem for Pairs.

Chong was partially supported by NUS grant WBS C146-000-025-00 for this research. Slaman's research was partially supported by NSF award DMS-1301659. Yang's research was partially supported by NUS research grant WBS 146-000-159-112 and NSFC (No. 11171031).

strength, the set of its arithmetic consequences is equal to the set of consequences of one of the well-known finitary principles, typically a principle of induction. Infinitary combinatorics, and Ramsey's Theorem in particular, provides a dramatic exception to this rule, as we will describe.

We work within the formalism of subsystems of second order arithmetic. These systems consist of arithmetic axioms, which are assertions concerning the natural numbers such as addition is commutative or instances of definable induction, and set comprehension axioms, which are assertions concerning the subsets of the natural numbers such as Ramsey's Theorem. Our base theory, RCA_0 consists of the usual first-order axioms for arithmetic operations and Σ^0_1-induction relative to parameters, together with the second-order recursive comprehension scheme $\exists X[(\forall x(x \in X \leftrightarrow \varphi(x))]$, for each Δ^0_1-formula φ (also with parameters). One should think of RCA_0 as axiomatizing the assertion that the subset of the natural numbers are closed under relative computation.

Let RT^2_2 denote Ramsey's Theorem for $k = 2$ and partitions F with range $\{0, 1\}$, that is for colorings of pairs by two colors. By Jockusch's theorem, RT^2_2 is not provable in RCA_0. Closely related to RT^2_2, and intuitively a more controlled coloring scheme, is stable Ramsey's Theorem for Pairs (SRT^2_2): If for any $x \in M$, all but finitely many $\{x, y\}$'s have the same color, then there is an infinite homogeneous set in $\mathcal{M}$. SRT^2_2 is also known to be unprovable from RCA_0.

The proof-theoretic strength of these two combinatorial principles has been investigated by various authors. Cholak, Jockusch and Slaman [1] showed that SRT^2_2, hence RT^2_2 as first established by Hirst [6], implies the Σ^0_2-bounding principle $B\Sigma^0_2$. By Slaman [9], $B\Sigma^0_2$ is equivalent to Δ^0_2-induction, so the arithmetic strength of RT^2_2 is at least as strong as that induction principle. Cholak et. al. also showed that RT^2_2 is Π^1_1-conservative over RCA_0 together with the Σ^0_2-induction $\mathrm{I}\Sigma^0_2$. That is, any Π^1_1-statement that is provable in $RT^2_2 + RCA_0 + I\Sigma^0_2$ is already provable in the system $RCA_0 + I\Sigma^0_2$. It follows immediately that any subsystem of $RT^2_2 + RCA_0 + I\Sigma^0_2$ (such as replacing RT^2_2 by SRT^2_2) is Π^1_1-conservative over $RCA_0 + I\Sigma^0_2$. [1] also showed that RT^2_2 is equivalent to $SRT^2_2 + COH$, where COH is the statement that for every sequence of sets (R_i), there is an infinite set G such that for each i, one of $R_i \cap G$ or $\overline{R_i} \cap G$ is finite.

Moreover, Chong, Slaman and Yang had demonstrated how to handle the two weaker principles separately. [4] showed that (among other things) one could preserve $B\Sigma_2$ while adding new sets to satisfy COH; [5] showed that by looking at a particular model, one could preserve $B\Sigma_2$ while adding new sets to satisfy SRT^2_2. Thus, $RCA_0 + SRT^2_2$ does not imply $I\Sigma_2$.

In this paper, we show how to combine the two constructions and conclude that $RCA_0 + RT^2_2$ does not imply $I\Sigma_2$. Though we fully expect that there is a natural axiomatization of the arithmetic consequences of RT^2_2, possibly as a collection of finite combinatorial principles, the situation as it stands now is quite mysterious.

About the proof, there is a significant amount of tension between adding sets to satisfy instances of SRT^2_2 and adding sets to satisfy instances of COH. When we act to satisfy SRT^2_2, we rely on an additional arithmetic property, BME, of the underlying model and on the existence of nonstandard numbers which code nonstandardly finite subsets of definable sets. When we act to satisfy COH, we show that BME is preserved. However, we do introduce set parameters relative to which there are definable sets with no such codes.

In the following, we show that there is a two-step way to finesse the above problem. We only need codes for subsets of definable sets which appear in our action to satisfy SRT^2_2. By employing the priority method in their construction, these sets admit an approximation that satisfies the hypothesis of the Chong-Mourad Coding Lemma. From this, we know that there is an auxiliary code that describes the dynamics of the SRT^2_2 construction relative to the sets we have already added. We can then conclude that we have the code needed in the SRT^2_2 argument by virtue of having it for a set defined from the auxiliary code. We run the coding argument twice, each time using a different coding mechanism, to show that the set for which we wish to have a code is indeed coded.

The rest of this paper is to carefully carry out all details of the above plan. To make it self-contained, we repeat certain segments of the constructions which are done in [4] and [5].

2. Preliminaries

For basic facts about recursion theory on nonstandard models, see [5] whose notations are followed here. We recall the notions and results that will be referred to in this paper. Let P^- denote the Peano axioms without mathematical induction. Let $I\Sigma^0_n$ denote the Σ^0_n-induction scheme, and $B\Sigma^0_n$ the Σ^0_n-bounding scheme. A model $\mathfrak{M}$ of second-order arithmetic is a structure of the form $\langle M, \mathbb{S}, +, \times, 0, 1\rangle$ where $\mathbb{S}$ is a collection of subsets of M. The following fact will be used implicitly throughout the paper.

Proposition 2.1. *If $\mathfrak{M} \models I\Sigma^0_n$, then every bounded $\Sigma^0_n(\mathfrak{M})$ set is $\mathfrak{M}$-finite.*

If $\mathfrak{M} \models B\Sigma^0_n$ but not $I\Sigma^0_n$, we call it a $B\Sigma^0_n$-model. In this case, there is a Σ^0_n-function mapping a Σ^0_n-definable cut (i.e. a Σ^0_n-cut) cofinally into M.

Given a model $\mathfrak{M}$ of RCA_0, and $G \subset M$, let $\mathfrak{M}[G]$ denote the structure generated by G over $\mathfrak{M}$ by closing under functions recursive in G, with parameters from $\mathfrak{M}$. By expanding the language of arithmetic to include set constants X, we may also consider $\mathfrak{M}[X]$ where X is a predicate denoting a subset of M.

The following lemma captures the essence of coding in $B\Sigma^0_n$-models of RCA_0 (see [3] for a discussion in the case of first-order theories). The generalization of the coding property to second order systems such as RCA_0 is straightforward.

Definition 2.2. Let A be a subset of M, where $\mathfrak{M} \models RCA_0$. A set $X \subseteq A$ is *coded on A* if there is an $\mathfrak{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.

Definition 2.3. Let A be a subset of M. We say that a set X is Δ^0_n *on A* if both $A \cap X$ and $A \cap \overline{X}$ are $\Sigma^0_n(\mathfrak{M})$.

Lemma 2.4 ([3])**.** *Let $\mathfrak{M}$ be a model of $RCA_0 + B\Sigma^0_n$ ($n \geq 2$) and let $A \subset M$. Then every bounded set that is Δ^0_n on A is coded on A.*

Lemma 2.4 may be viewed as the effective analog of the coding power of a saturated model of Peano arithmetic, and is applicable to any $B\Sigma^0_n$ model. It will be used in §4 to reduce the complexity of the generic set G to be constructed in Theorem 4.1.

2.1. The ground model $\mathfrak{M}_0$.

The ground model $\mathfrak{M}_0$ that we will be working with in this paper is a "Σ^0_1-reflection model" which was introduced in [5]. This model is a refinement of one that was constructed in [8].

Proposition 2.5. *There is a countable $B\Sigma^0_2$-model $\mathfrak{M}_0 = \langle M_0, +, \times, 0, 1\rangle$ with a Σ^0_2-function g such that:*

(1) $\mathfrak{M}_0$ is the union of a sequence of Σ_1-elementary end-extensions of models of PA:

$$\mathfrak{I}_0 \prec_{\Sigma_1,e} \mathfrak{I}_1 \prec_{\Sigma_1,e} \mathfrak{I}_2 \prec_{\Sigma_1,e} \cdots \prec_{\Sigma_1,e} \mathfrak{M}_0$$

(2) For each $i \in \omega$, $g(i) \in \mathfrak{I}_i$, and for $i > 0$, $g(i) \notin \mathfrak{I}_{i-1}$, and hence $\mathfrak{M}_0 \not\models I\Sigma^0_2$.

