Π_1^1-CONSERVATION OF COMBINATORIAL PRINCIPLES WEAKER THAN RAMSEY'S THEOREM FOR PAIRS

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ABSTRACT. We study combinatorial principles weaker than Ramsey's theorem for pairs over the system RCA_0 (Recursive Comprehension Axiom) with Σ_2^0-bounding. It is shown that the principles of Cohesiveness (COH), Ascending and Descending Sequence (ADS), and Chain/Antichain (CAC) are all Π_1^1-conservative over Σ_2^0-bounding. In particular, none of these principles proves Σ_2^0-induction.

Key words. Reverse mathematics, Π_1^1-conservation, RCA_0, Σ_2^0-bounding, combinatorial principles, Cohesiveness, Ascending/descending sequence, Chain/antichain.

1. Introduction

Our point of departure is the system RCA_0 which we take as our base theory throughout this paper. RCA_0 consists of the usual first-order axioms for arithmetic operations and Σ_1^0-induction relative to parameters, together with the second-order recursive comprehension scheme $\exists X[(\forall x(x \in X \leftrightarrow \varphi(x))]$, for each Δ_1^0-formula φ (also with parameters). Fix $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ to be a model of RCA_0, where $\mathbb{X}$ is the collection of subsets of M in $\mathcal{M}$. Ramsey's Theorem for pairs (RT_2^2) states that any partition in $\mathcal{M}$ of the two-element sets $\{x, y\}$ of M into two colors has an infinite monochromatic subset, i.e. an infinite $A \in \mathbb{X}$ all of whose two-element subsets have the same color. This set A is said to be homogeneous for the coloring. It is known that RT_2^2 is not provable in RCA_0. The strength of RT_2^2 in the context of subsystems of second order arithmetic has been a subject of major interest in reverse mathematics over the past several decades.

We thank Wei Wang for many useful discussions on the subject matter of this paper and Richard Shore for comments and suggestions for improvements on preliminary of our manuscript. Chong's research was partially supported by NUS research grant WBS 146-000-054-123. Slaman's research was partially supported by NSF award DMS-1001551 and by the John Templeton Foundation. Yang's research was partially supported by NUS research grant WBS 146-000-114-112.

Closely related to RT^2_2, and intuitively a more controlled coloring scheme, is stable Ramsey's Theorem for pairs (SRT^2_2): If for any $x \in M$, all but finitely many $\{x, y\}$'s have the same color, then there is an infinite homogeneous set in $\mathcal{M}$. SRT^2_2 is also known to be unprovable from RCA_0. The proof-theoretic strength of these two combinatorial principles has been investigated by various authors. Cholak, Jockusch and Slaman [1] showed that SRT^2_2, hence RT^2_2 as first established by Hirst [7], implies the Σ^0_2-bounding principle $B\Sigma^0_2$ (an induction scheme whose strength is known to lie strictly between Σ^0_1 and Σ^0_2-induction [8]), and that RT^2_2 is Π^1_1-conservative over RCA_0 together with the Σ^0_2-induction $\mathrm{I}\Sigma^0_2$, i.e. any Π^1_1-statement that is provable in $\mathsf{RT}^2_2 + \mathsf{RCA}_0 + I\Sigma^0_2$ is already provable in the system $\mathsf{RCA}_0 + I\Sigma^0_2$. It follows immediately that any subsystem of $\mathsf{RT}^2_2 + \mathsf{RCA}_0 + I\Sigma^0_2$ (such as replacing RT^2_2 by SRT^2_2) is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$.

There are several outstanding open problems relating to RT^2_2 and SRT^2_2, which provided the motivation for the problems studied in this paper. We list three of these: (1) Whether over RCA_0, RT^2_2 is strictly stronger than SRT^2_2; (2) whether RT^2_2 or even SRT^2_2 proves $I\Sigma^0_2$, given that they already imply $B\Sigma^0_2$, and (3) whether RT^2_2, or even SRT^2_2, is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$.

While these questions remain unsolved, similar or related questions for principles weaker than RT^2_2 or SRT^2_2 have been studied with some degree of success. First of all, Cholak, Jockusch and Slaman [1] introduced the principle COH and showed the equivalence of RT^2_2 with $\mathsf{COH}+\mathsf{SRT}^2_2$ over the system RCA_0. COH states that every array coded in $\mathcal{M}$ has a set in the model cohesive for the array (see §3 for the definition). Since COH is provable from RT^2_2, $\mathsf{COH}+I\Sigma^0_2$ is Π^1_1-conservative over $I\Sigma^0_2$. Secondly, Hirschfeldt and Shore [5] introduced two principles which they demonstrated to be strictly weaker than RT^2_2: The Chain and Antichain principle (CAC), which states that every infinite partially ordered set coded in $\mathcal{M}$ has an infinite chain or antichain in $\mathcal{M}$, and the Ascending or Descending Sequence principle (ADS), which asserts that every infinite linearly ordered set in $\mathcal{M}$ has an infinite ascending or descending sequence in $\mathcal{M}$. It is known classically that CAC implies ADS. In [5] the authors also introduced the stable versions of these two principles, denoted respectively as SCAC and SADS, and showed them to be strictly weaker than $\mathsf{CAC}/\mathsf{ADS}$. It follows that all of these systems weaker than RT^2_2 and/or SRT^2_2 are Π^1_1-conservative over $I\Sigma^0_2$ under the base theory RCA_0. On the other hand, a recent result of Chong, Lempp and Yang [2] implies that SADS, hence CAC, ADS and

SCAC, proves $B\Sigma^0_2$ over RCA_0 strengthening earlier results in [7] and [1] for RT^2_2 and SRT^2_2.

The aim of our work is to study the strength of $\mathsf{COH}, \mathsf{CAC}$ and ADS, as well as their stable versions, under (the strongest, as it turns out) first-order induction assumption provable by these principles. According to Slaman [10], $B\Sigma^0_n$ is equivalent to Δ^0_n-induction $I\Delta^0_n$ for all $n \geq 1$ (for $n = 1$, the proof requires the totality of the exponentiation function). Since $I\Delta^0_2$ lies strictly between Σ^0_1 and Σ^0_2-induction in proof-theoretic strength (see §2 below), one gets a sense of the logical link between a second order statement, such as CAC or ADS, and first order induction. In view of what was noted in the previous paragraph, we are in fact considering the combinatorial principles in the context of models of $\mathsf{RCA}_0 + B\Sigma^0_2$ where $I\Sigma^0_2$ fails. Here an interesting and technically challenging picture, requiring an analysis quite different from the situation where $I\Sigma^0_2$ is available, presents itself (see a discussion of the issues involved in [5]).

In general, the absence of Σ^0_2-induction in a model entails the existence of a cut in the model that is Σ^0_2-definable, and with it the fundamental task of ensuring, arising from a $\emptyset'$-recursive construction, a uniform bound for $\emptyset'$-recursive functions defined over (bounded) initial segments of the model. Of course there is no guarantee that this task is achievable every time. In this paper we consider a few constructions that do. The main theme, as the reader will observe, concerns the existence of extensions of a model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ to one that satisfies, additionally, an instance of the combinatorial principle being considered, by adjoining an appropriate subset of M. There are two conditions to meet: The subset has to be a solution to the instance of the principle (for example, one that is cohesive for a given array) and has to preserve $\mathsf{RCA}_0 + B\Sigma^0_2$ in the resulting extension. These two conditions are often conflicting requirements. The construction of a cohesive set, again to use this as an example, in the classical setting is known to be at best low$_2$ (see [11] and [1]) and does not adapt automatically to a model without $I\Sigma^0_2$. To demand the resulting cohesive set to preserve $B\Sigma^0_2$ introduces additional twist to the construction.

Our solution is to apply a two-step construction. Firstly through "internal forcing" (i.e. within the model) we define a $\emptyset'$-recursive tree in which every unbounded path is generalized low and has the cohesive property. Then working from "the outside", an external forcing operation is performed on the tree to obtain a path that preserves $B\Sigma^0_2$.

For ADS and CAC, we begin with their stable versions SADS and SCAC, where we build extensions of models of $\mathsf{RCA}_0 + B\Sigma^0_2$ to satisfy in addition an instance of each of these principles, by adjoining appropriate subsets of M. It turns out that via "internal forcing" there are solutions that are low in the model (relative to parameters), thus preserving $B\Sigma^0_2$. Coupled with the construction of cohesive sets discussed above, one gets solutions to instances of ADS and CAC. The approach answers questions (2) and (3) raised earlier, for ADS and CAC. It was shown in [5] that over RCA_0, CAC is strictly stronger than SCAC and ADS is strictly stronger than SADS. Our extension theorems imply that ADS and CAC, hence their stable versions, do not prove $I\Sigma^0_2$ over RCA_0. Furthermore, each of these principles is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$.

