Proposition

Let \( f : [0, 1] \rightarrow \mathbb{R}^k \) be a continuous curve. Then no hyperplane through 0 contains \( k \) points on the curve iff the determinants

\[
\det[ f(t_1) | \cdots | f(t_k) ] \quad (0 \leq t_1 < \cdots < t_k \leq 1)
\]

are either all positive or all negative.

Proof

Since \( \{ (t_1, \cdots, t_k) \in \mathbb{R}^k : 0 \leq t_1 < \cdots < t_k \leq 1 \} \subseteq \mathbb{R}^k \) is connected, its image \( \{ \det[ f(t_1) | \cdots | f(t_k) ] : 0 \leq t_1 < \cdots < t_k \leq 1 \} \subseteq \mathbb{R} \) is connected.

How can we discretize this result?
Alternating curves

Theorem (Gantmakher, Krein (1950); Schoenberg, Whitney (1951))

Let $x_1, \ldots, x_n \in \mathbb{R}^k$ span $\mathbb{R}^k$. Then the following are equivalent:

(i) the piecewise-linear path $x_1, \ldots, x_n$ crosses any hyperplane through 0 at most $k - 1$ times;

(ii) the sequence $(a^T x_1, \ldots, a^T x_n)$ changes sign at most $k - 1$ times for all $a \in \mathbb{R}^n$; and

(iii) the $k \times k$ minors of the $k \times n$ matrix $[x_1 | \cdots | x_n]$ are either all nonnegative or all nonpositive.

The set of such point configurations $(x_1, \ldots, x_n)$, modulo linear automorphisms of $\mathbb{R}^k$, is the totally nonnegative Grassmannian.

Can we characterize the maximum number of hyperplane crossings of the path $x_1, \ldots, x_n$ in terms of the $k \times k$ minors of $[x_1 | \cdots | x_n]$?
The Grassmannian $\text{Gr}_{k,n}$

- The Grassmannian $\text{Gr}_{k,n}$ is the set of $k$-dimensional subspaces $V$ of $\mathbb{R}^n$.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \text{Gr}_{2,4}$$

$$\Delta_{\{1,2\}} = 1, \Delta_{\{1,3\}} = 3, \Delta_{\{1,4\}} = 2, \Delta_{\{2,3\}} = 4, \Delta_{\{2,4\}} = 3, \Delta_{\{3,4\}} = 1$$

- Given $V \in \text{Gr}_{k,n}$ in the form of a $k \times n$ matrix, for $I \in \binom{[n]}{k}$ let $\Delta_I(V)$ be the $k \times k$ minor of $V$ with columns $I$. The Plücker coordinates $\Delta_I(V)$ are well defined up to multiplication by a global nonzero constant.

- We say that $V \in \text{Gr}_{k,n}$ is totally nonnegative if $\Delta_I(V) \geq 0$ for all $I \in \binom{[n]}{k}$, and totally positive if $\Delta_I(V) > 0$ for all $I \in \binom{[n]}{k}$. Denote the set totally nonnegative $V$ by $\text{Gr}_{k,n}^{\geq 0}$, and the set of totally positive $V$ by $\text{Gr}_{k,n}^{> 0}$. 
Sign variation

For \( v \in \mathbb{R}^n \), let \( \text{var}(v) \) be the number of sign changes in the sequence \((v_1, v_2, \cdots, v_n)\), ignoring any zeros.

\[
\text{var}(1, -4, 0, -3, 6, 0, -1) = \text{var}(1, -4, -3, 6, -1) = 3
\]

Similarly, let \( \overline{\text{var}}(v) \) be the maximum of \( \text{var}(w) \) over all \( w \in \mathbb{R}^n \) obtained from \( v \) by changing zero components of \( w \).

\[
\overline{\text{var}}(1, -4, 0, -3, 6, 0, -1) = 5
\]

Theorem (Gantmakher, Krein (1950))

Let \( V \in \text{Gr}_{k,n} \).