(3) Every $\mathfrak{M}_0$-arithmetical subset of ω is coded on ω.

We turn $\mathfrak{M}_0$ into a model of RCA_0 by letting the second-order elements of $\mathfrak{M}_0$ to consist of all the recursive subsets of M_0.

Definition 2.6. Given models $\mathfrak{M} = \langle M, \mathbb{S}, +, \times, 0, 1\rangle$ and $\mathfrak{M}^* = \langle M^*, \mathbb{S}^*, +, \times, 0, 1\rangle$ of RCA_0, we say that $\mathfrak{M}^*$ is an *M-extension of* $\mathfrak{M}$ if $M = M^*$ and $S \subseteq S^*$, i.e. only subsets of M are added to $\mathfrak{M}$ to form $\mathfrak{M}^*$.

For the rest of this paper, we fix $\mathfrak{M}_0$ and g to be the model and function defined in Proposition 2.5 (with the recursive sets as second-order elements of $\mathfrak{M}_0$). The model $\mathfrak{M}$ that we will construct in Theorem 4.1 to satisfy Corollary 4.2 will be an M_0-extension of $\mathfrak{M}_0$.

2.2. **Combinatorial principles.** We recall the combinatorial principles that are central to the discussion.

Let $R \in \mathfrak{M}$ be $\mathfrak{M}$-infinite and let $R_s = \{t|(s,t) \in R\}$. We say that a set G is *R-cohesive* if for all s, either $G \cap R_s$ is $\mathfrak{M}$-finite or $G \cap \overline{R_s}$ is $\mathfrak{M}$-finite.

Definition 2.7. (Principle of Cohesiveness (COH)) For every $R \in \mathfrak{M}$, there is an $\mathfrak{M}$-infinite $G \in \mathfrak{M}$ that is R-cohesive.

If $A \subseteq M$, then $[A]^2$ denotes the set of (unordered) pairs of numbers in A.

Definition 2.8. (Principle of Ramsey's Theorem for Pairs (RT^2_2)) Let $f : [M]^2 \to 2$ be a function in $\mathfrak{M}$. Then there is an $A \subseteq M$ in $\mathfrak{M}$ such that f is a constant on $[A]^2$.

We often refer to such an f as a 2-*coloring*. A 2-coloring $f : [M]^2 \to 2$ is *stable* if $\lim_s f(x, s)$ exists for each $x \in M$. The Principle of Stable Ramsey's Theorem for Pairs (SRT^2_2) is RT^2_2 restricted to stable 2-colorings.

The principles COH and SRT^2_2 originated in [1], where the next and final principle was also introduced:

Definition 2.9. The principle D^2_2 states: Given any Δ^0_2-set A, either A or its complement contains an infinite subset.

The following two propositions decompose Ramsey's Theorem for Pairs into two components which constitute the basic building blocks for the proof of our main result (Theorem 4.1).

Proposition 2.10 ([1])**.** *Over the base theory RCA_0, RT^2_2 is equivalent to $COH + SRT^2_2$.*

Proposition 2.11 ([2])**.** *Over the base theory RCA_0, SRT^2_2 is equivalent to D^2_2.*

2.3. **Seetapun disjunction and exit tree.** The construction of a $B\Sigma^0_2$ model for RT^2_2 begins with the decomposition given in Proposition 2.10. Sections 3 and 4 handle COH and SRT^2_2 respectively. The strategy for meeting the latter is to build successively sets G that solve instances of D^2_2 (applying Proposition 2.11) and are low relative respectively to predicates X that satisfy $B\Sigma^0_2$. Lowness is achieved by forcing the Σ^0_1-theory of G. Thus managing how $\Sigma^0_1(G)$-formulas are satisfied is central to the construction. The notions of a Seetapun disjunction and correspondingly an exit tree were introduced in [5] for this purpose. Not surprisingly, they continue to play a crucial role in this paper.

Let $\vec{o} = \{o_s\}$ be a (finite or infinite in the sense of $\mathfrak{M}_0$) sequence of pairwise disjoint $\mathfrak{M}_0$-finite sets such that for each s, $\max o_s < \min o_{s+1}$. We will only be concerned with X-recursive sequences $\vec{o}$.

The Seetapun tree S associated with $\vec{o}$ is the union of the $\mathfrak{M}_0$-finite trees S_s consisting of all the choice functions on $\{o_{s'}\}_{s'\leq s}$, i.e. $\mathfrak{M}_0$-finite functions h with domain $[0, s]$ such that for all $s' \leq s$, $h(s') \in o_{s'}$. We call $o_{s'}$ a *blob* and any subset of the range of a choice function h a *thread*.

Let X be a predicate and B_X an $\mathfrak{M}_0[X]$-finite block, synonymous for set, of (indices of) $\Sigma^0_1(X)$-formulas (which is also written as $B_{X,r}$ and $B_{X,b}$ for reason that will be clear from the definition below),

Definition 2.12. Let $\varepsilon = (\rho, \beta)$ be a pair of disjoint $\mathfrak{M}_0$-finite sets. A *Seetapun disjunction* (or S-disjunction for short) δ *for* $(B_{X,r}, B_{X,b})$ *with precondition* (ρ, β) is a pair $(\vec{o}, S)$ such that $S = S_s$ for some s, $\vec{o} = \{o_{s'}\}_{s'\leq s}$ and

(i) For each $s' < s$, $\mathfrak{M}_0 \models \psi_e(X, \rho * o_{s'})$ for some $e \in B_{X,r}$.
(ii) For each maximal branch τ of S, there exists an $\mathfrak{M}_0$-finite subset $\iota \subseteq \tau$ such that $\mathfrak{M}_0 \models \psi_d(X, \beta * \iota)$ for some $d \in B_{X,b}$.

The objective is to select $\rho \subseteq A$ and $\beta \subseteq \overline{A}$ for a given A that is $\Delta^0_2(X)$. We use the letters ρ and β to suggest the colors red and blue respectively. Given $\varepsilon = (\rho, \beta)$, define $B_{X,r}(\varepsilon)$ and $B_{X,b}(\varepsilon)$ to be respectively the set of formulas in B_X not *positively forced* by ρ and β. In other words, $B_{X,r}(\varepsilon) = B_{X,r} \setminus \{e : \mathfrak{M}_0[X] \models \psi_e(X, \rho)\}$ and $B_{X,b}(\varepsilon) = B_X \setminus \{d : \mathfrak{M}_0[X] \models \psi_d(X, \beta)\}$. The proof of the next lemma is in [5]. Recall that a set $A \subseteq M_0$ is amenable if $A \upharpoonright s$ is $\mathfrak{M}_0$-finite for each s.

Lemma 2.13. *Let X be a predicate and $\varepsilon = (\rho, \beta)$ be a pair of disjoint $\mathfrak{M}_0$-finite sets. Let $\delta = (\vec{o}, S)$ be an S-disjunction for $(B_{X,r}(\varepsilon), B_{X,b}(\varepsilon))$ with precondition (ρ, β). Let A be amenable such that $\rho \subseteq A$ and $\beta \subseteq \overline{A}$. Then one of the following applies:*

(i) There is an $o \in \vec{o}$ such that $\rho * o \subseteq A$ and $\psi_e(X, \rho * o)$ holds for some $e \in B_{X,r}(\varepsilon)$;
(ii) There is a $\tau \in S$ and a thread $\iota \subseteq \tau$ such that $\beta * \iota \subseteq \overline{A}$ and $\psi_d(X, \beta * \iota)$ holds for some $d \in B_{X,b}(\varepsilon)$.

Suppose that B_X is an $\mathfrak{M}_0$-finite set of $\Sigma^0_1(X)$-formulas. Lemma 2.13 leads to the notion of an exit tree. An exit tree is designed to manage in an effective manner all possible outcomes in the enumeration of S-disjunctions for B_X. Lemma 2.13 guarantees that for an amenable A, the "exits" in the tree are well-behaved, and BME (see the next subsection) implies that each exit tree is $\mathfrak{M}_0$-finite. Given A, (ρ, β) and S-disjunction δ that satisfy the hypothesis of Lemma 2.13, define an *exit taken by A from δ* to be a o or ι that satisfies the conclusion of the Lemma. An arbitrary o or ι in δ is simply called an exit. Beginning with $\rho = \beta = \emptyset$, one can enumerate an S-disjunction. Each pair of exits (ρ, β) may be used as a precondition to enumerate another S-disjunction. If ρ and β are exits taken by A, then applying Lemma 2.13 again produces a pair of exits taken by A which can be used as a precondition for another application of Lemma 2.13. The exit tree is obtained by assembling all the pairs of exits (not necessarily taken by A) generated during the process. Since the notion of an exit tree E was the source for the introduction of *BME* on which our main theorem relies, we recall formally its enumeration:

At stage 0, let $E[0]$ be the (code of the) empty set (as root of the tree), and search for an S-disjunction δ for $(B_{X,r}, B_{X,b})$ with the pair of preconditions $(\emptyset, \emptyset)$.