The organization of this paper is as follows: In §2 we summarize the basic notions and mathematical facts that will be used for the rest of the paper. In §3 we show that COH is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$ (Corollary 3.1). In §4 we show that ADS, and hence SADS, are similarly Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$ (Corollaries 4.2 and 4.3). Extending the results of §§3 and 4, we show in §5 that the same conclusions hold for CAC and SCAC (Corollaries 5.1 and 5.2). The paper concludes with a general question regarding combinatorial principles and the preservation of $B\Sigma^0_2$.

2. Preliminaries

We recall the basic notions and results that will be referred to in this paper.

A model $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ is a structure in the language of second order arithmetic. Let P^- denote the standard Peano axioms without mathematical induction. Let $I\Sigma^0_n$ denote the induction scheme for Σ^0_n formulas (with number and set variables), where $n \geq 0$. All models $\mathcal{M}$ considered in this paper satisfy $P^- + I\Sigma^0_1$. A bounded set in $\mathcal{M}$ is $\mathcal{M}$-finite if it is coded in $\mathcal{M}$, i.e., has a Gödel number in $\mathcal{M}$. Otherwise it is called $\mathcal{M}$-infinite. An unbounded set in $\mathcal{M}$ is necessarily $\mathcal{M}$-infinite, although the converse is not always true. It is known (Kirby and Paris [8]) that $I\Sigma^0_n$ is equivalent to the assertion that every Σ^0_n-definable set has a least element (although the result was only proved for models of first order theories, it extends to second order theories with a similar argument). We will use this fact implicitly throughout the paper.

Let $B\Sigma^0_n$ denote the scheme which states (over the base theory $B\Sigma_{n-1}$) that every Σ^0_n-definable (possibly with set or number parameters) function maps an $\mathcal{M}$-finite set onto an $\mathcal{M}$-finite set. In [8] it was shown that for all $n \geq 0$,

$$\cdots \to I\Sigma^0_{n+1} \to B\Sigma^0_{n+1} \to I\Sigma^0_n \to B\Sigma^0_n \to \cdots ,$$

and hence these induction schema generate a hierarchy of theories of increasing strength.

If $\mathcal{M}$ is a model of $B\Sigma^0_n$ or $I\Sigma^0_n$, then $\mathcal{M}$ may be viewed as a model of computation with restricted inductive power. For $n \geq 1$, there is a well-developed theory of computation. In particular, one may define notions of computability and Turing reducibility in $\mathcal{M}$. Thus, a set or a function is recursively (computably) enumerable if and only if it is Σ_1 definable (with no set parameters) over $\mathcal{M}$. It is recursive (computable) if both the set and its complement are recursively enumerable. If X and Y are subsets of M, then $X \leq_T Y$ ("X is *recursive in* Y") if there is an e such that for any $x \in M$, there exist $\mathcal{M}$-finite sets $P \subset Y$ and $N \subset \overline{Y}$ satisfying

$$x \in X \leftrightarrow \langle x, 1, P, N\rangle \in \Phi_e$$

and

$$x \in \overline{X} \leftrightarrow \langle x, 0, P, N\rangle \in \Phi_e,$$

where Φ_e is the eth r.e. set of quadruples. The following fact will be used implicitly throughout the paper.

Proposition 2.1. *If $\mathcal{M} \models I\Sigma^0_n$, then every bounded $\Sigma^0_n(\mathcal{M})$ set is $\mathcal{M}$-finite.*

A cut $I \subset M$ is a set that is closed downwards as well as under the successor function. I is a Σ^0_n-cut if it is Σ^0_n-definable over $\mathcal{M}$. It is known that $\mathcal{M} \models I\Sigma^0_n$ if and only if there is no proper (i.e. nonempty and bounded) Σ^0_n-cut. We consider only proper Σ^0_n-cuts in this paper.

If $\mathcal{M} \models B\Sigma^0_n$ but not $I\Sigma^0_n$, we call it a $B\Sigma^0_n$-model. In this case, there is a Σ^0_n-function mapping a Σ^0_n-cut cofinally into M.

The second order theory RCA_0 consists of the axiom system $P^- + I\Sigma^0_1$ (relative to parameters), and the recursive comprehension axiom which states that for any model $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ of the theory, every set Δ^0_1-definable over $\mathcal{M}$ (possibly with number and set parameters) is in $\mathbb{X}$ (see Simpson [9] for an introduction of the subject of reverse mathematics in general and RCA_0 in particular). Every model of RCA_0 is closed under Turing reducibility (meaning that if $\mathcal{M} \models \mathsf{RCA}_0$ and $A \subset M$ is in $\mathbb{X}$, then every set recursive in A is also in $\mathbb{X}$). If $X \subset M$, then $X \in \mathcal{M}$ is intended to mean $X \in \mathbb{X}$.

Given a model $\mathcal{M}$ of RCA_0, and $A \subset M$, let $\mathcal{M}[A]$ denote the structure generated from A over $\mathcal{M}$ by closing under functions recursive in A, with parameters from $\mathcal{M}$.

The following captures the essence of coding in $B\Sigma^0_n$-models of RCA_0. See [3] for details concerning models of $P^- + B\Sigma_n$. The generalization to second order theories such as RCA_0 is straightforward.

Definition 2.1. *Let A be a subset of M, where $\mathcal{M} \models \mathsf{RCA}_0$. A set $X \subseteq A$ is* coded on A *if there is an $\mathcal{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.*

Definition 2.2. *Let A be a subset of $\mathcal{M}$. We say that a set X is Δ^0_n* on A *if both $A \cap X$ and $A \cap \overline{X}$ are $\Sigma^0_n(\mathcal{M})$.*

Lemma 2.1 (Chong and Mourad [3])**.** *Let $\mathcal{M}$ be a model of $\mathsf{RCA}_0 + B\Sigma^0_n$ ($n \geq 2$) and let $A \subset M$. Then every bounded set that is Δ^0_n on A is coded on A.*

We next turn our attention to trees. By definition a tree T is a collection of $\mathcal{M}$-finite functions from an initial segment of M into M closed under pairwise intersection. T is *downward closed* if every substring of a member of T is a member of T. If T is downward closed, then it is recursively bounded if there is a recursive function f such that for all $x \in M$, there are at most $f(x)$ many elements in T of length x.

Definition 2.3. *Let $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ and $\mathcal{M} \subseteq \mathcal{M}^*$ be models of RCA_0. Then $\mathcal{M}^*$ is an M-extension of $\mathcal{M}$ if $M = M^*$, i.e. only subsets of M are added to $\mathcal{M}^*$.*

In the study of conservation results, it is convenient to have the notion of a *topped* model, which was first introduced by Cholak, Jockusch and Slaman in [1].

Definition 2.4. *We say that a model $\mathcal{M}$ is* topped *if there is a $Y \in \mathbb{X}$ of greatest Turing degree in $\mathcal{M}$. In this case, we say that $\mathcal{M}$ is* topped by *Y.*

3. COHESIVENESS

The principle of cohesiveness (COH) was introduced in [1], where the equivalence of RT^2_2 with $\mathsf{COH} + \mathsf{SRT}^2_2$ over the system RCA_0 was shown.

Definition 3.1. *Let $R \in \mathcal{M}$ be $\mathcal{M}$-infinite and let $R_s = \{t | (s, t) \in R\}$. We say that a set G is R-*cohesive *if for all s, either $G \cap R_s$ is $\mathcal{M}$-finite or $G \cap \overline{R_s}$ is $\mathcal{M}$-finite. The* cohesive principle COH *states that for every $R \in \mathcal{M}$, there is an $\mathcal{M}$-infinite $G \in \mathcal{M}$ that is R-cohesive.*

The aim of this section is to show that $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$. We show that any topped model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ has an $\mathcal{M}$-extension satisfying $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$. The proof splits into two parts, depending on whether $\mathcal{M}$ satisfies $I\Sigma^0_2$. The first part is relatively straightforward, and is implied by the following stronger result (see [1]):

Theorem 3.1. *Let $\mathcal{M}$ be a topped model of $\mathsf{RCA}_0 + I\Sigma^0_2$. Then there is an $\mathcal{M}$-extension of $\mathcal{M}$ that is a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + I\Sigma^0_2$.*

Thus for our purpose, we need concern ourselves only with preserving $B\Sigma^0_2$ in the absence of Σ^0_2-induction. From now on, we fix $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ to be a countable $B\Sigma^0_2$-model of RCA_0 with a Σ^0_2-cut I and topped by Y. Let $g : I \to M$ be a Σ^0_2-definable function with parameters (which we may assume to be the top set Y), so that g is strictly increasing and cofinal in M.

