LECTURES ON WAVE EQUATION

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Abstract. This is a note for the lectures given on Oct. 21st and 23rd, 2014 in lieu of D. Tataru, for the course MAT222 at UC Berkeley.

1. Wave equation

The purpose of these lectures is to give a basic introduction to the study of linear wave equation. Let $d \geq 1$. The *wave operator*, or the *d'Alembertian*, is a second order partial differential operator on $\mathbb{R}^{1+d}$ defined as

$$\Box := -\partial_t^2 + \partial_{x^1}^2 + \cdots + \partial_{x^d}^2 = -\partial_t^2 + \triangle, \tag{1.1}$$

where $t = x^0$ is interpreted as the *time* coordinate, and $x^1, \cdots, x^d$ are the coordinates for space. The corresponding PDE is given by

$$\Box\phi = F, \tag{1.2}$$

where ϕ and F are, in general, real-valued distributions on an open subset of $\mathbb{R}^{1+d}$. As usual, when the forcing term F is absent, we call (1.2) the *homogenous wave equation*. In general, (1.2) is referred to as the *inhomogeneous wave equation*.

As suggested by our terminology, the wave equation (1.2) is a evolutionary PDE, and a natural problem to ask is whether one can solve the *initial value* (or *Cauchy*) *problem*:

$$\begin{cases} \Box\phi = F, \\ (\phi, \partial_t\phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1). \end{cases} \tag{1.3}$$

We will use the notation Σ_t for the constant t-hypersurface in $\mathbb{R}^{1+d}$; hence $\Sigma_0 = \{t = 0\}$. We are being deliberately vague about the function spaces that ϕ, ϕ_0 and ϕ_1 live in; we will give a more concrete description as we go on.

Remark 1.1. Note that we prescribe not only $\phi(0)$ but also its time derivative $\partial_t\phi(0)$. This is necessary because (1.2) is second order in time. Observe that prescription of $\phi(0)$ and $\partial_t\phi(0)$ is enough to determine all derivatives of ϕ at Σ_0, and we can write down the formal power series of ϕ at each point on Σ. If ϕ_0, ϕ_1 and F are analytic, then these formal power series would converge and give a local solution to (1.3) by the Cauchy-Kowalevski theorem.

The wave equation models a variety of different physical phenomena, including:

- **Vibrating string.** It was for this example that (1.2) (with $F = 0$ and $d = 1$) was first derived by Jean-Baptiste le Rond d'Alembert.
- **Light in vacuum.** From Maxwell's equation in electromagnetism, it can be seen that each component of electric and magnetic fields satisfies (1.2) with $F = 0$ and $d = 3$.

- **Propagation of sound.** The wave equation (1.2) arises as the linear approximation of the compressible Euler equations, which describe the behavior of compressible fluids (e.g., air).
- **Gravitational wave.** A suitable geometric generalization of the wave equation (1.2) turns out to be the linear approximation of the Einstein equations, which is the basic equation of the theory of general relativity for gravity.

Needless to say, a good understanding of the *linear* operator (1.1) is fundamental for the study of any of the above topics in depth.

Our goal is to present basics of analysis of the d'Alembertian $\Box$. We will introduce three approaches:

(1) Fourier analytic method,
(2) Energy integral method,
(3) Approach using fundamental solution.

Each has its own strength and weakness, but nevertheless they all turn out to be useful in further studies.

For a systematic introduction to wave equations, it will be natural to have a discussion of the symmetries of (1.1) at this point. However, as this is a lecture with time constraint, we will be in favor of a quicker introduction and simply jump right into the analysis, deriving the symmetries of (1.1) that we need as we go on. By taking this route, it is hoped that the central role of the symmetries in the study of (1.1) would appear naturally.

2. Fourier analytic method

Note that (1.1) is a constant coefficient partial differential operator; therefore, translations in time and space commute with $\Box$, i.e.,

$$\begin{aligned} \Box\big(\phi(t+\Delta t, x^1, \ldots, x^d)\big) &= (\Box\phi)(t+\Delta t, x^1, \ldots, x^d), \\ \Box\big(\phi(t, x^1, \ldots, x^j + \Delta x^j, \ldots, x^d)\big) &= (\Box\phi)(t, x^1, \ldots, x^j + \Delta x^j, \ldots, x^d), \end{aligned} \tag{2.1}$$

This property suggests that Fourier analysis will be effective for studying $\Box$, since Fourier analysis exploits the global translation symmetries of $\mathbb{R}^{1+d}$. Indeed, the Fourier analytic method turns out to be the quickest of the three for solving (1.3), and it will be the subject of our discussion below.