At stage $s+1$, suppose that $E[s]$ is given. For each maximal branch $\varepsilon = (\rho, \beta)$ on $E[s]$ whose code is less than $s+1$, see whether there exists an S-disjunction δ for $(B_{X,r}(\varepsilon), B_{X,b}(\varepsilon))$ with (ρ, β) as a pair of preconditions. If no such δ is found, do nothing. Otherwise select the first S-disjunction enumerated over ε. Concatenate with ε each pair (ρ', β') of (the codes of) exits in δ, and also concatenate with ε pairs of the form $(\rho', \emptyset)$ and $(\emptyset, \beta')$ where ρ' and β' are exits in δ. Let the resulting tree be $E[s+1]$.

Observe that the enumeration of E is monotone in the sense that strings enumerated at stage $s+1$ sit on top of strings enumerated at stage s, a property that is prominent in the definition of *BME*.

2.4. The Principle of Bounded Monotone Enumeration (*BME*).

The Principle *BME* was introduced in [5]. Since it plays a crucial role in this paper, we restate its definition here. Recall that a tree T is a collection of $\mathfrak{M}$-finite functions (also called strings) from an initial segment of M into M closed under pairwise intersection. We often do not distinguish between a string σ and its range. T is *downward closed* if every initial segment of a member of T is a member of T. T is *recursively bounded* if there is a recursive function f such that for all $x \in M$, there are at most $f(x)$ many elements in T of length x.

Definition 2.14. Let E be a procedure to recursively enumerate a finite branching enumerable tree. We say that E is a *monotone enumeration* if and only if the following conditions apply to its stage-by-stage behavior.

(1) The empty sequence is enumerated by E during stage 0.
(2) Only $\mathfrak{M}$-finitely many sequences are enumerated by E during any stage.
(3) Suppose $T[s]$ is the tree enumerated by E at the end of stage s. Then any sequence enumerated by E during stage $s+1$ must extend some terminal node of $T[s]$.

The informal idea is that the tree enumerated by E always grows from leaves. Once a leaf becomes an interior node, no new sequence will be enumerated directly extending it, although indirect extensions through the new leaves extending it are allowed.

For an element τ enumerated by E, let k be the number of stages in the enumeration by E during which τ or an initial segment of τ is enumerated. Let $(\tau_i : i < k)$ be the stage-by-stage sequence of the maximal initial segments of τ associated with those stages. We say that τ has order k. The enumeration of E is said to be bounded by b if for each τ enumerated by E, it has order less than or equal to b.

Proposition 3.6 of [5] says that in general, given b in a $B\Sigma^0_2$-model $\mathfrak{M}$, there is no guarantee that a stage s exists bounding the stages where all sequences of order less than b are enumerated by E (or equivalently, the resulting tree T with height less than b reaches its limit by stage s). The principle BME ensures that such a stage always exists, even when one performs simultaneously an iteration of $\mathfrak{M}$-finitely many monotone enumeration operators.

The monotone enumeration operator E may be relativized to a string σ. $E(\sigma)$ is the $\mathfrak{M}$-finite tree enumerated with σ as an oracle. We follow the usual convention that if m is the maximum of the length of σ and its greatest element, then the evaluation of E relative to σ takes less than m steps and σ is queried only at arguments for which it is defined. The enumeration operator can also be relativized to a predicate X, resulting in an r.e. in X tree. E may also be relativized to the paths of a tree V, in which case the outcome will be a "forest"—that is, a collection of trees since incompatible paths on V may produce different trees.

In fact, the "tree" V mentioned above should be viewed as a procedure to compute a recursively bounded recursive tree. Similarly we may define V relative to a string τ, a predicate X, or even a tree T recursively enumerated by E. We will use σ and τ to denote respectively strings in the relativization of E and V. By abuse of notation, we will also use E or V to refer to the recursive or recursively enumerable trees defined by them. In particular, E is also referred to as an *exit tree*.

Definition 2.15. Suppose that V is the index for a recursively bounded recursive tree and suppose that E is a monotone enumeration procedure. For σ in the tree computed by V, say that σ is *E-expansionary* if in the enumeration of $E(\sigma)$ some new element is enumerated at stage $|\sigma|$. We say that a level ℓ in the tree computed by V is *E-expansionary* if there is an n such that ℓ is the least level in the tree computed by V at which every σ in that tree with $|\sigma| = \ell$ has at least n many E-expansionary initial segments.

We will first define a scaled down version of BME which is denoted BME_1. BME_1 is BME for "one-dimension" and is invoked when iteration is not involved in the process.

We say that $\mathfrak{M}$ satisfies BME_1 if and only if the following holds: For each $b \in M$, indices for a recursively bounded recursive tree V and a monotone enumeration procedure E, if E is bounded by b then there are only boundedly many E-expansionary levels in V. In other words, there is a level l^* such that for any $\sigma \in V$ of length greater than l^*, $E(\sigma)$ does not enumerate any new sequence τ of order less than or equal to b.

The general principle BME deals with iterated applications of monotone operators. In [5], the operator E_i carries out an enumeration of Seetapun disjunctions and "blobs", while the recursively bounded recursive tree V_i supplies the strings relative to which the next monotone operator E_{i+1} performs its enumeration. The sequence $\langle V_i, E_i\rangle_{1\leq i\leq k}$ is introduced to enable one to construct a sequence of generic sets solving instances of SRT^2_2 while satisfying the lowness property under any finite join. An outcome of the iteration process is that every infinite path on a V generates a tree and every tree generates a forest (of E's).

Definition 2.16. A *k-iterated monotone enumeration* is a sequence $\langle V_i, E_i\rangle_{1\leq i\leq k}$ with the following properties.

(1) Each V_i is an index for a relativized recursive recursively-bounded tree.

(2) Each E_i is an index for a monotone enumeration procedure.
(3) For each $1 \leq j \leq k$, if $\sigma \in V_j$ is E_j-expansionary, then for every new element τ enumerated in $E_j(\sigma)$, $V_{j+1}(\tau)$ is a proper E_{j+1}-expansionary extension of $V_{j+1}(\tau_0)$, where τ_0 is the longest initial segment of τ that had previously been enumerated in $E_j(\sigma)$, that is by a stage less than the length of σ.

Definition 2.17. A *k-path* of the k-iterated monotone enumeration $\langle V_i, E_i\rangle_{1\leq i\leq k}$ is a sequence $(\sigma_i, \tau_i)_{1\leq i\leq k}$ such that $\sigma_1 \in V_1$ and τ_1 is a maximal sequence in $E_1(\sigma_1)$, and for each j with $1 < j \leq k$, σ_j is a maximal sequence in $V_j(\tau_{j-1})$ and τ_j is a maximal sequence in $E_j(\sigma_j)$.

Definition 2.18. (1) A k-iterated monotone enumeration is *b-bounded* if and only if for every sequence enumerated in $E_k(\sigma_k)$ by some k-path of the k-iterated enumeration, its stage-by-stage enumeration has length less than or equal to b.
(2) We say that $\mathfrak{M}$ satisfies *bounding for iterated monotone enumerations* (*BME*) if and only if for every $k \in \omega$, every b in $\mathfrak{M}$ and every b-bounded k-iterated monotone enumeration, there are only boundedly many E_1-expansionary levels in V_1.
(3) If we restrict our attention to k-iterated monotone enumerations, we say that $\mathfrak{M}$ satisfies BME_k.

Proposition 2.19. *$\mathfrak{M}_0$ satisfies BME.*

Given predicates X, G and $j < \omega$, the Claim in Theorem 5.1 of [5] (where n is replaced by j here for the construction to be carried out in §4) asserts the existence of a $j+1$-enumeration procedure that amalgamates the collection C of all $g(j+1)$-bounded, k-iterated monotone enumerations (where $k \leq j+1$) relative to (X, G) whose indices are below $g(j+1)$. Suppose

$$C = \{\langle V_{e,i}, E_{e,i}\rangle_{1\leq i\leq k(e)} : e \leq e_0\}$$

for some e_0.