Theorem 3.2. *Every countable topped $B\Sigma^0_2$-model $\mathcal{M}$ has a countable M-extension satisfying $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$.*

Theorem 3.2 is a consequence of the following

Theorem 3.3. *Let $\mathcal{M}$ be countable $B\Sigma^0_2$-model topped by Y. For any R in $\mathcal{M}$, there is a $G \subset M$ which is R-cohesive such that the M-extension $\mathcal{M}[G]$ is a $B\Sigma^0_2$-model of RCA_0 topped by $G \oplus Y$.*

The construction consists of two parts. In the first part, we build within $\mathcal{M}$ a Y'-recursive tree T such that from every $\mathcal{M}$-infinite path on T one obtains a set which is generalized Y-low in a strong sense, i.e., $(G \oplus Y)' \equiv_T G \oplus Y'$, and R-cohesive. In the second part, we use T to select a path G through T that preserves $B\Sigma^0_2$. This G is constructed from *the outside* using the countability of $\mathcal{M}$.

Lemma 3.1 (Internal Forcing)**.** *There is a Y'-recursive tree T such that each $\mathcal{M}$-infinite path p on T yields a set G_p which is R-cohesive and generalized Y-low in a strong sense.*

Without loss of generality, we may assume $Y = \emptyset$ in the proof of Lemma 3.1 (the assumption on Y being a top element will be used only in the second part). In other words, we build a $\emptyset'$-recursive tree T such that each $\mathcal{M}$-infinite path p on T yields a set G_p which is R-cohesive and generalized low, i.e. $G'_p \equiv_T G_p \oplus \emptyset'$. The proof may be easily modified to incorporate a top set Y as parameter in the general situation (see also the remark after requirement Q_e below).

The outline of the proof is as follows. For each $e \in \mathcal{M}$, we have the cohesive requirement P_e and generalized lowness requirement Q_e as follows.

- P_e: $G \cap R_e$ is $\mathcal{M}$-finite or $G \cap \overline{R_e}$ is $\mathcal{M}$-finite.
- Q_e: (Deciding the jump) There is a string σ such that either $\Phi^{\sigma}_{e,|\sigma|}(e)\downarrow$ or for all $\tau \supset \sigma$ such that τ is available based on prior decisions concerning other P_d's, $\Phi^{\tau}_e(e)\uparrow$. We will refer to it as "$e \in G'$ is decided by σ".

(Remark: In the general situation when we have the parameter Y, Q_e takes the form: There is a string σ and a number n, such that either $\Phi^{\sigma\oplus Y\restriction n}_{e,|\sigma|}(e)\downarrow$ or for all $\tau \supset \sigma$, which are available in the above sense, for all $m > n$, $\Phi^{\tau\oplus Y\restriction m}_e(e)\uparrow$. Note that this may still be decided by Y'.)

For each $\mathcal{M}$-finite set B of indices, we first describe how to handle the two blocks of requirements $\{P_e : e \in B\}$ and $\{Q_e : e \in B\}$. We then use the method of blocking, in constructing the tree T, to alternate between the steps of satisfying cohesiveness and deciding the jump.

We first handle cohesiveness. Fix R.

Claim 1. For each $\mathcal{M}$-finite set B, there is a recursive tree T_B such that for every path $p \in [T_B]$, there is a set X_p associated with p with the following property:

(*) For every $e \in B$, either $X_p \cap R_e$ is $\mathcal{M}$-finite or $X_p \cap \overline{R}_e$ is $\mathcal{M}$-finite (informally, X_p is cohesive "for $e \in B$").

Furthermore, there is a path p such that X_p is $\mathcal{M}$-infinite.

Proof of Claim 1. Recall that given e, the e-state of x, for a number $x \in M$, is defined to be the $\mathcal{M}$-finite binary string ρ (in fact, $\rho(R, e, x)$) of length $e + 1$ such that for each $s \leq e$, $\rho(s) = 1$ if and only if $x \in R_s$.

Without loss of generality, we may assume that $B = \{e : e \leq b\}$ for some b. For each possible b-state $\eta \in 2^{b+1}$, let S_η be the set $\{x \in \mathcal{M} : \rho(x) = \eta\}$. We first argue that for some η, S_η is $\mathcal{M}$-infinite. Suppose for the sake of a contradiction that for every b-state η, the set S_η is $\mathcal{M}$-finite. Then

$$\forall \eta < 2^{b+1}\ \exists l_\eta\ \forall x > l_\eta\ (x \notin S_\eta).$$

By $B\Sigma^0_2$, there is a uniform upper bound l^* for all such l_η. Then any $x^* > l^*$ would satisfy $\rho(x^*) \neq \eta$ for all $\eta \in 2^{b+1}$, which is a contradiction.

Clearly, if S_η is $\mathcal{M}$-infinite, then it is cohesive for $e \in B$ because every element in S_η has the same b-state η. However, even $\emptyset'$ is unable to decide which S_η is $\mathcal{M}$-infinite. For the uniformity which is required in our later construction, we have to allow all possible approximations for S_η. Thus one forms the recursive tree T_B as follows: For each b-state η, let T_η be the set of binary strings σ which corresponds to an $\mathcal{M}$-finite subset of S_η, i.e.,

$$T_\eta = \{\sigma \in 2^{<\mathcal{M}} : \forall x < |\sigma| \ (\sigma(x) = 1 \Rightarrow \rho(x) = \eta)\}.$$

Clearly for each η, T_η is a recursive binary tree, in fact it is uniformly recursive in η. When S_η is $\mathcal{M}$-finite, all unbounded paths on T_η are eventually all zeros, hence isolated. When S_η is $\mathcal{M}$-infinite, T_η is isomorphic to $2^{\mathcal{M}}$. Form the recursive disjoint union T_B of T_η, for $\eta \in 2^{b+1}$, with the empty string λ as its root. Then each path p on T_B is also a path on T_η for some η. Define $X_p = \{x : \exists \sigma \subset p : \sigma(x) = 1\}$. It is easy to see that T_B satisfies (*). This proves Claim 1.

Next we handle a block of generalized lowness requirements.

Claim 2. There is a $\emptyset'$-recursive function $h : \mathcal{M} \times \mathcal{M} \to 2^{<\mathcal{M}}$ with the following property: for each (canonical index of an) $\mathcal{M}$-finite set $B \subset \mathcal{M}$ and (an index of) a recursive tree S, $\sigma = h(B, S)$ is a string on S which either decides "$e \in G'$" for all $e \in B$; or $q = \{\sigma\hat{\ }0^n : n \in \mathcal{M}\}$ is an isolated path on S.

Proof of Claim 2. Without loss of generality, we may assume $B = \{e : e \leq b\}$. The proof is essentially the construction of a low set under $I\Sigma^0_1$.

Fix a recursive tree S. We do a finite injury argument. The generalized lowness requirement Q_e says: There is a σ such that either $\Phi^\sigma_{e,|\sigma|}(e)\downarrow$ or for all $\tau \supseteq \sigma$, if $\tau \in S$ then $\Phi^\tau_e(e)\uparrow$. We also want the last digit of σ to be 1 (so that any infinite path corresponds to an infinite set).

We search for a sequence of strings $\sigma^0 \subseteq \sigma^1 \subseteq \cdots \subseteq \sigma^b$, together with a nested sequence of trees $U^e = \{\tau \in S : \sigma^e \subseteq \tau \vee \tau \subseteq \sigma^e\}$ such that σ^e is the witness string for Q_e and the last digit of σ^e is 1. Initially we set $U^0_0 = U^1_0 = \cdots = U^b_0 = S$ and $\sigma^0_0 = \sigma^1_0 = \cdots = \sigma^b_0 =$ the root of S. We use another finite three-branching tree $V = \{-1, 0, 1\}^b$ of height $b+1$ to help us organize the construction. The intended interpretations of outcomes are as follows: -1 indicates no string ending with 1 is found, 0 indicates the Π_1-outcome "for all τ on U_e extending σ, $\Phi^\tau_e(e)\uparrow$" and 1 indicates a string σ ending with 1 such that $\Phi^\sigma_{e,|\sigma|}(e)\downarrow$ is found. The strings ν on V can be viewed as a record of the injuries to the requirements.