Applying Fourier transform[1] in x to (1.2), we obtain the equation

$$\partial_t^2 \widehat{\phi}(t,\xi) + |\xi|^2 \widehat{\phi}(t,\xi) = \widehat{F}(t,\xi). \tag{2.2}$$

Fix $\xi \in \mathbb{R}^d$ such that $\xi \neq 0$; then the preceding equation is a second order ODE in t. We easily checked that

$$\{e^{it|\xi|}, e^{-it|\xi|}\}$$

forms a fundamental system for this ODE. Using the variation of constants formula, we see that a solution to (2.2) for each ξ is given by

$$\widehat{\phi}(t,\xi) = c_+ e^{it|\xi|} + c_- e^{-it|\xi|} + \int_0^t \Big(e^{i(t-s)|\xi|} \widehat{F_+}(s,\xi) + e^{-i(t-s))|\xi|} \widehat{F_-}(s,\xi) \Big) \, \mathrm{d}s, \tag{2.3}$$

[1]We are using the convention $\widehat{f}(\xi) = \int f(x) e^{-ix\cdot\xi} \, \mathrm{d}x$ and $f(x) = \int \widehat{f}(\xi) e^{ix\cdot\xi} \frac{\mathrm{d}\xi}{(2\pi)^d}$ for the Fourier transform.

where $c_\pm$ are to be determined from the initial data $(\widehat{\phi}_0, \widehat{\phi}_1)$, and $\widehat{F}_\pm$ can be computed from $\widehat{F}$. Carrying out the algebra using Euler's identity

$$e^{\pm it|\xi|} = \cos(t|\xi|) \pm i \sin(t|\xi|),$$

we can rewrite the preceding formula as follows:

$$\widehat{\phi}(t,\xi) = \cos(t|\xi|)\widehat{\phi}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{\phi}_1(\xi) + \int_0^t \frac{\sin((t-s)|\xi|)}{|\xi|}\widehat{F}(s,\xi)\,\mathrm{d}s. \tag{2.4}$$

The formula (2.4) describes the evolution of a single Fourier mode $\widehat{f}(\xi)$ under the wave equation (1.2) for every $\xi \neq 0$. Combining this result for different ξ's under the assumption that[2] $(\phi_0, \phi_1) \in H^k \times H^{k-1}$ and $F \in L^1_t([0,T]; H^{k-1}_x)$ (which is natural in view of the Plancherel theorem), we obtain the following solvability result for the wave equation:

Theorem 2.1 (Solvability of wave equation)**.** *Let $k \in \mathbb{Z}_+ := \{1, 2, \dots\}$ and $T > 0$. The initial value problem* (1.3) *is solvable on $[0,T] \times \mathbb{R}^d$ for $(\phi_0, \phi_1) \in H^k \times H^{k-1}$ and $F \in L^1_t([0,T]; H^{k-1}_x)$ with a unique solution $\phi(t,x) \in C_t([0,T]; H^k_x) \cap C^1_t([0,T]; H^{k-1}_x)$. The spatial Fourier transform $\widehat{\phi}(t,\xi)$ of $\phi(t,x)$ is described by the formula* (2.4).

Proof. The existence of a solution follows from simply verifying that ϕ given by (2.4) solves the equation (1.2). The fact that the solution ϕ belongs to $C_t([0,T]; H^k_x) \cap C^1_t([0,T]; H^{k-1}_x)$ is a consequence of the following *energy inequality*:

$$\|(\phi, \partial_t \phi)(t)\|_{H^k_x \times H^{k-1}_x} \leq C \|(\phi_0, \phi_1)\|_{H^k_x \times H^{k-1}_x} + C \int_0^t \|F(s)\|_{H^{k-1}_x}\,\mathrm{d}s. \tag{2.5}$$

This inequality easily follows from (2.4), triangle inequality, Minkowski's inequality and Plancherel. Uniqueness is then a consequence of the uniqueness of solutions to the ODE (2.2), applied to almost every $\xi \in \mathbb{R}^d$. $\square$

Given any function $a : \mathbb{R}^d \to \mathbb{C}$, define the *multiplier* operator $a(D)$ by the formula

$$(\widehat{a(D)f})(\xi) = a(\xi)\widehat{f}(\xi).$$

We refer to the function $a(\xi)$ as the symbol of the operator $a(D)$. Then (2.4) can be also written in the following form:

$$\phi(t,x) = \cos(t|D|)\phi_0(x) + \frac{\sin(t|D|)}{|D|}\phi_1(x) + \int_0^t \frac{\sin((t-s)|D|)}{|D|}F(s,x)\,\mathrm{d}s. \tag{2.6}$$