Proposition 2.20. *There exists a $g(j+1)$-bounded, $j+1$-iterated monotone enumeration $\langle \hat{V}_i, \hat{E}_i\rangle_{1\leq i\leq j+1}$ such that*

- *for each $e \leq e_0$, i, σ and τ, $0 * e * \sigma \in \hat{V}_i(\tau)$ if and only if $\sigma \in V_{e,i}(\tau)$, and $\tau \in \hat{E}_i(0 * e * \sigma)$ if $\tau \in E_{e,i}(\sigma)$.*

Proposition 2.20 allows one to handle, at each stage of constructing a generic set G, to handle only one sequence of monotone enumeration operators which is an amalgamation of an $\mathfrak{M}$-finite block of such operators.

2.5. **Forcing.** The notion of forcing we require for the proof of Theorem 4.1 was introduced in [5]:

Definition 2.21. Let X be a predicate. The partial order $P = \langle p, \leq\rangle$ of forcing conditions p satisfies:

(1) $p = (\varepsilon, U)$ where $\varepsilon = (\rho, \beta)$ is a pair of disjoint $\mathfrak{M}_0$-finite strings of the same length and U is (an index of an) X-recursively bounded recursive increasing tree such that the maximum number appearing in either ρ or β is less than the minimum number appearing in U (note that U may enumerate an $\mathfrak{M}_0$-finite tree).

(2) We say that $q = (\varepsilon_q, U_q)$ is *stronger* than $p = (\varepsilon_p, U_p)$ (written $p \geq q$) if and only if the following conditions hold.
 (i) Let $\varepsilon_p = (\rho_p, \beta_p)$ and $\varepsilon_q = (\rho_q, \beta_q)$. Then $\rho_p \preceq \rho_q$ and $\beta_p \preceq \beta_q$;
 (ii) $(\forall \sigma \in U_q)(\exists \tau \in U_p)(range(\sigma) \subseteq range(\tau))$.

Given a $\Sigma^0_1(X)$-formula ψ with a free set variable $\check{G}$ of the form $\exists s\varphi(s, X, \check{G})$, we say that p *red forces* ψ (written $p \Vdash_r \psi$) if

$$\mathfrak{M}_0[X] \models \exists s \leq \max(\rho_p)\varphi(s, X, \rho_p).$$

Define *blue forcing* similarly, except that ρ_p is replaced by β_p and $\Vdash_r$ by $\Vdash_b$. Also we say that p *red forces* $\neg\psi$ (written $p \Vdash_r \neg\psi$) if for all $\tau \in U_p$, for all $o \subseteq \tau$,

$$(*) \quad \mathfrak{M}_0[X] \models \forall s \leq \max(\tau)\neg\varphi(s, X, \rho_p * o).$$

Define $p \Vdash_b \neg\psi$ similarly, replacing ρ_p by β_p. [For consistency of notation with that for an S-disjunction, we use ι in place of o in $(*)$ above for $p \Vdash_b \neg\psi$.]

3. Preserving $BME + COH + B\Sigma^0_2$

Fix the ground model $\mathfrak{M}_0$ (Proposition 2.5). Throughout this section, we work in the Cantor space so that a string σ is an element of $2^{<M_0}$. Let X be a predicate on $\mathfrak{M}_0$ such that $\mathfrak{M}_0[X]$ satisfies $RCA_0 + B\Sigma^0_2$ and BME. In $\mathfrak{M}_0$, ω is a Σ^0_2-cut on which a cofinal Σ^0_2-function $g : \omega \to M_0$ is defined. The aim of this section is to present the construction of a G that solves an instance of COH so that $\mathfrak{M}_0[X, G]$ preserves $RCA_0 + BME + B\Sigma^0_2$.

The construction is based on the idea implemented in [4] with the new ingredient of preserving BME. Thus steps which are mere repetitions of the earlier construction will be omitted. Note that we do not need to build a topped model as this was carried out in [4] to establish Π^1_1-conservation of COH over $RCA + B\Sigma^0_2$. We will prove the following theorem.

Theorem 3.1. *For any R in $\mathfrak{M}_0[X]$, there is a $G \subset M_0$ which is R-cohesive such that the M_0-extension $\mathfrak{M}_0[X, G]$ is a $B\Sigma^0_2$-model of $RCA_0 + BME$.*

The proof consists of two parts. In the first part, we build within $\mathfrak{M}_0[X]$ an X'-recursive X'-recursively bounded tree T such that from each $\mathfrak{M}_0$-infinite path p on T one computes a set G_p which is generalized X-low in the sense that $(G_p \oplus X)' \equiv_T G_p \oplus X'$. Furthermore, G_p is R-cohesive, and $\mathfrak{M}_0[X, G_p]$ preserves BME.

Lemma 3.2 (Internal Forcing)**.** *There is an X'-recursive X'-recursively bounded tree T such that each $\mathfrak{M}_0[X]$-infinite path p on T yields a set G_p which is R-cohesive, generalized X-low and $\mathfrak{M}_0[X, G_p]$ satisfies BME.*

In the second part, we select a path p through T such that G_p preserves $B\Sigma^0_2$. This path p is constructed from *the outside* using the countability of $\mathfrak{M}_0[X]$.

Lemma 3.3 (External Forcing)**.** *There is an unbounded path p on T such that $\mathfrak{M}_0[X, G_p] \models B\Sigma^0_2$.*

Since the proof of Lemma 3.3 is essential the same as that of Lemma 3.2 in [4], we will omit it and devote the rest of this section to the proof of Lemma 3.2. To simplify notations, we only prove the case when $X = \emptyset$. The general case follows upon replacing $\emptyset'$ by X' and $\mathfrak{M}_0$ by $\mathfrak{M}_0[X]$. At the few places where extra care is required, we will make appropriate comments on the changes to be made.

To build the tree T, for each $e \in M_0$, we have the cohesive requirement P_e, generalized lowness requirement Q_e and the e-th instance BME to satisfy. For each path p on T and for each $G = G_p$, we have

- P_e: $G \cap R_e$ is $\mathfrak{M}_0$-finite or $G \cap \overline{R_e}$ is $\mathfrak{M}_0$-finite.
- Q_e: (Deciding the jump) There is an initial segment σ of p such that either $\Phi^{\sigma}_{e,|\sigma|}(e)\downarrow$ or for all $\tau \succ \sigma$, if $\tau \in T$ then $\Phi^{\tau}_{e}(e)\uparrow$. When such a σ is found, we say that "$e \in G'$ is decided by σ".

Remark. When $X \neq \emptyset$, Q_e takes the form: There is a string σ and a number n such that either $\Phi^{\sigma\oplus X\restriction n}_{e,|\sigma|}(e)\downarrow$ or for all $\tau \succ \sigma$ on T, for all $m > n$, $\Phi^{\tau\oplus X\restriction m}_{e}(e)\uparrow$. Note that this is decided by X'.

We organize the set of requirements P_e and Q_e into blocks D_n, $n \in \omega$. As the Σ^0_2-cut is ω, one may define $D_n = \{e : e \leq g(n)\}$. By Proposition 2.20, BME may be decomposed into ω many instances of the principle which can be preserved individually in ω steps in the course of the construction.

For any recursive subset Z of M_0, we can naturally associate with it a binary recursive tree T_Z so that $\sigma \in T_Z$ if and only if $\{i : \sigma(i) = 1\} \subseteq Z$. Note that when Z is $\mathfrak{M}_0$-infinite, then T_Z is recursively isomorphic to the perfect full binary tree $2^{<M_0}$. When Z is $\mathfrak{M}_0$-finite, T_Z has an initial segment isomorphic to the $\mathfrak{M}_0$-finite perfect tree $2^{|Z|}$ followed by $2^{|Z|}$ many isolated paths consisting of only extensions by 0. Note that the T_Z above has $\emptyset$ as its root. As we will see below, it is convenient to have a notion "T_Z on top of a string σ" or "T_Z with root σ", which is a tree consisting of the strings τ such that $\tau(n) = \sigma(n)$ if $n \leq |\sigma|$ and $\{i > |\sigma| : \tau(i) = 1\} \subseteq Z$. For a tree T and a node $\sigma \in T$, we use the notation $T[\sigma]$ to denote the subtree $\{\tau \in T : \tau$ is comparable with $\sigma\}$.