(i) \( V \) is totally nonnegative iff \( \text{var}(v) \leq k - 1 \) for all \( v \in V \).
(ii) \( V \) is totally positive iff \( \overline{\text{var}}(v) \leq k - 1 \) for all nonzero \( v \in V \).

e.g. \[
\begin{bmatrix}
1 & 0 & -4 & -3 \\
0 & 1 & 3 & 2
\end{bmatrix} \in \text{Gr}_{2,4}^>.
\]

Note that every \( V \in \text{Gr}_{k,n} \) contains a vector \( v \) with \( \text{var}(v) \geq k - 1 \).
A history of sign variation and total positivity

- Descartes’s rule of signs (1637): The number of positive real zeros of a real polynomial \[ \sum_{i=0}^{n} a_i t^i \] is at most \( \text{var}(a_0, a_1, \ldots, a_n) \).
- Pólya (1912) asked when a linear map \( A : \mathbb{R}^k \to \mathbb{R}^n \) \( \text{diminishes variation} \), i.e. satisfies \( \text{var}(Ax) \leq \text{var}(x) \) for all \( x \in \mathbb{R}^k \). Schoenberg (1930) showed that an injective \( A \) diminishes variation iff for \( j = 1, \ldots, k \), all nonzero \( j \times j \) minors of \( A \) have the same sign.

Formations. The problem of characterizing such transformations was attacked by Schoenberg in 1930 with only partial success.

- Gantmakher, Krein (1935): The eigenvalues of a \textit{totally positive} square matrix (whose minors are all positive) are real, positive, and distinct.
A history of sign variation and total positivity

- Whitney (1952): The totally positive matrices are dense in the totally nonnegative matrices.
- Aissen, Schoenberg, Whitney (1952): Let $r_1, \cdots, r_n \in \mathbb{C}$. Then $r_1, \cdots, r_n$ are all nonnegative reals iff $s_\lambda(r_1, \cdots, r_n) \geq 0$ for all partitions $\lambda$.
- Lusztig (1994) constructed a theory of total positivity for $G$ and $G/P$.

One of the main tools in our study of $G_{\geq 0}$ and $G_{>0}$ is the theory of canonical bases in [L1]. Thus, our proof of the fact that $G_{\geq 0}$ is closed in $G$ (Theorem 4.3) is based on the positivity properties of the canonical bases (in the simply-laced case), proved in [L1],[L2], which is a non-elementary statement, depending ultimately on the Weil conjectures. The Rietsch (1997) and Marsh, Rietsch (2004) developed the theory for $G/P$.

- Fomin and Zelevinsky (2000s) introduced cluster algebras.
- Postnikov (2006) and others studied the combinatorics of $\text{Gr}_{k,n}^{\geq 0}$.
- Kodama, Williams (2014): A $\tau$-function $\tau = \sum_{I \in \binom{[n]}{k}} \Delta_I(V)s_{\lambda(I)}$ associated to $V \in \text{Gr}_{k,n}$ gives a regular soliton solution to the KP equation iff $V$ is totally nonnegative.
How close is a subspace to being totally positive?

Can we determine $\max_{v \in V} \operatorname{var}(v)$ and $\max_{v \in V \setminus \{0\}} \overline{\operatorname{var}}(v)$ from the Plücker coordinates of $V$?

**Theorem (Karp (2015))**

Let $V \in \text{Gr}_{k,n}$ and $s \geq 0$. Then $\overline{\operatorname{var}}(v) \leq k - 1 + s$ for all nonzero $v \in V$ iff

$$\overline{\operatorname{var}}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq s$$

for all $J \in \binom{[n]}{k-1}$ such that the sequence above is not identically zero.

e.g. Let $V := \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 2 & 1 & 1 \end{bmatrix} \in \text{Gr}_{2,4}$ and $s := 1$. The fact that $\overline{\operatorname{var}}(v) \leq 2$ for all $v \in V \setminus \{0\}$ is equivalent to the fact that the sequences $(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (2, 1, 1)$, $(\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (1, 4, -6)$, $(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (2, 4, -8)$, $(\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (1, -8, -6)$ each change sign at most once.
How close is a subspace to being totally nonnegative?

**Theorem (Karp (2015))**

Let \( V \in \text{Gr}_{k,n} \) and \( s \geq 0 \).

(i) If \( \text{var}(v) \leq k - 1 + s \) for all \( v \in V \), then

\[
\text{var}((\Delta_{J \cup \{i\}}(V))_{i \notin J}) \leq s \quad \text{for all } J \in \binom{[n]}{k-1}.
\]

The converse holds if \( V \) is generic (i.e. \( \Delta_I(V) \neq 0 \) for all \( I \)).