We would like to record a consequence of (2.4), which is one of the fundamental properties of (1.2). Consider a solution $\phi \in C_t(\mathbb{R}; H^1_x) \cap C_t(\mathbb{R}; L^2_x)$ to the homogeneous wave equation $\Box \phi = 0$ with $(\phi, \partial_t \phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1)$. Then we have

$$\begin{aligned} |\xi|\widehat{\phi}(t,\xi) &= \cos(t|\xi|)|\xi|\widehat{\phi}_0(t,\xi) + \sin(t|\xi|)\widehat{\phi}_1(t,\xi) \\ \partial_t \widehat{\phi}(t,\xi) &= -\sin(t|\xi|)|\xi|\widehat{\phi}_0(t,\xi) + \cos(t|\xi|)\widehat{\phi}_1(t,\xi) \end{aligned}$$

Hence, an easy computation shows that

$$|\xi|^2 |\widehat{\phi}(t,\xi)|^2 + |\partial_t \widehat{\phi}(t,\xi)|^2 = |\xi|^2 |\widehat{\phi}_0(\xi)|^2 + |\widehat{\phi}_1(\xi)|^2 \tag{2.7}$$

[2]The Sobolev norm $\|\cdot\|_{H^k}$ is defined as $\|\phi\|^2_{H^k} := \sum_{\ell=1}^k \|\nabla^{(\ell)}\phi\|^2_{L^2}$, and the space $H^k = H^k(\mathbb{R}^d)$ is the completion of $C^\infty_0(\mathbb{R}^d)$ with respect to this norm. See [2, Chapter 5] for more about Sobolev spaces.

for each $t \in \mathbb{R}$. Integrating this identity in ξ and using Plancherel, we arrive at the following result.

Proposition 2.2 (Conservation of energy)**.** *Let $\phi \in C_t(\mathbb{R}; H^1_x) \cap C_t(\mathbb{R}; L^2_x)$ be a solution to the homogeneous wave equation $\Box\phi = 0$. Then for any $t \in \mathbb{R}$, we have*

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d} &|\partial_t\phi(t,x)|^2 + |\partial_1\phi(t,x)|^2 + \cdots |\partial_d\phi(t,x)|^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t\phi(0,x)|^2 + |\partial_1\phi(t,x)|^2 + \cdots |\partial_d\phi(0,x)|^2 \,\mathrm{d}x.\end{aligned} \tag{2.8}$$

The time-independent or *conserved* quantity

$$E[\phi](t) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t\phi(t,x)|^2 + |\nabla_x\phi(t,x)|^2 \,\mathrm{d}x,$$

is called the *energy* of the solution ϕ at time t. It corresponds to the notion of energy in physical interpretations of the wave equation. Here $|\nabla_x\phi(t,x)|^2$ is a shorthand for

$$|\nabla_x\phi(t,x)|^2 := |\partial_1\phi(t,x)|^2 + \cdots |\partial_d\phi(t,x)|^2.$$

3. Energy integral method

Next, we present another technique for studying the wave equation, namely, the energy integral method. In the nutshell, this method consists of two parts:

(1) **Method of multipliers:** Multiply the equation $\Box\phi = F$ by $X\phi$, where X is an appropriate vector field on $\mathbb{R}^{1+d}$, and integrate by parts to derive bounds.
(2) **Method of commutators:** Commute $\Box$ with the infinitesimal symmetries (or near symmetries) to derive higher order bounds.

In this lecture, due to time constraint, we only give the simplest application of these methods, namely, an alternative proof of conservation of energy (2.8) and the energy inequality (2.5). The strength of the energy integral method lies in its robustness; hence it has proved to be effective for dealing with highly nonlinear equations. We refer the reader to the book [1] for a systematic introduction to this method.

Alternative proof of Proposition 2.2. It suffices to prove (2.8) for $t = T > 0$. We multiply $\Box\phi = 0$ by $\partial_t\phi$, and integrate over the set $(0,T) \times \mathbb{R}^d$. We compute

$$\begin{aligned}0 &= \int_0^T \int_{\mathbb{R}^d} \Box\phi\partial_t\phi \,\mathrm{d}t\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} \partial_t^2\phi\partial_t\phi - \triangle\phi\partial_t\phi \,\mathrm{d}t\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} \partial_t^2\phi\partial_t\phi + \nabla_x\phi\cdot\nabla_x\partial_t\phi \,\mathrm{d}t\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} \frac{1}{2}\partial_t(\partial_t\phi)^2 + \frac{1}{2}\partial_t|\nabla\phi|^2 \,\mathrm{d}t\mathrm{d}x,\end{aligned}$$

where ∇_x denotes the spatial gradient operator (with d components). Note that the integration by parts in x is justified thanks to the assumption $\phi \in C_t(\mathbb{R}; H^1_x)$. Applying the fundamental theorem of calculus to the t-integral, (2.8) follows. $\Box$

Next, we give an alternative proof of (2.5). Here, we use the method of commutators.