For $x \in M_0$, let the e-state of x be the $\mathfrak{M}_0$-finite binary string ν (in fact, $\nu(R, e, x)$) of length $e + 1$ such that for each $s \leq e$, $\nu(s) = 1$ if and only if $x \in R_s$. We make the following claims:

Claim 1. For each T_Z and e-state ν, there is a recursive subtree T_{Z_ν} such that $Z_\nu \subseteq Z$ and every element in Z_ν has e-state ν.
Proof of Claim 1. Let $Z_\nu = \{x \in Z : \nu(R, e, x) = \nu\}$ and T_{Z_ν} to be the tree associated with Z_ν. Note that Z_ν may be $\mathfrak{M}_0$-finite or even empty.

For Q_e, we have the same Claim 2 as in [4], whose proof we will not repeat.

Claim 2. There is a $\emptyset'$-recursive function $h : M_0 \times M_0 \to 2^{<M_0}$ with the following property: for each (canonical index of an) $\mathfrak{M}_0$-finite set $D \subset M_0$ and (an index of) a recursive tree S of the form T_Z for some recursive set Z, $\sigma = h(B, S)$ is a string on S which either decides "$e \in G'$" for all $e \in D$, or $q = \{\sigma * 0^s : s \in M_0\}$ is an isolated path on S.

Claim 3. For each recursive set Z and b-bounded k-iterated instance $\langle V_i, E_i\rangle_{1\leq i\leq k}$ of BME_k, there is a string $\sigma^* \in T_Z$ such that for any path p on T_Z extending σ^*, $\mathfrak{M}_0[G_p]$ satisfies the instance of BME. Furthermore such a σ^* may be computed using $\emptyset'$.

Proof of Claim 3. Given a recursive tree T_Z, let V_0 be T_Z. Define a monotone enumeration procedure E_0 as follows. E_0 is described in terms of $E_0(\sigma)$ with an $\mathfrak{M}_0$-finite string σ as the oracle. When applied to an infinite path λ, $E_0(\lambda)$ is naturally the union of $E_0(\sigma)$ for $\sigma \prec \lambda$. Moreover, when applied to a tree V (in particular V_0 here), $E_0(V)$ is a "forest", i.e., a union of trees where incomparable nodes on V may produce different $\mathfrak{M}_0$-finite trees. Given σ, the enumeration procedure $E_0(\sigma)$ can be viewed as selecting initial segments $\eta_m \prec \sigma$ one by one, so that $V_1(\eta_m)$ has exactly m many E_1-expansionary levels. More precisely, enumerate $\eta_0 = \emptyset$ as the root. Suppose that we have enumerated $\eta_m \prec \sigma$. Search for an η such that $\eta_m \prec \eta \prec \sigma$ and there is an $\mathfrak{M}_0$-finite subtree W_{m+1} of $V_1(\eta)$ whose leaves are either a dead end on $V_1(\eta_m)$ or has exactly $m+1$ many E_1-expansionary levels. It no such η exists, then $E_0(\sigma)$ stops at η_m. If such an η exists, let η_{m+1} be the shortest such η.

By definition, E_0 is monotone. To satisfy the given instance $\langle V_i, E_i\rangle_{1\leq i\leq k}$ of BME_k, we apply BME_{k+1} over the ground model and conclude that the E_0-expansionary levels on V_0 are bounded. Thus for some $\sigma^* \in T_Z$ and some number m^*, any p extending σ^* on T_Z will have exactly m^* many E_1-expansionary stages. Furthermore $\emptyset'$ is able to compute such a σ^* and m^*, by enumerating $g(n)$ and checking each node on T_Z of length $g(n)$ to see if there is an E_1-expansionary stage (the latter is a Σ^0_1-process). This completes the proof of Claim 3.

We now return to the proof of Lemma 3.2. We use $\emptyset'$ as oracle to construct a sequence $\langle T_n\rangle_{n\in\omega}$ of $\emptyset'$-recursively bounded recursive trees. Each recursive tree T_n can be decomposed into two parts: The first part is an $\mathfrak{M}_0$-finite tree, whose dead ends are listed as $\sigma_1, \ldots, \sigma_s$; the rest of T_n are the parts extending each σ_j, if σ_j is not a dead end of T_n then it is extended by a tree T_{Z_j} associated with a set $Z_j \subseteq M_0$. We refer to a tree like T_n as an *amalgamation tree.* Informally, an amalgamation tree is a "union" of $\mathfrak{M}_0$-finitely many subtrees each of which has a string σ_j as root and, if σ_j is not a dead end, is a tree of the form T_Z on top of σ_j, for some set Z.

In defining T_{n+1} from T_n, we do not redefine the strings σ_j. Instead, we use $\emptyset'$ to turn each of the perfect trees T_{Z_j} extending σ_j into a new amalgamation tree (as was done in the proofs of the three claims above). Therefore the $\mathfrak{M}_0$-finite tree described above as the first part of T_{n+1} is an end-extension of that of T_n. The union of those $\mathfrak{M}_0$-finite trees is the tree T desired in Lemma 3.2.

Step 0. Begin with the full binary tree T_0 in $\mathfrak{M}_0$.

Step $n+1$. Suppose that T_n is an amalgamation tree consisting of σ_j and T_{Z_j} for $1 \leq j \leq s$. We work on each σ_j to modify T_{Z_j} by applying the claims.

First we handle a block of cohesive requirements P_e for $e \in D_n$. For each $g(n)$-state ν, add an extension of σ_j with label ν (so that different ν's have different extensions), apply Claim 1 to obtain T_{Z_ν}, and "place it on top of" the node labelled ν. Note that by the same argument as in [4], $B\Sigma_2$ implies that at least one of the Z_ν's must be $\mathfrak{M}_0$-infinite.

Next consider a block of generalized low requirements Q_e for $e \in D_n$. On each T_{Z_ν}, search for a string η that decides $e \in G'$ for all $e \in D_n$. Some care has to be exercised to preempt the possibility that η consists only of 0's, for otherwise the set G eventually obtained would be an $\mathfrak{M}$-finite set and hence not cohesive. However, this can be taken care of quite easily by including a "1" in η. If Z_ν is not $\mathfrak{M}$-finite, one can always find a branching node τ on T_{Z_ν}. This is then followed by "cutting off" the left half of the tree above τ and moving the construction to nodes in T_{Z_ν} extending $\tau * 1$. If Z_ν is $\mathfrak{M}$-finite, then there is no need to consider the tree. More precisely, the procedure searches for the first branching point τ (if any) on T_{Z_ν}, stops extending any node extending $\tau * 0$ and applies Claim 2 to the tree $T_{Z_\nu}[\tau * 1]$ consisting of strings in T_{Z_ν} extending $\tau * 1$ to obtain $\eta = h(D_n, T_{Z_\nu}[\tau * 1])$. The tree $T_{Z_\nu}[\eta]$ consists of nodes extending η will be further trimmed to satisfy *BME*.

Finally let $\langle \hat{V}_i, \hat{E}_i \rangle_{1 \leq i \leq n+1}$ be the instance of *BME* that is the $g(n+1)$-bounded, $n+1$-iterated monotone enumeration in Proposition 2.20. For each $T_{Z_\nu}[\eta]$ constructed above, apply Claim 3 to obtain the σ^*, which we now denote as σ^*_ν, for $\langle \hat{V}_i, \hat{E}_i \rangle_{1 \leq i \leq n+1}$. Constructions from step $n+2$ onwards will only apply to the tree $T_{Z_\nu}[\sigma^*_\nu]$ consisting of nodes extending σ^*_ν.

This concludes Step $n+1$ of the construction. Note that the resulting tree is still an amalgamation tree.

We verify that T has the desired properties. First by construction there is at least one $\mathfrak{M}_0$-infinite path on T. Every $\mathfrak{M}_0$-infinite path p on T necessarily corresponds to an $\mathfrak{M}_0$-infinite set G_p. Clearly, every $\mathfrak{M}_0$-infinite path on T eventually has the same e-state, and so the cohesiveness requirements are satisfied. Let p be an $\mathfrak{M}_0$-infinite path on T and G_p be the corresponding $\mathfrak{M}_0$-infinite set. G_p is generalized low as argued in [4]. Moreover $M_0[G_p]$ satisfies *BME* by the proof of Claim 3, completing the proof of Lemma 3.2.

4. Σ^0_2-INDUCTION

Theorem 4.1. *Let* $\mathcal{M}_0[X] \models RCA_0 + BME + B\Sigma^0_2$. *Let* A *be* $\Delta^0_2(X)$. *Then there is an* $\mathfrak{M}_0[X]$*-infinite* $G \subseteq M_0$ *such that*

(1) $G \subseteq A$ *or* $G \subseteq \overline{A}$;
(2) $G' \leq_T X'$ *and hence* $\mathfrak{M}_0[X, G] \models RCA_0 + B\Sigma^0_2$;
(3) $\mathcal{M}_0[X, G] \models BME$.