(ii) We can perturb \( V \) into a generic \( W \) with \( \max_{v \in V} \text{var}(v) = \max_{v \in W} \text{var}(v) \).

- e.g. Consider \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix} 
\sim \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0.1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 1 & 0.01 \\
0 & 1 & 0.1 & 1.001
\end{bmatrix}.
\]

The 4 sequences of Plücker coordinates are

\[
(\Delta_{\{1,2\}}, \Delta_{\{1,3\}}, \Delta_{\{1,4\}}) = (1, \emptyset, 1), \quad (\Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \Delta_{\{3,4\}}) = (\emptyset, -1, 1),
\]

\[
(\Delta_{\{1,2\}}, \Delta_{\{2,3\}}, \Delta_{\{2,4\}}) = (1, -1, \emptyset), \quad (\Delta_{\{1,4\}}, \Delta_{\{2,4\}}, \Delta_{\{3,4\}}) = (1, \emptyset, 1).
\]

- Note: \( \text{var} \) is **increasing** while \( \overline{\text{var}} \) is **decreasing** with respect to genericity.
Oriented matroids

- An *oriented matroid* is a combinatorial abstraction of a real subspace, which records the Plücker coordinates up to sign, or equivalently the vectors up to sign.

- These results generalize to oriented matroids.
Amplituhedra

Let $Z : \mathbb{R}^n \rightarrow \mathbb{R}^{k+m}$ be a linear map, and $Z_{\text{Gr}} : \text{Gr}_{k,n}^{\geq 0} \rightarrow \text{Gr}_{k,k+m}$ the map it induces on $\text{Gr}_{k,n}^{\geq 0}$. In the case that all $(k + m) \times (k + m)$ minors of $Z$ are positive, the image $Z_{\text{Gr}}(\text{Gr}_{k,n}^{\geq 0})$ is called a \textit{tree amplituhedron}.

e.g. Let $Z := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix}$ and $k := 1$. Then $Z_{\text{Gr}}(\text{Gr}_{1,5}^{\geq 0})$ equals

$$\left\{ (1 : -2a - b + d + 2e : a, b, c, d, e \geq 0, \\ 4a + b + d + 4e) : a + b + c + d + e = 1 \right\} \subseteq \mathbb{P}^2.$$
When \( k = 1 \), amplituhedra are precisely cyclic polytopes. Cyclic polytopes achieve the maximum number of faces (in every dimension) in Stanley’s upper bound theorem (1975).

Lam (2015) proposed relaxing the positivity condition on \( Z \), and called the more general class of images \( Z_{Gr}(Gr_{k,n}^{\geq 0}) \) Grassmann polytopes. When \( k = 1 \), Grassmann polytopes are precisely polytopes.

Arkani-Hamed and Trnka (2013) introduced amplituhedra in order to study scattering amplitudes, which they compute as an integral over the amplituhedron \( Z_{Gr}(Gr_{k,n}^{\geq 0}) \) when \( m = 4 \).

A scattering amplitude is a complex number whose modulus squared is the probability of observing a certain scattering process, e.g. a process involving \( n \) gluons, \( k + 2 \) of negative helicity and \( n - k - 2 \) of positive helicity.
When is $Z_{Gr}$ well defined?

- Recall that $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ is a linear map, which induces a map $Z_{Gr} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ on $Gr_{k,n}^{\geq 0}$. How do we know that $Z_{Gr}$ is well defined on $Gr_{k,n}^{\geq 0}$, i.e. $\dim(Z_{Gr}(V)) = k$ for all $V \in Gr_{k,n}^{\geq 0}$?

- Note: $\dim(Z_{Gr}(V)) = k \iff Z(v) \neq 0$ for all nonzero $v \in V$.

**Lemma**

$$\bigcup Gr_{k,n}^{\geq 0} = \{ v \in \mathbb{R}^n : \text{var}(v) \leq k - 1 \}.$$  

- $\subseteq$ follows from Gantmakher and Krein's theorem. $\supseteq$ is an exercise.

$$(2, 0, 5, -1, -4, -1, 3) \in \begin{bmatrix} 2 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \in Gr_{3,7}^{\geq 0}$$
When is $Z_{\text{Gr}}$ well defined?