Alternative proof of of (2.5). Applying a similar argument as above to $\Box\phi = F$, we obtain

$$\frac{1}{2}\int_{\Sigma_t}(\partial_t\phi)^2 + |\nabla_x\phi|^2\,\mathrm{d}x = \frac{1}{2}\int_{\Sigma_0}(\partial_t\phi)^2 + |\nabla_x\phi|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^d} F(s,x)\partial_t\phi(s,x)\,\mathrm{d}s\mathrm{d}x. \tag{3.1}$$

for any $t \in \mathbb{R}$. Applying the Cauchy-Schwarz inequality, it is not difficult to prove the energy inequality

$$\sup_{t\in\mathbb{R}}\Big(\int_{\Sigma_t}(\partial_t\phi)^2 + |\nabla_x\phi|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \leq C\Big(\int_{\Sigma_0}(\partial_t\phi)^2 + |\nabla_x\phi|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} + C\int_0^t \|F(s,x)\|_{L^2_x}\,\mathrm{d}s \tag{3.2}$$

for some $C > 0$. (**Exercise:** Prove it!) Finally, as $\partial_t, \partial_{x^j}$ commute with $\Box$ (by translation invariance of $\Box$), we can apply the preceding method to the *commuted* equation

$$\Box(\partial_t^{\alpha_0}\partial_{x^1}^{\alpha_1}\cdots\partial_{x^d}^{\alpha_d}\phi) = \partial_t^{\alpha_0}\partial_{x^1}^{\alpha_1}\cdots\partial_{x^d}^{\alpha_d}F.$$

Combining the last observation with (3.2), we obtain an alternative proof of (2.5). $\Box$

4. Fundamental solution for d'Alembertian

Finally, we present yet another approach for studying the wave equation, namely that of the fundamental solution to the d'Alembertian.

4.1. **The case of $\mathbb{R}^{1+1}$.** As a warm-up, we first consider the $(1+1)$-dimensional case. This case is simple to analyze, but nevertheless gives us intuition about what to expect in the more difficult case of $\mathbb{R}^{1+d}$ for $d \geq 2$.

In $\mathbb{R}^{1+1}$, the d'Alembertian takes the form

$$\Box = \partial_t^2 - \partial_x^2. \tag{4.1}$$

We can formally factor $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$. It will be convenient if we find a different coordinate system in which $\partial_t - \partial_x$ and $\partial_t + \partial_x$ are coordinate derivatives. To this end, we consider the *null coordinates*

$$u = t - x, \quad v = t + x. \tag{4.2}$$

Then we have

$$\partial_u = \frac{1}{2}(\partial_t - \partial_x), \quad \partial_v = \frac{1}{2}(\partial_t + \partial_x).$$

Hence the d'Alembertian (4.1) becomes

$$\Box = 4\partial_u\partial_v. \tag{4.3}$$

Moreover, the δ_0 distribution transforms as

$$\delta_{(t,x)=(0,0)} = 2\delta_{(u,v)=(0,0)};$$

we refer to Lemma A.1 and Corollary A.2 for a proof.

We seek a fundamental solution to $\Box$, i.e., a solution E to the equation

$$\partial_u\partial_v E = \frac{1}{2}\delta_0. \tag{4.4}$$

By the factorization $\Box = \partial_u\partial_v$, we can impose the ansatz that $E(u,v)$ is the tensor product $\frac{1}{2}E_1(u)E_2(v)$ as distributions, where

$$\partial_u E_1 = \delta_{u=0}, \quad \partial_v E_2 = \delta_{v=0}.$$

We know solutions to $\partial_u E_1 = \delta_{u=0}$ are of the form

$$E_1(u) = H(u) + c_1$$

where H is the Heaviside function and a constant $c_u \in \mathbb{R}$. Similar statement applies to $E_2(v)$. Hence

$$E(u,v) = \frac{1}{2}(H(u) + c_1)(H(v) + c_2).$$

How do we choose the constants $c_1, c_2 \in \mathbb{R}$? We look for the *forward* fundamental solution, i.e., a solution E_+ to (4.4) which is supported in the half-space $\{t \geq 0\}$. Then we see that we are forced to choose $c_1 = c_2 = 0$, and we arrive at $E_+(u,v) = \frac{1}{2}H(u)H(v)$, or

$$E_+(t,x) = \frac{1}{2}H(t-x)H(t+x). \tag{4.5}$$

Next we proceed to show how to use E_+ to solve the initial value problem (1.3). We do not gain much by restricting to $\mathbb{R}^{1+1}$ for this procedure; hence, in the following section, we will simply give a general discussion on the uses of E_+, and return to $\mathbb{R}^{1+1}$ to give a concrete example.