Initially $\nu_0 = (-1)^b$. At stage s, we say that a requirement Q_e *requires attention* if either (1) $\nu_s(e) = -1$ and there is a (least under a canonical order) string $\tau \in U^e_s$ whose last digit is 1; or (2) $\nu_s(e) = 0$ and there is a (least) string $\sigma \in U^e_s$ such that $|\sigma| \le s$ and $\Phi^\sigma_{e,|\sigma|}(e)\downarrow$ and the last digit of σ is 1. If no requirement requires attention at stage s, then go to stage $s+1$. Otherwise, pick the least such e such that Q_e requires attention. If (1) is true, we set $U^e_{s+1} = \{\tau' \in U^e_s : \tau'$ is compatible with $\tau\}$, $\sigma^e_{s+1} = \tau$, update the string $\nu_{s+1} = (\nu_s \upharpoonright e)\hat{}0\hat{}(-1)^{b-e-1}$, and for each $i > e$, define $U^i_{s+1} = U^e_{s+1}$, $\sigma^i_{s+1} = \sigma^e_{s+1}$. In the case that (2) is true, set $U^e_{s+1} = \{\tau' \in U^e_s : \tau'$ is compatible with $\sigma\}$, $\sigma^e_{s+1} = \sigma$, for each $i > e$, $U^i_{s+1} = U^e_{s+1}$, $\sigma^i_{s+1} = \sigma^e_{s+1}$ and update the string $\nu_{s+1} = (\nu_s \upharpoonright e)\hat{}1\hat{}(-1)^{b-e-1}$. This ends the construction.

By $I\Sigma_1$, $\{\eta : (\exists s)\ \eta = \nu_s\}$ is an $\mathcal{M}$-finite subset of V. Let η^* be the largest string with respect to the lexicographic order in that set and s^* be the stage at which $\eta^* = \nu_{s^*}$. The construction implies that no more injury occurs after s^*. Then $\sigma^b_{s^*}$ is the required string, which may be computed using $\emptyset'$ as an oracle. This ends the proof of Claim 2.

As a final step, we use the blocking technique to mix the cohesiveness and generalized lowness requirements together to obtain the tree T in Lemma 3.1, as follows.

We use $\emptyset'$ to enumerate two sequence $\{b_j\}_{j\in J}$ and $\{c_j\}_{j\in J}$ for some cut J. The sequence $\{b_j\}$ is used to dynamically determine the blocks; and c_j is used to determine the initial segments of T (i.e., we use $\emptyset'$ to decide whether a string σ of length $\le c_j$ belongs to $T \upharpoonright c_j$).

The sequences $\{b_j\}$, and $\{c_j\}$ are defined inductively. Let $b_0 = g(0)$ and $B_0 = \{x : 0 \le x < b_0\}$. By Claim 1, there is a recursive tree T_0, which is the disjoint union of $T_\eta, \eta \in 2^{b_0+1}$, such that any path on T_η is cohesive for $e \in B_0$. For each $\eta \in 2^{b_0+1}$, apply the recursive in $\emptyset'$ function h to B_0 and T_η, to get a string $\sigma_\eta \in T_\eta$ which either decides "$e \in G'$" for all $e \in B_0$ or realizes σ_η corresponds to an isolated path on T_η. Let $c_0 = \max\{|\sigma_\eta| : \eta < 2^{b_0+1}\}$. We determine $T \upharpoonright c_0$ by trimming T_η one by one as follows: For each $\eta < 2^{b_0+1}$, if $h(B_0, T_\eta)$ decides "$e \in G'$", then keep strings with length less than or equal to c_0 which are compatible with $h(B_0, T_\eta)$ in $T \upharpoonright c_0$; if $h(B_0, T_\eta)$ corresponds to an isolated path then terminate T at $h(B_0, T_\eta)$. In general, suppose b_j, c_j and $T \upharpoonright c_j$ are defined. Let b_{j+1} be the least $g(i) > \max\{c_j, b_j\}$. Let $B_{j+1} = \{x : b_j \le x < b_{j+1}\}$. Consider each string $\sigma \in T \upharpoonright c_j$ of height c_j that is not terminal, with $\sigma \in T_\eta$ for some b_j-state η, and apply Claim 1 to T_η to obtain a subtree which is cohesive for B_{j+1}. The resulting tree will be a disjoint union of trees T_μ, where μ is a b_{j+1}-state extending η. Then applying the function h in Claim 2 to B_{j+1} and each T_μ, we get strings $\sigma_\mu \in T_\mu$ which either decides "$e \in G'$" for all $e \in B_{j+1}$ or realizes σ_μ corresponds to an isolated path on T_μ.

Let $c_{j+1} = \max\{|\sigma_\mu| : \mu < 2^{b_{j+1}+1}\}$. We obtain $T \restriction c_{j+1}$ by trimming T_μ one by one as in the case $j = 0$.

Let $J = \{j : b_j \text{ is defined}\}$. Then J is a Σ_2-cut and $\{b_j\}_{j\in J}$ is unbounded in $\mathcal{M}$. Arguing along J, we see that for every j, there is a perfect tree U_j with stem σ_j such that U_j satisfies all cohesive requirements in B_j in the sense of Claim 1; and σ_j decides "$e \in G'$" for all $e \in B_j$. Since $\sigma_j \subseteq \sigma_{j+1}$ for all $j \in J$, there is at least one $\mathcal{M}$-infinite path on T. Every $\mathcal{M}$-infinite path necessarily corresponds to an $\mathcal{M}$-infinite set, since we have chopped off all isolated paths. Clearly, every $\mathcal{M}$-infinite path on T eventually has the same e-state, and so the cohesiveness requirements are satisfied. Let p be an $\mathcal{M}$-infinite path on T and G_p be the corresponding $\mathcal{M}$-infinite set. Say $G_p = \{x_0 < x_1 < \dots\}$. We check that $G'_p \leq_T G_p \oplus \emptyset'$ as follows. Fix e, say $b_j < e \leq b_{j+1}$. First use $\emptyset'$ to find b_0, then find the b_0-state of x_0, say η_0. Form T_{η_0} and use $\emptyset'$ to find the string σ_0 on T_{η_0} which decides $e \in G'$ for all $e < b_0$. Suppose $b_{j'}$, $j' \leq j$, is defined. Use $\emptyset'$ to recover the construction and find $b_{j'+1}$. Find an element $x_{j'+1} \in G_p$ such that $x_{j'+1} > b_{j'+1}$ and use it to obtain the $b_{j'+1}$-state of $x_{j'+1}$. Then the value of $G'(e)$ is determined by $\emptyset'$. This procedure eventually reached $j' = j$, which ends the proof of Lemma 3.1.

Lemma 3.2 (External Forcing)**.** *There is an unbounded path G on T such that $\mathcal{M}[G] \models B\Sigma^0_2$.*

Proof. Since $\mathcal{M}$ is countable, let $\{\exists s\ \varphi_n(x, s, \vec{X}, G) | n < \omega\}$ be a list of all Σ^0_1-formulas with a distinguished set variable G and $\vec{X}$ is a finite set of set parameters. Since $\mathcal{M}$ is topped by Y, every parameter in $\vec{X}$ is $\Delta_1(Y)$. We may assume the above list is in fact of the form $\{\exists s\ \varphi_n(x, s, Y, G) | n < \omega\}$. Let $\{D_n | n < \omega\}$ be a list of all $\mathcal{M}$-finite sets. We work from the outside and choose a path G on T so that $\mathcal{M}[G]$ is a model of $B\Sigma^0_2$.

Let U_{-1} be T and $\sigma_{-1} = \emptyset$. Assume that σ_n and $U_n \subset T$ are defined so that U_n is $\mathcal{M}$-infinite, and all strings in U_n extend σ_n. Let $n + 1 = (k, m)$. For $x \in D_m$, let

$$P_x = \{\sigma \in U_n | \ (\forall s \leq |\sigma|)\ \neg\varphi_k(x, s, Y, \sigma \oplus Y')\}.$$

P_x is uniformly recursive in x and Y'.

Case (i). For all $x \in D_m$, P_x is bounded. Then there is a u_x such that for all $\sigma \in U_n$ of length greater than u_x, $\sigma \notin P_x$, and hence $\exists s \leq |\sigma|\ \varphi_k(x, s, Y, \sigma \oplus Y')$. By $B\Sigma_2$ on $\mathcal{M}$, there is a uniform upper bound u such that for all $x \in D_m$ and all $\sigma \in U_n$ of length greater than u, $\exists s \leq |\sigma|\ \varphi_k(x, s, Y, \sigma \oplus Y')$.

Choose σ_{n+1} to extend σ_n and of length greater than $\max\{u, u_{i_n}\}$ such that $U_n[\sigma_{n+1}] = \{\tau | \tau \supset \sigma_{n+1} \ \&\ \tau \in U_n\}$ is unbounded. Let U_{n+1} be $U_n[\sigma_{n+1}]$.