The proof of Theorem 4.1 parallels those of Theorems 4.1 and 5.1 in [5], with a major difference deserving elaboration. We begin by retracing the steps taken to produce a model $\mathfrak{M}$ of $RCA_0 + SRT^2_2 + B\Sigma^0_2$ in Theorem 4.1 of [5]. The key idea was to arrange for the second order elements of $\mathfrak{M}$ to consist only of low sets. The construction of G was in fact $\emptyset''$-recursive, but the arithmetical saturation of the ground model $\mathfrak{M}_0$ allowed one to argue that the set G constructed was $\emptyset'$-recursive. This approach fails in the current setting. Since RT^2_2 is equivalent to $COH + SRT^2_2$ over RCA_0 (Proposition 2.10), the obvious strategy would be to build a $B\Sigma^0_2$-model $\mathfrak{M}$ of $RCA_0 + RT^2_2$ by successively satisfying instances of COH (as in Section 3) and D^2_2 (which is equivalent to SRT^2_2 over RCA_0 by Proposition 2.11). However, since the construction in Section 3 of an R-cohesive set X is highly noneffective, one cannot expect arithmetical saturation relative to X to automatically hold, i.e. it is not necessarily true that any subset of ω that is arithmetical relative to X is coded on ω. Thus constructing a set over $\mathfrak{M}_0[X]$ that solves an instance of D^2_2 following the steps in Theorem 4.1 of [5] only produces a G that is recursive in X''. There need not be a code available for reducing the complexity of G by one Turing jump relative to X, a step required to show that $B\Sigma^0_2$ is preserved relative to (X, G). Of course there is the additional issue of preserving BME at each stage to enable a successful iteration of the construction to generate a countable sequence of generic sets that will form the desired model.

We now describe in some detail how the desired reduction from X'' to X' should be achieved. We will view the X''-recursive construction as approximated by an X'-recursive construction. We satisfy an instance of D^2_2 over $\mathcal{M}_0[X]$ (a model of $RCA_0 + B\Sigma^0_2$) by an X'-recursive construction, and implement an X'-recursive construction involving finite injury and forcing so that for $j \le n$, the approximation $G[g(n)] \upharpoonright g(j)$ of $G \upharpoonright g(j)$ at stage n is defined. In the end, $G \upharpoonright g(j) = \lim_{n\to\omega} G[g(n)] \upharpoonright g(j)$. The outcome of the entire construction is compressed into a set that is $\Delta^0_2(X)$ on $\omega \times \omega \times 2$. Lemma 2.4 ensures that this set is coded on $\omega \times \omega \times 2$. This code replaces any reference to X and allows us to invoke arithmetical saturation of $\mathfrak{M}_0$ on the coded set, yielding another code that is needed to argue that $G \le_T X'$.

Let A be $\Delta^0_2(X)$. The construction is carried out in ω many stages. We define an X'-recursive sequence of (indices of) forcing conditions $\{p_{j,n}\}_{j\le n<\omega}$, where conditions are defined as in Definition 2.21, such that $p_{j,n} = \langle \varepsilon_{j,n}, U_{i,n}\rangle$, $\varepsilon_{j,n} = (\rho_{j,n}, \beta_{j,n})$ with $\rho_{j,n} \subseteq A$ and $\beta_{j,n} \subseteq \overline{A}$, and $p_{j+1,n} \le p_{j,n}$ for each n and j. Here $p_{j,n}$ is the X'-recursive approximation of the forcing condition p_j at stage n. For each j, $p_{j,n} \ne p_{j,n+1}$ for only finitely many n. In particular, $p_j = \lim_{n\to\omega} p_{j,n}$ is defined with U_j $\mathfrak{M}_0$-infinite. The generic set G will be an $\mathfrak{M}_0[X]$-infinite sequence that is the union of either $\{\rho_j\}_{j<\omega}$ or $\{\beta_j\}_{j<\omega}$.

Proof of Theorem 4.1. We begin by setting $p_{-1,n} = \langle (\emptyset,\emptyset), Id\rangle$, where Id is the set whose sth element is the number s. We regard Id as a tree whose elements are strings $\sigma \in M_0^{<M_0}$ so that $\sigma(s) = s$. Let $j_{-1} = \hat{i}_{-1} = 0$.

Stage $n+1$: Assume j_n and $\hat{i}_n \ge n$ are defined. The values j_n and $\hat{i}_n$ are computed by X' and $g(\hat{i}_n)$ is a stage where $U_{j,n}$ is not seen to be $\mathfrak{M}_0[X]$-finite for $j \le j_n$. Perform $g(\hat{i}_n + 1)$ steps of computation to search for the least $j \le n$, denoted j^*, such that $U_{j,n}$ is $\mathcal{M}_0[X]$-finite. Let $j^* = n+1$ otherwise. Assume by induction hypothesis that uniformly in X', $p_{j,n}$ is defined for all $j < j^*$, and that if $0 \le j < j^*$, then:

1. (1) $\varepsilon_{j,n} = (\rho_{j,n}, \beta_{j,n})$ and $\rho_{j,n} \subseteq A, \beta_{j,n} \subseteq \overline{A}$;
2. (2) There is a $c \in \{r, b\}$ such that for all $\Sigma^0_1(X)$-formulas ψ with parameters below $g(j)$, either $p_{j,n} \Vdash_c \psi$ or $p_{j,n} \Vdash_c \neg\psi$;
3. (3) For $k \le j$, let $BME_{k,j}$ denote BME_k relative to the predicate X restricted to the $g(j)$-bounded, k-iterated monotone enumerations, with indices below $g(j)$. Then $BME_{k,j}$ has been ensured with the following additional conclusion: For any instance $\langle V_i, E_i\rangle_{1\le i\le k}$ of $BME_{k,j}$, for any $\mathfrak{M}_0$-finite subset Y of a string in $U_{j,n}$, no E_1-expansionary level in V_1 relative to $(X, \rho_{j,n} * Y)$ (or relative to $(X, \beta_{j,n} * Y)$, depending on whether $U_{j,n}$ was obtained previously through skipping or thinning, see below) is enumerated unless it was already enumerated relative to $(X, \rho_{j,n})$ (respectively, $(X, \beta_{j,n})$).

By Proposition 2.20, we may assume that (3) was achieved by working with the amalgamated monotone enumeration $\langle \hat{V}_j, \hat{E}_i\rangle_{1\le i\le j}$.

Let $B_{X,j} = B_{X,r,j} = B_{X,b,j}$ be the collection of $\Sigma^0_1(X)$-formulas with free variable $\check{G}$ and parameters less than $g(j)$. Let $\psi(\ell, X, \check{G})$ be the $\Sigma^0_1(X)$-formula statting that there exists an s where ℓ-many $\hat{E}_1$-expansionary levels are enumerated at stage s.

Let

$$\hat{B}_{X,j} = B_{X,j} \cup \{\psi(\ell, X, \check{G}) : \ell \geq 1\}.$$

Define $\hat{B}_{X,r,j}$ and $\hat{B}_{X,b,j}$ similarly. Let $\hat{V}_0 = U_{j^*-1,n}$ and let $\hat{E}_0$ be (an index of the) exit tree (see §2.3) over the pair of preconditions $(\rho_{j^*-1,n}, \beta_{j^*-1,n})$ for formulas in $\hat{B}_{X,j^*}$ (hence also formulas that enumerate $\hat{E}_1$-expansionary levels) along any $\sigma \in \hat{V}_0$. If $U_{j^*-1,n}$ is $\mathfrak{M}_0$-infinite, then Lemma 2.13 and BME in $\mathfrak{M}_0[X]$ yields a maximal pair (ρ, β) of exits in $\hat{E}_0$ such that $\rho \subset A$, $\beta \subset \overline{A}$, $\rho_{j^*-1} \prec\prec \rho$ and $\beta_{j^*-1} \prec \beta$. However, the tree enumerated as $U_{j^*-1,n}$ may be $\mathfrak{M}_0$-finite, so that $\hat{E}_0$'s enumeration may terminate prematurely before a "genuinely" maximal pair (ρ, β) is enumerated to satisfy (2). We resolve this difficulty by a series of approximations to arrive at the "correct U".