4.2. **Uses of the forward fundamental solution.** Recall from Section 4.1 that a forward fundamental solution E_+ on $\mathbb{R}^{1+1}$ satisfied the following properties (with $d = 1$):

$$\Box E_+ = \delta_0 \tag{4.6}$$

$$\operatorname{supp} E_+ \subseteq \{(t,x) \in \mathbb{R}^{1+d} : 0 \leq |x| \leq t\}. \tag{4.7}$$

Assume, for the moment, that we have constructed E_+ exists on $\mathbb{R}^{1+d}$ for $d \geq 1$. In what follows, we explain how to use E_+ to study the wave equation.

One consequence of the fact that E_+ exists is, amusingly, that it must be the unique forward fundamental solution. In fact, we have the following statement.

Proposition 4.1. *Suppose that a forward fundamental solution E_+ with the properties* (4.6), (4.7) *exists. Then it is the unique forward fundamental solution, i.e., any fundamental solution E with* $\operatorname{supp} E \subseteq \{t \geq 0\}$ *equals E_+.*

Proof. Let E be a forward fundamental solution, i.e., $\Box E = \delta_0$ and $\operatorname{supp} E \subseteq \{t \geq 0\}$. Then we compute

$$E = \delta_0 * E = \Box E_+ * E = E_+ * \Box E = E_+.$$

The crucial fact here is (4.7), which allows us to define the convolution of E_+ and E (or more generally, some derivatives of E_+ and E). We leave the justification of the above chain of identities as an exercise. ☐

Next, we show how to derive a *representation formula* for a solution ϕ to the inhomogeneous wave equation (1.2) using E_+. The procedure we are about to describe is quite general, and hence useful in other situations when we know the existence of a fundamental solution. Before we begin, we present a similar analysis in a simpler model to motivate our computation.

Example 4.2 (Fundamental theorem of calculus via forward fundamental solution)**.** Consider the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathbb{R}$, whose forward fundamental solution is $H(x) = 1_{\{x \geq 0\}}(x)$. Given

$f, g \in C^\infty(\mathbb{R})$ that satisfy $\frac{\mathrm{d}}{\mathrm{d}x} f = g$, we may derive a representation formula for $f(x)$ on $\{x > 0\}$ as follows:

$$\begin{aligned} f(x) =& \delta_0 * (f 1_{\{x \geq 0\}})(x) \\ =& (\frac{\mathrm{d}}{\mathrm{d}x} 1_{\{x \geq 0\}}) * (f 1_{\{x \geq 0\}})(x) \\ =& 1_{\{x \geq 0\}} * (f \delta_0)(x) + 1_{\{x \geq 0\}} * (g 1_{\{x \geq 0\}})(x) \\ =& f(0) + \int_0^x g(y) \, \mathrm{d}y. \end{aligned}$$

As we see, the trick is to stick in $H(x)$ with f to access the information at the endpoint 0. Note that replacing f by $f 1_{\{x \geq 0\}}$ also allows us to justify the convolution for any $f \in C^\infty(\mathbb{R})$. (**Exercise:** Justify the computation $H * (f \delta_0)(x) = f(0)$!) Of course, the resulting representation formula is nothing but the *fundamental theorem of calculus.*

Now we return to the wave equation on $\mathbb{R}^{1+d}$. Let $\phi, F \in C^\infty(\mathbb{R}^{1+d})$ solve the equation

$$\Box \phi = \partial_t^2 \phi - \triangle \phi = F.$$

Let $(t, x) \in \mathbb{R}^{1+d}$ with $t > 0$. We compute

$$\begin{aligned} \phi(t, x) =& \delta_0 * \phi 1_{\{t \geq 0\}} \\ =& \Box E_+ * \phi 1_{\{t \geq 0\}} \\ =& \partial_t^2 E_+ * \phi 1_{\{t \geq 0\}}(t) - E_+ * \triangle \phi 1_{\{t \geq 0\}} \\ =& \partial_t^2 E_+ * \phi 1_{\{t \geq 0\}} - E_+ * (\partial_t^2 \phi) 1_{\{t \geq 0\}} + E_+ * F 1_{\{t \geq 0\}} \end{aligned}$$

where 1_X is denotes the characteristic function of a set X. We then formally compute

$$\begin{aligned} & \partial_t^2 E_+ * \phi 1_{\{t \geq 0\}} - E_+ * (\partial_t^2 \phi) 1_{\{t \geq 0\}} \\ & = \partial_t E_+ * (\partial_t \phi) 1_{\{t \geq 0\}} + \partial_t E_+ * \phi \delta_{t=0} - E_+ * (\partial_t^2 \phi) 1_{\{t \geq 0\}} \\ & = E_+ * (\partial_t^2 \phi) 1_{\{t \geq 0\}} + E_+ * (\partial_t \phi) \delta_{t=0} + \partial_t E_+ * \phi \delta_{t=0} - E_+ * (\partial_t^2 \phi) 1_{\{t \geq 0\}} \\ & = E_+ * (\partial_t \phi) \delta_{t=0} + \partial_t E_+ * \phi \delta_{t=0}. \end{aligned}$$