Case (ii). P_x is not bounded for some $x \in D_m$. Let $\sigma_{n+1} = \sigma_n$ and U_{n+1} be the set of all strings

$$\{\sigma \in U_n : (\forall s \leq |\sigma|)\ \neg\varphi_k(x, s, Y, \sigma \oplus Y')\}.$$

Notice that U_{n+1} is unbounded and Y'-recursive for each n. Let G be the set whose characteristic function is $\bigcup_n \sigma_n$. Then G is generalized Y-low in a strong sense, R-cohesive by Lemma 3.1 and $B\Sigma^0_1(G \oplus Y')$ holds by the above. Since $G' \leq_T G \oplus Y'$, this implies that $B\Sigma^0_1(G')$ is true, and hence $\mathcal{M}[G]$ satisfies $\mathsf{RCA}_0 + B\Sigma^0_2$. Moreover it is topped by $G \oplus Y$. This proves Theorem 3.3. □

Proof of Theorem 3.2.

Proof. Let $h : \omega \to \omega \times \omega$ be a bijection such that for all n, if $h(n) = (m, k)$ then $m < n$. Iterate Theorem 3.3 as follows: Let $\mathcal{M}_0 = \mathcal{M}$. If $\mathcal{M}_n$ is defined, let $R \in M_m$ be the kth set in $\mathcal{M}_m$ and apply Theorem 3.3 to get a G that is R-cohesive while preserving $B\Sigma^0_2$. Let $\mathcal{M}_{n+1} = \mathcal{M}_n[G]$. Let $\mathcal{M}^* = \bigcup_n \mathcal{M}_n$. Then $\mathcal{M}^* \models \mathsf{RCA}_0 + \mathsf{COH} + B\Sigma_2$. □

Corollary 3.1. $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2$.

Proof. Otherwise, there is a Π^1_1 sentence $\forall X\varphi$ (where φ is arithmetic) provable in $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ but not in $\mathsf{RCA}_0 + B\Sigma^0_2$. Then there is a countable model $\mathcal{M}$ of the latter in which $\exists X\neg\varphi$ is true. Furthermore, as was done in [1], we may assume that $\mathcal{M}$ is topped by X. Apply Theorem 3.2 to get a model $\mathcal{M}^*$ of which $\mathcal{M}$ is an M-submodel and $\mathcal{M}^* \models \mathsf{RCA}_0 + \mathsf{COH}$. Then $\mathcal{M}^* \models \mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ in which $\exists X\neg\varphi$ holds. But this is not possible by assumption. □

4. Ascending or Descending Sequence

The principle ADS of increasing or descending sequence states that every infinite linearly ordered set contains an infinite subsequence that is either increasing or decreasing. Hirschfeldt and Shore [5] showed that ADS is strictly weaker than Ramsey's Theorem for pairs RT^2_2 and that both COH and $B\Sigma^0_2$ are consequences of ADS over RCA_0. In [5], a principle strictly weaker than ADS, called *stable* ascending or descending sequence (SADS), was introduced:

Definition 4.1 (SADS). *Every linear ordering $\leq_L$ such that for each x, either $\{y|y \leq_L x\}$ is finite or $\{y|y \geq_L x\}$ is finite has an infinite ascending or descending subsequence.*

Although SADS is strictly weaker than ADS over RCA_0, these two principles have equal first order strength in the Kirby-Paris hierarchy. In fact, it was shown in [2] that SADS implies $B\Sigma^0_2$ over RCA_0, and results in this section imply that ADS, hence SADS, does not imply $I\Sigma^0_2$.

We show that $\mathsf{SADS} + B\Sigma^0_2$ is Π^1_1-conservative over $B\Sigma^0_2$, and derive as a corollary that so is ADS. Since SADS implies $B\Sigma^0_2$ over RCA_0, any consideration of the conservation power of SADS will have to be carried out over models with at least this level of first order inductive strength. The main theorem of this section states:

Theorem 4.1. *Let $\mathcal{M}$ be a topped model of $\mathsf{RCA}_0 + B\Sigma^0_2$. Then there is an M-extension of $\mathcal{M}$ satisfying $\mathsf{RCA}_0 + \mathsf{SADS} + B\Sigma^0_2$.*

By a linear ordering $\langle M, \leq_L\rangle$ in $\mathcal{M}$, we mean a set $\leq_L \subset M \times M$ in $\mathbb{X}$ satisfying the following properties:

(1) For all x, y, either $(x, y) \in \leq_L$ or $(y, x) \in \leq_L$. They both belong to $\leq_L$ if and only if $x = y$;
(2) For all x, y, z, if $(x, y) \in \leq_L$ and $(y, z) \in \leq_L$ then $(x, z) \in \leq_L$.

We write $x \leq_L y$ if $(x, y) \in \leq_L$. $\leq_L$ is stable if $\langle M, \leq_L\rangle$ is a stable linear ordering. As for COH in the previous section, Theorem 4.1 follows immediately from

Theorem 4.2. *Let $\mathcal{M}$ be a model of $\mathsf{RCA}_0 + B\Sigma^0_2$ topped by Y. If $\langle M, \leq_L\rangle$ is an infinite stable linear ordering in $\mathcal{M}$, then there is an $\mathcal{M}$-infinite $G \subset \leq_L$ such that $\langle G, \leq_L\rangle$ is either an ascending or descending sequence and $\mathcal{M}[G] \models \mathsf{RCA}_0 + B\Sigma^0_2$. Furthermore, $\mathcal{M}[G]$ is topped by $Y \oplus G$.*

As explained in §3, Theorem 3.1 takes care of the case when $\mathcal{M}$ further satisfies $I\Sigma^0_2$. Thus in the following we consider only models of $\mathsf{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$ for Theorem 4.2:

Proof. It is sufficient to assume that $\mathcal{M}$ has only recursive sets as second order members. At the end of the construction, $\mathcal{M}[G]$ will be a model of $\mathsf{RCA}_0 + B\Sigma^0_2$ topped by G, and G is either an ascending or descending sequence that solves the given stable linear ordering $\leq_L$. The general case of a topped model follows by straightforward relativization, where the top set Y is used. Let I be a Σ^0_2-cut in $\mathcal{M}$ with increasing cofinal function $g : I \to M$ whose graph is $\emptyset'$-recursive. Define

$$X_0 = \{x | \text{There are only } \mathcal{M}\text{-finitely many } y <_L x\}$$

and

$$X_1 = \{x | \text{There are only } \mathcal{M}\text{-finitely many } y >_L x\}.$$

Observe that both X_0 and X_1 are Δ^0_2 since they are clearly Σ^0_2, and the stability of $\langle M, \leq_L\rangle$ implies that $M = X_0 \cup X_1$, so that M is split into a disjoint union of two Δ^0_2-sets.

We make the following claim.

Claim . One of the following holds:

(1) Either there is an $\mathcal{M}$-infinite recursive descending sequence in $\langle M, \leq_L\rangle$, or for each c, there are at least c-many elements in X_0;
(2) Either there is an $\mathcal{M}$-infinite recursive ascending sequence in $\langle M, \leq_L\rangle$, or for each c, there are at least c-many elements in X_1.

Proof of Claim . We only prove (1) as the proof of (2) is similar. Suppose there is a c such that for all $x \in X_0$ there are less than c-many y with $y <_L x$. Observe that for any $b \in X_1$, since $\{y | y >_L b\}$ is $\mathcal{M}$-finite, the set $\{y | y <_L b\}$ must be $\mathcal{M}$-infinite. Recursively define an $\mathcal{M}$-infinite descending sequence in X_1 as follows: Let x_0 be any fixed element in X_1. Assume that x_s is defined. Enumerate $\leq_L$ to find the first $y <_L x_s$ such that $\{z | z <_L y\}$ has at least c elements. Let x_{s+1} be this y. Note that by the choice of c, $x_{s+1} \in X_1$. Thus $\{x_s | s \in M\}$ is what we wanted.

Thus assume that there is no $\mathcal{M}$-infinite recursive ascending or descending sequence in $\langle M, \leq_L\rangle$. Then for each c, there are at least c-many elements in X_0 and X_1. We will show that there is an $\mathcal{M}$-infinite ascending sequence $G \subset X_0$ such that $\mathcal{M}[G] \models \mathsf{RCA}_0 + B\Sigma^0_2$. In fact, G will be a low set.