Enumerate, recursively in X, $\hat{E}_0[\sigma]$ for each $\sigma \in U_{j^*-1,n}$. This is achieved by enumerating S-disjunctions along σ for formulas in $\hat{B}_{X,j^*}$. Note that X' is able to decide if there is a new S-disjunction to be enumerated, and hence if there will be a new $\hat{E}_0$-expansionary level to be enumerated on $U_{j^*-1,n}$. Then there is a step $s^* \geq g(\hat{i}_n + 1)$ where one of the following holds:

(a) Every string in $U_{j^*-1,n}$ has length bounded by s^* and hence $U_{j^*-1,n}$ is $\mathcal{M}_0[X]$-finite;
(b) There is a string in $U_{j^*-1,n}$ of length greater than s^* and by BME_{k,j^*} in $\mathfrak{M}_0[X]$, no new $\hat{E}_0$-expansionary level is enumerated on $U_{j^*-1,n}$ after s^*.

We may require that in the computation above, s^* is in the range of g. Now if (a) holds, return to the beginning of the construction at stage $n+1$ and reset j^* to be the least $j \leq n$ for which $U_{j,n}$ is $\mathcal{M}_0[X]$-finite after s^* steps of computation and repeat the above to arrive at a new s^* in the range of g where either (a) or (b) holds, using X' as oracle. If (a) holds at (the new) step s^*, the process is repeated one more time. This cycle of resetting j^* and s^* must, however, end after i rounds, for some $i < \omega$, since each time j^* is reset, it assumes a smaller standard natural number, and $U_{0,n}$ is $\mathcal{M}_0[X]$-infinite. Let i_{n+1} be the least such i and $\hat{i}_{n+1}$ be chosen such that $g(\hat{i}_{n+1})$ is the step s^* where i_{n+1} is computed by X'. At step $g(\hat{i}_{n+1})$, (b) holds.

Let j_{n+1} be the j^* at step $g(\hat{i}_{n+1})$. We will define $U_{j_{n+1},n+1}$ through a series of "T_a analysis" as follows. For each $\sigma \in U_{j_{n+1}-1,n}$, let $\#_\delta\sigma$ be the number of S-disjunctions δ enumerated in $|\sigma|$ steps along σ for formulas in $\hat{B}_{X,j_{n+1}}$. Let

$$T^\delta_{a,j_{n+1}} = \{\sigma \in U_{j_{n+1}-1,n} : \#_\delta\sigma \leq a\}.$$

Condition (b) guarantees that there is a largest a, denoted $a_{\delta,j_{n+1}}$ and computable by X', for which the supremum of $\{|\sigma| : \#_\delta\sigma \leq a\}$ is bounded in $U_{j_{n+1}-1,n}$. Let $\sigma_{\delta,j_{n+1}}$ be the (canonically least) string σ such that $\#_\delta\sigma = a_{\delta,j_{n+1}}$ and for every $\tau \in U_{j_{n+1}-1,n}$, there is a $\sigma \in U_{j_{n+1}-1,n}$ extending $\sigma_{\delta,j_{n+1}}$ satisfying $|\sigma| = |\tau|$ and $\#_\delta\sigma = a_{\delta,j_{n+1}}$. The objective here is to select $\sigma_{\delta,j_{n+1}}$ so that the subtree of $T^\delta_{a_{\delta,j_{n+1}}+1,j_{n+1}}$ extending $\sigma_{\delta,j_{n+1}}$ is unbounded in the event that $T^\delta_{a_{\delta,j_{n+1}}+1,j_{n+1}}$ is unbounded. Let

$$\hat{U}_{j_{n+1}} = \{\sigma \in U_{j_{n+1}-1,n} : \sigma_{\delta,j_{n+1}} \prec \sigma \wedge \#_\delta\sigma = a_{\delta,j_{n+1}}\}.$$

For $j < j_{n+1}$, let $p_{j,n+1} = p_{j,n}$. Let $\varepsilon_{j_{n+1},n+1} = (\rho_{j_{n+1},n+1}, \beta_{j_{n+1},n+1})$ be a maximal string in $\hat{E}_0$ so that $\rho_{j_{n+1},n+1} \subseteq A$ and $\beta_{j_{n+1},n+1} \subseteq \overline{A}$.

For each $\sigma \in \hat{U}_{j_{n+1}}$, compute $|\sigma|$ steps for an increasing sequence $\{o_s\}$ of blobs such that $\min o_{s+1} > \max o_s$, $\min o_s > \max \rho_{j_{n+1},n+1}$ and $\psi(X, \rho_{j_{n+1},n+1}{}^\frown o_s)$ holds for some $\psi \in \hat{B}_{X,j_{n+1}}(\varepsilon_{j_{n+1},n+1})$. Let $\#_o\sigma$ be the number of blobs so enumerated and $T^o_{a,j_{n+1}} = \{\sigma : \#_o\sigma \leq a\}$. There are two cases to consider.

Case 1 (Skipping). At step $\hat{i}_{n+1}$, there is a largest a, denoted $a_{o,j_{n+1}}$, for which the supremum of $\{|\sigma| : \#_o\sigma \leq a\}$ is bounded in $\hat{U}_{j_{n+1}}$. Let $U_{j_{n+1},n+1}$ be (the canonical index of) the X-recursively enumerated increasing recursive tree consisting of strings $\tau \succ \sigma_{o,j_{n+1}}$ from $\hat{U}_{j_{n+1}}$ satisfying $\#_o\tau = a$. $U_{j_{n+1},n+1}$ may be viewed as the subtree of $\hat{U}_{j_{n+1}}$ restricted to the strings τ extending $\sigma_{o,j_{n+1}}$ with $\#_o\tau = a_{o,j_{n+1}}$. Let $p_{j_{n+1},n+1} = \langle \varepsilon_{j_{n+1},n+1}, U_{j_{n+1},n+1}\rangle$.

Case 2 (Thinning). At step $\hat{i}_{n+1}$, there is no largest a where the supremum of $\{|\sigma| : \#_o\sigma \leq a\}$ is bounded in $\hat{U}_{j_{n+1}}$.

We do thinning of $\hat{U}_{j_{n+1}}$ by following the construction in §4.6 of [5] (conditions (1) and (2) before Lemma 4.8). This is carried out by using the blobs o enumerated uniformly along strings σ in $\hat{U}_{j_{n+1}}$ to determine an index of an X-recursively bounded increasing X-recursive tree S enumerated as follows.

Stage -1. Let $S[-1]$ be $\emptyset$ (the root).

Stage $v+1$. Suppose that $S[v]$ is the amalgamation of the choice functions on blobs enumerated by $\sigma \in T^o_v$. In other words, $S[v]$ satisfies the following conditions:

(i) If τ is a node of $S[v]$ with length v, then there is a node $\sigma \in T^o_{v,j_{n+1}}$ such that τ is a choice function on blobs enumerated along σ.

(ii) If σ is a node in $T^o_{v,j_{n+1}}$ which enumerates v many blobs along σ, say $\vec{o}$, and f is a choice function on $\vec{o}$, then there is a unique maximal branch $\tau \in S[v]$ such that $\tau = f$.

To define $S[v+1]$, examine all the maximal branches σ in $T^o_{v+1,j_{n+1}}$ such that $\#_o(\sigma) = v+1$. Let the blob-sequence enumerated along σ be $\vec{o}$. (Note that it is not necessary to consider other maximal branches in $T^o_{v+1,j_{n+1}}$ as these must be dead ends in $\hat{U}_{j_{n+1}}$.) Now each choice function f on $\vec{o}$ necessarily extends some choice function f' on the first v blobs of $\vec{o}$.. By condition (ii) for v, f' is τ for some unique $\tau \in S[v]$. Enumerate f into $S[v+1]$ extending τ, provided f has not been enumerated into $S[v+1]$ earlier. (different branches on $\hat{U}_{j_{n+1}}$ may enumerate identical blob-sequences.) This ensures that all choice functions on blob-sequences of length $v+1$ along any node on T^o_{v+1} are included. It follows that (i) and (ii) remain valid for $S[v+1]$.

Let

$$U_{j_{n+1},n+1} = \{\sigma \in S : (\forall t < \max(\sigma))(\forall \iota \subseteq \sigma)(\forall e \in B_{X,j_{n+1},b}(\varepsilon_{j_{n+1},n+1}))\ \mathfrak{M}_0[X] \models \neg\psi_e(t, X, \beta_{n+1} * \iota)\}$$

(we write ψ_e to be the eth $\Sigma^0_1(X)$-formula with free variable $\check{G}$). Then $U_{j_{n+1},n+1}$ is an X-recursively bounded recursive increasing tree as it is generated from S. Pursuing the same argument as that in [5] for U_{n+1} (before Lemma 4.8), one concludes that $U_{j_{n+1},n+1}$ is $\mathfrak{M}_0[X]$-infinite if and only if S is. Let $p_{j_{n+1},n+1} = \langle \varepsilon_{j_{n+1},n+1}, U_{j_{n+1},n+1}\rangle$.