In fact, the expressions $(E_+ * \phi \delta_{t=0})(t, x)$ and $(\partial_t E_+ * \phi \delta_{t=0})(t, x)$ always make sense in $\{t > 0\}$, and hence the above computation is justified. It suffices to show that for every $t_0 > 0$ and $\psi \in C^\infty(\mathbb{R}^d)$, we can make sense of the following expressions:

$$\langle E_+ \mid \delta_{t=t_0} \psi \rangle, \quad \langle \partial_t E_+ \mid \delta_{t=t_0} \psi \rangle.$$

The preceding statement is a consequence of the fact that $\partial_t^2 E_+(t, x) = \triangle E_+(t, x)$ on $\{t > 0\}$, which implies that E_+ is C^2 in t with values in $\mathcal{D}'(\mathbb{R}^d)$; see Lemma A.3.

For $\phi \in C^\infty(\mathbb{R}^{1+d})$, note that

$$\phi(t, x) \delta_{t=0} = \phi \restriction_{\{t=0\}} (x) \delta_{t=0}, \quad \partial_t \phi(t, x) \delta_{t=0} = \partial_t \phi \restriction_{\{t=0\}} (x) \delta_{t=0}.$$

Let us write $(\phi_0, \phi_1) = (\phi, \partial_t \phi) \restriction_{\{t=0\}}$. Putting everything together, we arrive at the following proposition.

Proposition 4.3 (Representation formula)**.** *Suppose that a forward fundamental solution E_+ with the properties* (4.6), (4.7) *exists. Then given any solution ϕ to the equation $\Box\phi = F$ with $\phi, F \in C^\infty(\mathbb{R}^{1+d})$, we have the formula*

$$\phi = E_+ * \phi_1 \delta_{t=0} + \partial_t E_+ * \phi_0 \delta_{t=0} + E_+ * F 1_{\{t \geq 0\}}. \tag{4.8}$$

where $(\phi_0, \phi_1) := (\phi, \partial_t \phi) \restriction_{\{t=0\}}$.

Thanks to the support property (4.7), we have the following corollary.

Corollary 4.4 (Finite speed of propagation)**.** *Suppose that a forward fundamental solution E_+ with the properties* (4.6), (4.7) *exists. Let $\phi \in C^\infty(\mathbb{R}^{1+d})$ solve the inhomogeneous wave equation $\Box\phi = F$ with initial data $(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (\phi_0, \phi_1)$, and consider a point $(t, x) \in \mathbb{R}^{1+d}$ such that $t > 0$. If*

$$\begin{aligned} F(s, y) =& 0 \quad \textit{in } \{(s, y) : 0 < s < t,\ |y - x| \leq t - s\}, \\ (\phi_0, \phi_1)(y) =& (0, 0) \quad \textit{in } \{y : |y - x| \leq t\} \end{aligned}$$

then $\phi(t, x) = 0$.

The regularity hypothesis for ϕ and F in both Proposition 4.3 and Corollary 4.4 can be weakened considerably, we leave this task as an exercise to the reader. We also remark that analogous statements can be proved in the negative time direction, simply by reversing the time coordinate $t \mapsto -t$.

Finally, we show that the representation formula (4.8) can be used to solve the initial value problem (1.3). The idea, of course, is to simply take (4.8) as a *definition* of a solution ϕ, noting that the right-hand side only involves the data of (1.3). It is easy to see that ϕ solves $\Box\phi = F$; we are left to verify that ϕ obeys the initial condition, i.e.,

$$\lim_{t \to 0} (\phi, \partial_t \phi)(t, x) = (\phi_0, \phi_1).$$

For this purpose, it is convenient to define the time-dependent distribution $E_+(t)$ in the space $C^2((0, \infty); \mathcal{D}'(\mathbb{R}^d))$ by the formula

$$\langle E_+(t_0) \mid \psi \rangle := \langle E_+ \mid \delta_{t=t_0} \psi \rangle \quad \text{for every } \psi \in C_0^\infty(\mathbb{R}^d). \tag{4.9}$$

Then for every $\phi \in C_0^\infty(\mathbb{R}^{1+d})$, we have the identity

$$\langle E_+ \mid \phi \rangle = \int_0^\infty \langle E_+(t) \mid \phi(t) \rangle \, \mathrm{d}t. \tag{4.10}$$

This identity clearly holds for ϕ supported in the half-space $\{t > 0\}$. Next, by the support properties of E_+ and $E_+(t)$, it is easy to see that it holds for ϕ supported in $\mathbb{R}^{1+d} \setminus \{(0, 0)\}$ as well. The full identity then follows by noting that both sides are homogeneous of the same degree.