Forcing. Define a notion of forcing $\mathcal{F}$ as follows: A condition is an $\mathcal{M}$-finite string $\sigma = \langle y_0, y_1, \ldots, y_{k-1}\rangle$ such that for any $0 \leq i < j < k$, $y_i <_L y_j$. We use $|\sigma|$ to denote the length of σ which is k and $\max_L(\sigma)$ to denote the $<_L$-maximal number y_k in the range of σ. Conditions τ extends σ (written $\tau \leq_{\mathcal{F}} \sigma$) if $\sigma \subseteq \tau$ and $\tau \restriction |\sigma| = \sigma$. An $\mathcal{M}$-finite string σ is said to be contained in X_0 if its range is a subset of X_0. Strings of arbitrary length contained in X_0 exist by our assumption on the "unbounded size" of X_0, and they form an $\mathcal{M}$-infinite Δ^0_2-set. The idea is to build G to be 1-generic with respect to X_0 in the following sense:

(3) For each e, there is a σ contained in X_0 such that either $\Phi_e^\sigma(e)\downarrow$ or for all extensions σ' of σ, if $\Phi_e^{\sigma'}(e)\downarrow$, then σ' is not contained in X_0.

Expand the language of Peano arithmetic by adding a second order set variable G. Let $\{\varphi_e|e \in M\}$ be a recursive union of all bounded formulas. Define

(4) $\sigma \Vdash \varphi$ for a bounded formula $\varphi(G)$ if $\mathcal{M} \models \varphi(G)$ when G is interpreted as σ;
(5) $\sigma \Vdash \exists x\varphi(x, G)$ if for some $c \leq \max_L(\sigma)$, $\sigma \Vdash \varphi(c, G)$;
(6) $\sigma \Vdash \neg\varphi(G)$ if there is no $\tau \leq_{\mathcal{F}} \sigma$ contained in X_0 such that $\tau \Vdash \varphi(G)$.

Note that while (4) appears to contradict (6) in general, for example, some condition σ contained in X_0 may force $\neg\varphi(G)$ in the sense of (6) and yet some $\tau \leq_F \sigma$ with τ not being contained in X_0 may force $\varphi(G)$. However, in our construction we only consider conditions which are contained in X_0. In such a situation, consistency is preserved.

The generic set G will be constructed in J-many steps, where $J \subseteq I$ is a Σ^0_2-cut to be determined dynamically in the course of the construction, as the union of a $\emptyset'$-recursive sequence $\langle\sigma_i\rangle_{i\in J}$. Let σ_{-1} be the empty string and $\hat{g}(-1)$ be undefined. Assume that $\sigma_i \subset X_0$ and $\hat{g}(i)$ is defined and for all $e \leq \hat{g}(i)$, $\sigma_i \Vdash \varphi_e$ or $\sigma_i \Vdash \neg\varphi_e$. Let $\hat{g}(i + 1) = \max\{\max_L(\sigma_i), g(i + 1)\}$.

Construction of σ_{i+1}.

For each $\mathcal{M}$-finite subset $D \subseteq \hat{g}(i + 1)$, define

$$S_{D,i+1} = \{\tau|\tau \leq_{\mathcal{F}} \sigma_i \ \&\ \forall e \in D[\tau \Vdash \varphi_e]\}.$$

Claim 1. For each $D \subset \hat{g}(i+1)$, $\emptyset'$ is able to decide uniformly if $S_{D,i+1}$ has an element contained in X_0.

Proof of Claim 1. Fix i and σ_i as parameters. Also fix a recursive enumeration of $S_{D,i+1}$, which can be chosen uniformly in D. Recursively enumerate a sequence τ_s in $S_{D,i+1}$ in descending order of $\max_L(\tau_s)$ as follows. Suppose for all $t < s$, τ_t has been defined, let τ_s be the first string (with respect to the fixed enumeration of $S_{D,i+1}$) $\tau \in S_{D,i+1}$ such that $\max_L(\tau) <_L \min\{\max_L(\tau_t) : t < s\}$ if such τ exists; undefined otherwise. Note if $S_{D,i+1} = \emptyset$ then $\langle\tau_s\rangle$ is empty sequence. By $I\Sigma_1$, either for all $s \in \mathcal{M}$, τ_s is defined; or there is some s_0 which is the least s such that τ_s is undefined. We rule out the first possibility as follows. Suppose for all $s \in \mathcal{M}$, τ_s is defined. Then by the definition of X_0, $\max_L(\tau_s)$ has to be contained in X_1, which gives us an $\mathcal{M}$-infinite descending sequence in X_1, contradicting our assumption. Now define a function H by setting $H(D) = s_0$ which is the least t such that τ_t is undefined. H is recursive in $\emptyset'$. Finally to decide if $S_{D,i+1}$ has an element contained in X_0, we first use H to find s_0. If $s_0 \neq 0$, use $\emptyset'$ again to see if $\max_L(\tau_{s_0-1}) \in X_0$.

We can extract more information from H. Let $h_{i+1} : 2^{\hat{g}(i+1)} \to \{-1\} \cup I$ be such that

$$h_{i+1}(D) = \begin{cases} -1, & \text{either } H(D) = 0 \text{ or } \max_L(\tau_{H(D)}) \in X_1; \\ \max_L(\tau_{H(D)}), & \text{otherwise.} \end{cases}$$

Notice that h_{i+1} is a Σ^0_2 function with $\mathcal{M}$-finite domain, hence by $B\Sigma^0_2$, (the graph of) h_{i+1} is $\mathcal{M}$-finite.

Claim 2. There is a $\hat{D} \subset \hat{g}(i+1)$ and $\hat{\tau} \leq_{\mathcal{F}} \sigma_i$ contained in X_0 such that $\hat{\tau} \in S_{\hat{D},i+1}$ and $h_{i+1}(\hat{D}) \neq -1$, and if $\tau \leq_{\mathcal{F}} \hat{\tau}$ is contained in X_0, then $\{e | e \leq \hat{g}(i+1) \ \& \ \tau \Vdash \varphi_e\} = \hat{D}$.

Proof of Claim 2. Observe that the set

$$\mathcal{D} = \{D' | D' \subset \hat{g}(i+1) \ \& \ h_{i+1}(D') \neq -1\}$$

is $\mathcal{M}$-finite. $\mathcal{D}$ is partially ordered under inclusion. Let $\hat{D}$ be a maximal element in $\mathcal{D}$. Let $\hat{\tau}$ be the least τ such that $\max_L(\tau) \leq h_{i+1}(\hat{D})$ with $\hat{D} = \{e | e \leq \hat{g}(i+1) \ \& \ \tau \Vdash \varphi_e\}$.

We say that $(\hat{D}, \hat{\tau})$ is $\hat{g}(i+1)$-maximal for σ_i if it satisfies the conclusion of Claim 2. Note that $\emptyset'$ is able to identify those $(\hat{D}, \hat{\tau})$'s which are $\hat{g}(i+1)$-maximal for σ_i. Indeed the collection of pairs $(\hat{D}, \hat{\tau})$ which are $\hat{g}(i+1)$-maximal for σ_i, with $\max_L(\hat{\tau}) \leq \max \{h_{i+1}(D) | D \subset \hat{g}(i+1)\}$, is $\mathcal{M}$-finite.

Let σ_{i+1} be the least $\hat{\tau} \leq_{\mathcal{F}} \sigma_i$ for which there is a $\hat{D}$ so that $(\hat{D}, \hat{\tau})$ belongs to this collection.

Claim 3. Let $J = \{i \in I | \hat{g}(i) \text{ is defined}\}$. Then $\hat{g}[J] = \{\hat{g}(i) | i \in J\}$ is cofinal in M.

Proof of Claim 3. The construction ensures that if $\hat{g}(i)$ is defined, then so is $\hat{g}(i+1)$. Hence J is a cut. If $\hat{g}[J]$ is bounded in M, say by $g(i^*)$, then the construction may be carried out within $g(i^*)$, recursively in $\emptyset'$. In particular, the set

$$E = \{(i, \sigma_i, \hat{g}(i)) | i \in J\}$$

is Δ^0_2 on $J \times \hat{g}[J] \times \hat{g}[J]$. By Lemma 2.1, there is an $\mathcal{M}$-finite set E^* whose intersection with $J \times \hat{g}[J] \times \hat{g}[J]$ is E. Then if $(i, \sigma_i, \hat{g}(i)) \in E^*$ and $i \notin J$, it can be identified by $\emptyset'$. This implies that J is a Δ^0_2-cut. By $I\Delta^0_2$ which is equivalent to $B\Sigma^0_2$ (see [10]), J has a largest element, which is a contradiction.