**Theorem (Karp (2015))**

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ have rank $k + m$, and $W \in \text{Gr}_{k+m,n}$ be the row span of $Z$. The following are equivalent:

(i) the map $Z_{\text{Gr}}$ is well defined, i.e. $\dim(Z_{\text{Gr}}(V)) = k$ for all $V \in \text{Gr}_{k,n}^0$;

(ii) $\var(v) \geq k$ for all nonzero $v \in \ker(Z) = W^\perp$; and

(iii) $\var((\Delta_{J \setminus \{i\}}(W))_{i \in J}) \leq m$ for all $J \in \binom{[n]}{k+m+1}$ with $\dim(W_J) = k+m$.

- e.g. Let $Z := \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 3 \end{bmatrix}$, so $n = 4$, $k + m = 2$. The 4 relevant sequences of Plücker coordinates (as $J$ ranges over $\binom{[4]}{3}$) are

$(\Delta_{\{2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{1,2\}}) = (-1, -3, 5)$, $(\Delta_{\{3,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,3\}}) = (4, 5, -3)$,

$(\Delta_{\{2,4\}}, \Delta_{\{1,4\}}, \Delta_{\{1,2\}}) = (-5, 5, 5)$, $(\Delta_{\{3,4\}}, \Delta_{\{2,4\}}, \Delta_{\{2,3\}}) = (4, -5, -1)$.

The maximum number of sign changes among these 4 sequences is 1, which is at most $2 - k$ iff $k \leq 1$. Hence $Z_{\text{Gr}}$ is well defined iff $k \leq 1$. 
When is $Z_{\text{Gr}}$ well defined?

**Theorem (Karp (2015))**

Let $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ have rank $k + m$, and $W \in \text{Gr}_{k+m,n}$ be the row span of $Z$. The following are equivalent:

(i) the map $Z_{\text{Gr}}$ is well defined, i.e. $\dim(Z_{\text{Gr}}(V)) = k$ for all $V \in \text{Gr}_{k,n}^\geq$;

(ii) $\text{var}(v) \geq k$ for all nonzero $v \in \ker(Z) = W^\perp$; and

(iii) $\text{var}((\Delta_{J \setminus \{i\}}(W))_{i \in J}) \leq m$ for all $J \in \binom{[n]}{k+m+1}$ with $\dim(W_J) = k + m$.

- If $m = 0$, then (ii) $\iff$ (iii) is a ‘dual version’ of Gantmakher and Krein’s theorem: $V \in \text{Gr}_{k,n}$ is totally positive iff $\text{var}(v) \geq k$ for all $v \in V^\perp \setminus \{0\}$.
- Arkani-Hamed and Trnka’s condition on $Z$ (for $Z$ to define an amplituhedron) is that its $(k + m) \times (k + m)$ minors are all positive. In this case, $Z_{\text{Gr}}$ is well defined by either (ii) or (iii).
- Lam’s condition on $Z$ (for $Z$ to define a Grassmann polytope) is that $W$ has a totally positive $k$-dimensional subspace. This is sufficient by (ii).
- Open problem: is Lam’s condition also necessary?
Further directions

- Is there an efficient way to test whether a given $V \in \text{Gr}_{k,n}$ is totally positive using the data of sign patterns? (For Plücker coordinates, in order to test whether $V$ is totally positive, we only need to check that some particular $k(n-k)$ Plücker coordinates are positive, not all $\binom{n}{k}$.)

- Is there a simple way to index the cell decomposition of $\text{Gr}_{k,n}^{\ge 0}$ using the data of sign patterns?

- Is there a nice stratification of the subset of the Grassmannian

$$\{ V \in \text{Gr}_{k,n} : \var(v) \leq k - 1 + s \text{ for all } v \in V \},$$

for fixed $s$? (If $s = 0$, this is $\text{Gr}_{k,n}^{\ge 0}$.)

- I determined when $Z_{\text{Gr}}$ is well defined on the totally positive Grassmannian $\text{Gr}_{k,n}^{>0}$. When is $Z_{\text{Gr}}$ well defined on a given cell of $\text{Gr}_{k,n}^{>0}$?

Thank you!