We may now rewrite the representation formula (4.8) as

$$\begin{aligned} \phi(t) =& (\partial_t E_+)(t) * \phi_0 + E_+(t) * \phi_1 + \int_0^t E_+(t - s) * F(s) \, \mathrm{d}s \\ =& \partial_t (E_+(t) * \phi_0) + E_+(t) * \phi_1 + \int_0^t E_+(t - s) * F(s) \, \mathrm{d}s, \end{aligned} \tag{4.8$'$}$$

where all convolutions are only with respect to the spatial coordinates x.

We claim that

$$E_+(+0) = 0, \quad \partial_t E_+(+0) = \delta_0, \quad \partial_t^2 E_+(+0) = 0. \tag{4.11}$$

as distributions in $\mathcal{D}'(\mathbb{R}^d)$.

That $E''_+(+0) = 0$ follow from $E_+(+0) = 0$ and $\partial_t^2 E_+(t) = \triangle E_+(t)$ for $t > 0$ (**Exercise:** Verify!). Hence, it suffices to prove $E_+(+0) = 0$ and $\partial_t E_+(+0) = \delta_0$. Indeed, for $\phi \in C_0^\infty(\mathbb{R}^{1+d})$, we have

$$\begin{aligned}
\phi(0,0) =& \langle E_+ \mid \Box\phi\rangle \\
=& \int_0^\infty \langle E_+(t) \mid \Box\phi(t)\rangle \,\mathrm{d}t \\
=& \int_0^\infty \langle E_+(t) \mid \partial_t^2\phi(t)\rangle \,\mathrm{d}t + \int_0^\infty \langle -\triangle E_+(t) \mid \phi(t)\rangle \,\mathrm{d}t \\
=& \int_0^\infty \langle E_+(t) \mid \partial_t^2\phi(t)\rangle \,\mathrm{d}t - \int_0^\infty \langle \partial_t^2 E_+(t) \mid \phi(t)\rangle \,\mathrm{d}t \\
=& \langle \partial_t E_+(+0) \mid \phi(0)\rangle - \langle E_+(+0) \mid \partial_t\phi(0)\rangle.
\end{aligned}$$

Given $\psi \in C_0^\infty(\mathbb{R}^d)$, choosing ϕ so that $(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (\psi, 0)$ and $(\phi, \partial_t\phi)\restriction_{\{t=0\}} = (0, \psi)$, the desired conclusion follows for $E'_+(+0)$ and $E_+(+0)$, respectively.

We have proved the following proposition.

Proposition 4.5 (Solvability of the wave equation)**.** *Suppose that a forward fundamental solution E_+ with the properties* (4.6), (4.7) *exists. Given $\phi_0, \phi_1 \in C^\infty(\mathbb{R}^d)$ and $F \in C^\infty(\mathbb{R}^{1+d})$, there exists a unique solution ϕ to the initial value problem* (1.3) *defined by the formula* (4.8).

We end this section by applying the general theory we developed to the case $\mathbb{R}^{1+1}$, where we already know the form of the forward fundamental solution. Recall that $E_+(t,x) = \frac{1}{2}H(t-x)H(t+x)$. We compute

$$\begin{aligned}
E_+ * \phi_1\delta_{t=0} =& \frac{1}{2}\langle H(t-s-(x-y))H(t-s+x-y) \mid \phi_1(y)\delta_0(s)\rangle_{y,s} \\
=& \frac{1}{2}\langle H(t-(x-y))H(t+x-y) \mid \phi_1(y)\rangle_y \\
=& \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y)\,\mathrm{d}y,
\end{aligned}$$

and

$$\begin{aligned}
E_+ * F1_{\{t\geq 0\}} =& \frac{1}{2}\langle H(t-s-(x-y))H(t-s+x-y) \mid F(s,y)H(s)\rangle_{y,s} \\
=& \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} F(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Furthermore,

$$\begin{aligned}\partial_t E_+ * \phi_0 \delta_{t=0} =& \partial_t (E_+ * \phi_0 \delta_{t=0}) \\ =& \partial_t \Big(\frac{1}{2} \int_{x-t}^{x+t} \phi_0(y) \, \mathrm{d}y \Big) \\ =& \frac{1}{2} (\phi_0(x+t) + \phi_0(x-t)).\end{aligned}$$