Let $G = \bigcup_{i \in J} \sigma_i$. Then G is $\emptyset'$-recursive and 1-generic with respect to X_0. Clearly $G \Vdash \varphi_e$ if and only if $\sigma_i \Vdash \varphi_e$ where i is the least such that $e \leq \hat{g}(i)$. This implies that $G' \leq_T \emptyset'$ and hence low, completing the proof of Theorem 4.2. □

Corollary 4.1. $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{SADS}$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2$.

We highlight in particular a consequence of the above corollary. Since $\mathsf{RCA}_0 + \mathsf{SADS}$ implies $B\Sigma^0_2$ (see [2]), one obtains a sharp bound on the first order strength of SADS:

Corollary 4.2. $\mathsf{RCA}_0 + \mathsf{SADS}$ *does not prove* $I\Sigma^0_2$.

Corollary 4.3. *Every countable topped model* $\mathcal{M}$ *of* $\mathsf{RCA}_0 + B\Sigma^0_2$ *has an* M*-extension that is a model of* $\mathsf{RCA}_0 + \mathsf{ADS} + B\Sigma^0_2$.

Proof. It was proved in [5] that over RCA_0, ADS is equivalent to $\mathsf{COH} + \mathsf{SADS}$.[1] Now starting with a countable topped model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$, we may expand it to an M-extension that is a model of $\mathsf{RCA}_0 + \mathsf{ADS} + B\Sigma^0_2$. First of all, the proof of Lemma 9.5 of [1] allows the expansion of every countable topped model of $\mathsf{RCA}_0 + I\Sigma^0_2$ to a topped model that satisfies additionally the principle COH. This model can then be further expanded to satisfy SADS while preserving $\mathsf{RCA}_0 + \mathsf{COH} + I\Sigma^0_2$, hence ADS. On the other hand, if $\mathcal{M}$ does not satisfy $I\Sigma^0_2$, then one may apply the constructions used for Theorem 3.2 and Theorem 4.1 to obtain an M-extension that is a model of $\mathsf{RCA}_0 + \mathsf{COH} + \mathsf{SADS} + B\Sigma^0_2$, hence of ADS. □

Corollary 4.4. $\mathsf{RCA}_0 + \mathsf{ADS} + B\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2$.

[1] The direction that $\mathsf{COH} + \mathsf{SADS}$ proves ADS over RCA_0 is not explicitly stated in [5] but follows from Propositions 2.7, 2.9 and 2.10.

5. The Chain and Antichain Principle CAC

CAC states that every infinite partially ordered set $\langle M, \leq_L\rangle$ has an infinite chain or antichain. A refinement of CAC is the stable chain antichain principle (SCAC) which asserts CAC for partial orders for which one of the following conditions holds:

(i) For all x, either all but finitely many y's are $\leq_L$-above x, or all but finitely many y's are $\leq_L$-incomparable with x;
(ii) For all x, either all but finitely many y's are $\leq_L$-below x, or all but finitely many y's are $\leq_L$-incomparable with x.

In this section, we show that neither CAC nor SCAC proves $I\Sigma^0_2$ over RCA_0. In fact both are Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$. The key component of the proof is the following M-extension theorem:

Theorem 5.1. *Let $\mathcal{M}$ be a countable topped model of $\mathsf{RCA}_0 + B\Sigma^0_2$. Then $\mathcal{M}$ has an M-extension that is a model of $\mathsf{RCA}_0 + \mathsf{SCAC} + B\Sigma^0_2$.*

This theorem is an immediate consequence of

Theorem 5.2. *Let $\mathcal{M}$ be a model of $\mathsf{RCA}_0 + B\Sigma^0_2$ topped by $Y \in \mathbb{X}$. If $\langle M, \leq_L\rangle$ is an $\mathcal{M}$-infinite stable partially ordered set in $\mathcal{M}$, then there is an $\mathcal{M}$-infinite $G \subset M$ that is low relative to Y such that $\langle G, \leq_L\rangle$ is either a chain or an antichain, and $\mathcal{M}[G] \models \mathsf{RCA}_0 + B\Sigma^0_2$. Furthermore, $\mathcal{M}[G]$ is topped by $Y \oplus G$.*

Proof. As in the previous section, it is sufficient to consider the case when $\mathbb{X}$ consists only of recursive sets, as the rest follows by relativization. In the situation we are considering, G will be a low set.

As in the previous sections, the case when $\mathcal{M}$ satisfies $I\Sigma^0_2$ has been taken care of by Theorem 3.1. We present here a construction of G when only $B\Sigma^0_2$ holds in $\mathcal{M}$.

Assume that for every x, either all but $\mathcal{M}$-finitely many y's are $\leq_L$-above x, or all but $\mathcal{M}$-finitely many y's are $\leq_L$-incomparable with x. The proof for the other case is similar. Let Q be the set of all x such that all but $\mathcal{M}$-finitely many y's are $\leq_L$-incomparable with x. Notice that Q is upward closed.

Suppose that Q has no $\mathcal{M}$-infinite recursive subset. We show that there is an $\mathcal{M}$-infinite low set G contained in $M \setminus Q$. As in previous sections, we build a $\emptyset'$-recursive sequence of strings $\langle \sigma_i\rangle_{i\in J}$ for some dynamically determined Σ^0_2-cut J, so that G is 1-generic with respect to a notion of forcing which we now define. As before, I denotes a Σ^0_2-cut and g is a Σ^0_2-function from I cofinally into M.

Forcing. We define a notion of forcing $\mathcal{F}$ in a similar way as in §4. Expand the language of Peano arithmetic to include a set variable G. A condition is an $\mathcal{M}$-finite string $\sigma = \langle y_0, y_1, \ldots, y_{k-1}\rangle$ such that for all $0 \leq i < j < k$, $y_i <_L y_j$. Again, we use $|\sigma|$ to denote k which is the length of σ and $\max_L(\sigma)$ to denote y_{k-1} which is the $<_L$ largest element in the range of σ. Note that the set of conditions is recursive and $\mathcal{M}$-infinite. The first assertion is obvious by the way a condition is defined. The second assertion follows from the assumption that there is no $\mathcal{M}$-infinite recursive antichain: Suppose x_0 is such that for all $x < y$ above x_0 in the natural ordering, either $y <_L x$ or they are $\leq_L$-incomparable. Since by assumption all but $\mathcal{M}$-finitely many y's are $\leq_L$-above x or incomparable with x, we can recursively enumerate a sequence of pairwise $\leq_L$-incomparable elements $x_1 < x_2 < \cdots$. Now the length of this sequence has to be M, since otherwise the sequence, being recursive, has to be $\mathcal{M}$-finite and may be extended to a longer sequence with pairwise $\leq_L$-incomparable elements. But if $\{x_1 < x_2 < \cdots\}$ has length M, then it contradicts our assumption on the nonexistence of an $\mathcal{M}$-infinite recursive antichain.

A condition τ extends another condition σ (written $\tau \leq_{\mathcal{F}} \sigma$) if $\sigma \subseteq \tau$ and $\tau \upharpoonright |\sigma| = \sigma$. Define

(7) $\sigma \Vdash \varphi(G)$ if φ is a bounded formula and $\mathcal{M} \models \varphi(G)$ when G is interpreted as σ;

(8) $\sigma \Vdash \exists x\varphi(x, G)$ if for some c, $\sigma \Vdash \varphi(c, G)$ where φ is bounded;

(9) $\sigma \Vdash \neg\exists x\varphi$ if φ is bounded and for all $\tau <_{\mathcal{F}} \sigma$, if $\tau \Vdash \exists x\varphi$ then $\max_L(\tau) \in Q$.

We remark that although (7) and (9) appear contradictory to each other for σ such that $\max_L(\sigma) \in Q$, our construction of G will consider only strings σ whose range is a subset of $M \setminus Q$ (this is equivalent to requiring $\max(\sigma) \in M \setminus Q$). In such a situation, (7) and (9) are consistent.

Let $\{\varphi_e\}$ be a recursive list of all bounded formulas (with free variable G) and $\sigma_{-1} = \emptyset$ and let $\hat{g}(-1)$ be undefined. Suppose that σ_i is defined and 1-generic with respect to $e \leq \hat{g}(i)$: For all $e \leq \hat{g}(i)$, either $\sigma_i \Vdash \exists x\varphi_e(x, G)$ or $\sigma_i \Vdash \neg\exists x\varphi_e(x, G)$.

Construction of σ_{i+1}:
Let $\hat{g}(i+1) = \max\,\{\max_L(\sigma_i), g(i+1)\}$. For $D \subset \hat{g}(i+1)$, let

$$S_{D,i+1} = \{\tau | \tau \leq_{\mathcal{F}} \sigma_i \;\&\; \forall e \in D[\tau \Vdash \varphi_e]\}.$$

Claim 1. For each $D \subset \hat{g}(i{+}1)$, $\emptyset'$ decides if $S_{D,i+1}$ contains a condition in $M \setminus Q$.