This completes the construction at stage $n+1$. Note that the entire construction is recursive in X'.

Claim 1. For each j, $p_j = \lim_n p_{j.m} = \langle \varepsilon_j, U_j \rangle$ exits. Then U_j is $\mathfrak{N}[X]$-infinite and $p_j \geq p_{j+1}$.

Proof of Claim 1. This is proved by induction. By construction, $U_{-1,n} = Id$ for all $n \geq 0$ and is $\mathfrak{M}[X]$-infinite. Hence $p_{-1} = \langle (\emptyset, \emptyset), Id \rangle$.

Assume that $p_{j',n} = p_{j',n_j}$ for all $j' \leq j$ and $n \geq n_j$. Then $U_{j,n}$ is $\mathfrak{M}_0[X]$-infinite for such j and n., If $j_{n+1} > j+1$ for all $n \geq n_j$, then $U_{j+1,n}$ is $\mathfrak{M}_0$-infinite for such n's and is equal to U_{j+1} which was obtained at stage $n_j + 1$ through thinning. If $j_{n+1} = j+1$ for some $n \geq n_j$, then $U_{j+1,n}$ is $\mathfrak{M}_0[X]$-finite as computed by X' at stage $n+1$ under Case 1. At stage $n+1$, we perform skipping of $U_{j+1,n}$. Then $U_{j+1,n'} = U_{j+1,n+1}$, and hence $p_{j+1,n'} = p_{j+1,n+1}$ for all $n' \geq n+1$.

Given p_j, the choice of $\varepsilon_j = (\rho_j, \beta_j)$ ensures that $\rho_j \subseteq A$ and $\beta_j \subseteq \overline{A}$. Furthermore, there is a $c \in \{r, b\}$ such that $p_j \Vdash_c \psi$ or $p_j \Vdash_c \neg\psi$ for each $\psi \in B_{X,j}$. This yields (1) and (2) for p_j. If $p_{j',n'} = p_{j'}$ for all $n' \geq n_j$ and $j' \leq j$, then each $U_{j',n}$ for $j' \leq j$ is $\mathfrak{M}[X]$-infinite, and one argues as in Theorem 5.1 of [5] that (3) holds for p_j. This gives

Claim 2. For each j, p_j satisfies (1), (2) and (3).

If the range of $\bigcup_j \rho_j$ is $\mathfrak{M}_0[X]$-infinite, let $G = \bigcup_j \rho_j$. Otherwise, let G be $\bigcup_j \beta_j$. The next claim reduces the complexity of G and is the final step in establishing our main result.

Claim 3. $G \leq_T X'$.

Proof of Claim 3. Let $Y = \{(j,n) : j_n = j\}$. Then $Y \subset \omega \times \omega$ is $\Delta^0_2(X)$. Since $\mathfrak{M}[X] \models B\Sigma^0_2$, by Lemma 2.4 there is an $\mathfrak{M}_0[X]$-finite (hence $\mathfrak{M}_0$-finite) set $\hat{Y}$ such that $\hat{Y} \cap (\omega \times \omega) = Y$. For $j \in \omega$, define $h(j) = \max\{n : n \in \omega \wedge j = j_n\}$. Then $h \subset \hat{Y} \cap (\omega \times \omega)$ is coded on $\omega \times \omega$ since it is definable from $\hat{Y}$ over the arithmetically saturated structure $\mathfrak{M}_0$.

Let $\hat{h}$ be $\mathfrak{M}_0$-finite such that $\hat{h} \cap (\omega \times \omega) = h$. Then for each j, $p_{j,n} = p_j$ whenever $n > \hat{h}(j)$. Assume that $G = \bigcup_j \rho_j$ (the argument for $G = \bigcup_j \beta_j$ is the same). Then $\rho_j = \rho_{j,\hat{h}(j)+1}$. Since the map $j \mapsto \rho_{j,\hat{h}(j)+1}$ is $\Delta^0_2(X)$, we have $G \leq_T X'$. In fact $G' \leq_T X'$ since the Σ^0_1-theory of G is decidable by X'.

Thus $G \subseteq A$ or $G \subseteq \overline{A}$ and is $\mathfrak{M}[X]$-infinite. Furthermore, $\mathfrak{M}[X,G]$ is a model of $RCA_0 + BME$ and preserves $B\Sigma^0_2$. This completes the proof of Theorem 4.1. □

Corollary 4.2. $RCA_0 + RT^2_2 \not\vdash I\Sigma^0_2$.

Proof. We prove this by applying alternately Theorem 3.1 and Theorem 4.1. Let $\mathfrak{M}_0$ be as before. Since $\mathfrak{M}_0$ is countable, we can successively define M_0-extensions $\mathfrak{M}_i$ $(1 \leq i < \omega)$ such that $\mathfrak{M}_i = \mathfrak{M}_0[G_1, \ldots, G_i] \models RCA_0 + BME + B\Sigma^0_2$ and each $\mathfrak{M}_i$ resolves an instance of COH or D^2_2 with parameters in some $\mathfrak{M}_j$ where $j < i$. Furthermore, each instance of COH and D^2_2 with parameters in an $\mathfrak{M}_j$ is resolved in some $\mathfrak{M}_i$ where $i > j$. Let $\mathfrak{M} = \bigcup_i \mathfrak{M}_i$. Then $\mathfrak{M} \models RCA_0 + RT^2_2 + \neg I\Sigma^0_2$.

□

5. Questions

We end this paper with three questions.

The proof of Corollary 4.2 uses a $B\Sigma^0_2$ model that satisfies BME. It is unlikely that this is true for all $B\Sigma^0_2$ models.

Question 5.1. *Is $RCA_0 + RT^2_2 + B\Sigma^0_2$ Π^1_1-conservative over $B\Sigma^0_2$?*

Question 5.2. *Does $RCA_0 + RT^2_2 + B\Sigma^0_2$ imply BME?*

The work in [5] and this paper was motivated by the original question of separating SRT^2_2 from RT^2_2. That question remains open, for ω-models and more generally models of $I\Sigma^0_2$. It is made even more interesting in view of what one knows today.

Question 5.3. *Is there an ω-model of Peano arithmetic in which SRT^2_2 but not RT^2_2 holds? More generally, does $RCA_0 + SRT^2_2 + I\Sigma^0_2$ imply RT^2_2?*

References

[1] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.

[2] C. T. Chong, Steffen Lempp, and Yue Yang. On the role of the collection principle for Σ^0_2-formulas in second-order reverse mathematics. *Proc. Amer. Math. Soc.*, 138(3):1093–1100, 2010.

[3] C. T. Chong and K. J. Mourad. The degree of a Σ_n cut. *Ann. Pure Appl. Logic*, 48(3):227–235, 1990.

[4] C. T. Chong, Theodore A. Slaman, and Yue Yang. Π^1_1-conservation of combinatorial principles weaker than Ramsey's theorem for pairs. *Advances in Mathematics*, 230:1060–1077, 2012.

[5] C. T. Chong, Theodore A. Slaman, and Yue Yang. The metamathematics of Stable Ramsey's Theorem for Pairs. *J. Amer. Math. Soc.*, 27(3):863–892, 2014.

[6] J. L. Hirst. *Combinatorics in Subsystems of Second Order Arithmetic*. PhD thesis, The Pennsylvania State University, 1987.

[7] Carl G. Jockusch, Jr. Ramsey's theorem and recursion theory. *J. Symbolic Logic*, 37:268–280, 1972.

[8] Kenneth McAloon. Completeness theorems, incompleteness theorems and models of arithmetic. *Trans. Amer. Math. Soc.*, 239:253–277, 1978.

[9] Theodore A. Slaman. Σ_n-bounding and Δ_n-induction. *Proc. Amer. Math. Soc.*, 132(8):2449–2456 (electronic), 2004.

Department of Mathematics, National University of Singapore, Singapore 119076
E-mail address: chongct@math.nus.edu.sg

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840 USA
E-mail address: slaman@math.berkeley.edu

Department of Mathematics, National University of Singapore, Singapore 119076
E-mail address: matyangy@nus.edu.sg