Proof of Claim 1. We note that if $S_{D,i+1}$ is $\mathcal{M}$-infinite, then it contains a τ such that $\max_L(\tau) \in M \setminus Q$. Otherwise, every $\tau \in S_{D,i+1}$ satisfies $\max_L(\tau) \in Q$. Now recursively one may enumerate a sequence $\{\tau_0, \dots, \tau_t, \dots\}$ such that $\tau_t \Vdash \varphi_e$ for each $e \in D$ and extends σ_i. Then $\max_L(\tau_t) \in Q$ and it is straightforward to trim the sequence down to a subsequence which forms a recursive antichain in Q, contradicting our assumption.

Thus recursively in $\emptyset'$, one is able to decide if $S_{D,i+1}$ is $\mathcal{M}$-finite (and within this $\mathcal{M}$-finite set whether there is a τ such that $\max_L(\tau) \in M \setminus Q$), or enumerate a $\tau \in S_{D,i+1}$ that satisfies $\max_L(\tau) \in M \setminus Q$. This proves the Claim.

Define $h_{i+1} : 2^{\hat{g}(i+1)} \to \{-1\} \cup I$ such that $h_{i+1}(D) = -1$ if $S_{D,i+1}$ has no element τ such that $\max_L(\tau) \in M \setminus Q$, and otherwise equals the least $j \in I$ such that $g(j)$ bounds the minimum of

$$\{\max_L(\tau) | \max_L(\tau) \in M \setminus Q \ \& \ \tau \in S_{D,i+1}\}.$$

Then h_{i+1} is a function recursive in $\emptyset'$ defined on an $\mathcal{M}$-finite domain. By $B\Sigma^0_2$ its graph is $\mathcal{M}$-finite.

The proof of the following Claim is similar to that of Claim 2 in Theorem 4.2.

Claim 2. There is a $\hat{D} \subset \hat{g}(i+1)$ and a $\hat{\tau} \leq_{\mathcal{F}} \sigma_i$ such that $(\hat{D}, \hat{\tau})$ is $g(i+1)$-maximal for σ_i, i.e. $\hat{\tau} \in S_{\hat{D},i+1}$, $\max_L(\hat{\tau}) \in M \setminus Q$, and if $D' \supsetneq \hat{D}$ and $\tau \Vdash \varphi_e$ for each $e \in D'$ then $\max_L(\tau) \in Q$ or $\tau \not\leq_{\mathcal{F}} \hat{\tau}$.

Recursively in $\emptyset'$, choose the least pair $(\hat{D}, \hat{\tau})$ that is $\hat{g}(i+1)$-maximal for σ_i. Let $\sigma_{i+1} = \hat{\tau}$.

The above construction shows that $J = \{i | \sigma_i \text{ is defined}\}$ is closed under the successor function and therefore forms a cut. An argument similar to that for Claim 3, Theorem 4.2 yields the following:

Claim 3. $\hat{g}[J]$ is cofinal in M.

Let G be the set whose characteristic function is $\bigcup_{i \in I} \sigma_i$. G is recursive in $\emptyset'$ and for each e, $\mathcal{M}[G] \models \exists x \varphi_e$ if and only if $\sigma_i \Vdash \exists x \varphi_e$, and $\mathcal{M}[G] \models \neg\exists x \varphi_e$ if and only if $\sigma_i \Vdash \neg\exists x \varphi_e$, where $i \in J$ is the least such that $e \leq \hat{g}(i)$. Thus G is low and $\{x | G(x) = 1\}$ forms a chain under $\leq_L$, completing the proof of Theorem 5.2. $\square$

Corollary 5.1. $\mathsf{RCA}_0 + \mathsf{SCAC} + B\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + B\Sigma^0_2$. *In particular,* $\mathsf{RCA}_0 + \mathsf{SCAC}$ *does not prove* $I\Sigma^0_2$.

The final corollary follow from the above:

Corollary 5.2. $\mathsf{RCA}_0+\mathsf{CAC}+B\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0+B\Sigma^0_2$.

Proof. It was shown in [5] that $\mathsf{RCA}_0 \vdash \mathsf{CAC} \leftrightarrow \mathsf{SCAC} + \mathsf{ADS}$. By Corollary 4.3 every countable topped model $\mathcal{M}$ of $\mathsf{RCA}_0 + B\Sigma^0_2$ has an M-extension that satisfies additionally ADS. In fact, applying the constructions in this and previous sections, it is possible to obtain an M-extension of $\mathcal{M}$ with a top element satisfying instances of COH, SADS and SCAC. Alternating these steps in the construction yields a countable model that satisfies $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{CAC}$. □

We end with the following general question: In what way are combinatorial principles linked to first-order theoretic complexity? More precisely, the combinatorial principles that follow from RT^2_2, including those considered in this paper, all imply $B\Sigma^0_2$ (over the base theory RCA_0). Hence, working over models $\mathcal{M}$ of $B\Sigma^0_2$ is the most natural setting for the study of these principles. Nonetheless, constructing M-extensions to yield models of these principles while preserving $B\Sigma^0_2$ has not always been successful. For example, one would like to have either a proof from RT^2_2 of $I\Sigma^0_2$ or an M-extension theorem for RT^2_2, or even SRT^2_2, but none exists so far.

A typical combinatorial principle is a Π^1_2-statement $(\forall X)(\exists Y)P(X,Y)$, e.g. every array in $\mathcal{M}$ has a cohesive set in $\mathcal{M}$. There is a heuristic correspondence between a recursion theoretic conclusion that for every X there is a Y such that $X^{(n)} \geq_T Y^{(n)}$ and $P(X,Y)$, and a model theoretic conclusion that one can exhibit M-extensions satisfying $I\Sigma^0_n$ and $(\forall X)(\exists Y)P(X,Y)$. However, we have been unable to formulate a similar heuristic for the principle $B\Sigma^0_n$. One obstruction to making a simple correspondence comes from [6], in which it is shown that $I\Sigma^0_2$ follows from $B\Sigma^0_2$ and the existence of sufficiently generic Cohen reals. Thus, one cannot exhibit M-extentions of arbitrary models of $B\Sigma^0_2$ which satisfy $B\Sigma^0_2$ and the statement, "For all X, there is a Y such that Y is (sufficiently) Cohen generic relative to X", despite the fact that Cohen reals are low according to most recursion theoretic criteria. So, we are left with the following question: If lowness is the recursion theoretic expression of conservation over principles of Σ^0_n-induction, what is the recursion theoretic expression of conservation over principles of Σ^0_n-bounding?

References

[1] Peter Cholak, Carl Jockusch and Theodore A. Slaman. On the strength of Ramsey's Theorem. *J. Symbolic Logic*, **66** (2001): 1–55.
[2] C. T. Chong, Steffen Lempp and Yue Yang. On the role of the Σ^0_2 collection principle in second order reverse mathematics. *Proc. Amer. Math. Soc.*, **138** (2010):1093—1100.
[3] C. T. Chong and K. J. Mourad. The degree of a Σ_n cut. *Ann. Pure Appl. Logic*, **48**(3) (1990): 227–235.
[4] Rod Downey, Denis Hirschfeldt, Steffen Lempp and Reed Solomon.. A Δ^0_2 set with no infinite low subset in either it or its complement. *J. Symbolic Logic*, **66** (2001): 1371—1381.
[5] Denis Hirschfeldt and Richard Shore. Combinatorial Principles Weaker than Ramsey's Theorem for Pairs. *J. Symbolic Logic*, **72** (2007): 171–206.
[6] Denis Hirschfeldt, Theodore A. Slaman and Richard Shore. The atomic model theorem and type omitting. *Trans. Amer. Math. Soc.* **361** (2009), 5805-5837.
[7] Jeff Hirst. *Combinatorics in Subsystems of Second Order Arithmetic*. PhD thesis, Pennsylvania State University, 1987.
[8] L. A. Kirby and J. B. Paris. Σ_n-collection schemas in arithmetic. In *Logic Colloquium '77*, North–Holland Publishing Co., 199–209, Amsterdam, 1978.
[9] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Heidelberg, Springer Verlag, 1999.
[10] Theodore A. Slaman. Σ_n-bounding and Δ_n-induction. *Proc. Amer. Math. Soc.*, **132** (2004):2449-2456.
[11] Frank Stephan and Carl G. Jockusch, Jr. A Cohesive Set which is not High. *Math. Log. Quarterly*, **39** (1993): 515-530.

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