Hence we arrive at *d'Alembert's formula* in $\mathbb{R}^{1+1}$:

Theorem 4.6 (d'Alembert's formula)**.** *Let ϕ be a solution to the equation $\Box\phi = F$ with $\phi, F \in C^\infty(\mathbb{R}^{1+1})$. Then we have the formula*

$$\phi(t,x) = \frac{1}{2}(\phi_0(x-t) + \phi_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y)\,\mathrm{d}y + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} F(s,y)\,\mathrm{d}y\mathrm{d}s. \tag{4.12}$$

where $(\phi_0, \phi_1) = (\phi, \partial_t \phi)\restriction_{\{t=0\}}$.

Conversely, given any initial data $(\phi_1, \phi_2) \in C^\infty(\mathbb{R})$ and $F \in C^\infty(\mathbb{R}^{1+1})$, there exists a unique solution ϕ to the initial value problem (1.3) *defined by the formula* (4.12)*.*

4.3. **General dimension.** Our goal now is to construct the forward fundamental solution E_+ to the d'Alembertian on $\mathbb{R}^{1+d}$ for every $d \geq 1$. Although the existence of the forward fundamental solution can be deduced by more abstract means (see, e.g., Remark **??**), there is no systematic way to explicitly construct E_+. To find an explicit formula, we will make an *ansatz* (i.e., an educated guess) of the form of E_+, based on the symmetries of the d'Alembertian $\Box$.

Symmetries: Rotation, Lorentz boosts, scaling. We have already seen that $\Box$ is invariant under translations; however, these symmetries will not be useful for the purpose of finding a solution to $\Box E_+ = \delta_0$, since δ_0 is *not* invariant under translations. We need to determine the symmetries of $\Box$ which fix the origin.

Such symmetries turn out to be precisely the linear transformations $L : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ which leave invariant the scalar quantity[3]

$$s^2(t,x) := t^2 - |x|^2. \tag{4.13}$$

These transformations are called *Lorentz transformations.* (**Exercise:** From the defining property $s^2(t,x) = s^2(L(t,x))$, show that $\Box(\phi \circ L) = (\Box\phi) \circ L$.) The Lorentz transformations form a group (by composition), which we will denote by $O(1,d)$. The group $O(1,d)$ is generated by the following elements:

(1) **Rotations.** Linear transformation $R : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ represented by the matrix

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widetilde{R} & \\ 0 & & & \end{pmatrix} \tag{4.14}$$

where $\widetilde{R} \in O(d)$ is a $d \times d$ orthonormal matrix.

[3]This quantity, of course, has a geometric meaning. It is precisely the 'space-time distance' from the origin to the event (t,x) in special relativity.

(2) **Reflection.** Linear transformation $\rho_k : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ $(k = 0, \dots, d)$ defined by

$$(x^0, \cdots, x^d) \mapsto (x^0, \cdots, -x^k, \cdots, x^d). \tag{4.15}$$

(3) **Lorentz boosts.** These symmetries correspond to choosing another frame of reference, which travels at a constant velocity compared to the original frame. If the frame moves at speed $\gamma \in (0,1)$ in the x_1 direction, then its matrix representation is

$$\Lambda_{01}(\gamma) = \begin{pmatrix} \frac{1}{\sqrt{1-\gamma^2}} & -\frac{\gamma}{\sqrt{1-\gamma^2}} & 0 & \cdots & 0 \\ \frac{-\gamma}{\sqrt{1-\gamma^2}} & \frac{1}{\sqrt{1-\gamma^2}} & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \mathrm{Id}_{d-1 \times d-1} & \\ 0 & 0 & & & \end{pmatrix} \tag{4.16}$$

All Lorentz boosts then take the form $\Lambda(\gamma) = cR\Lambda_{01}(\gamma)R^{-1}$ for some constant $c \neq 0$ and rotation R.

For more on Lorentz transformations, we refer to [4, Chapter 9].

Although it is not exactly a symmetry of $\Box$, we also point out that $\Box$ transforms in a simple way under *scaling*, i.e.,

$$\Box(\phi(t/\lambda, x/\lambda)) = \lambda^{-2}(\Box\phi)(t/\lambda, x/\lambda) \quad \text{for } \lambda > 0.$$

In particular, if ϕ is homogeneous of degree a, then $\Box\phi$ is homogeneous of degree $a - 2$.

Fundamental solution using distribution theory. Now we construct the forward fundamental solution to the d'Alembertian, which we refer to as E_+. From the scaling symmetry of $\Box$, it is natural to look for E_+ which is homogeneous. From the equation

$$\Box E_+ = \delta_0,$$

observe that the right-hand side, being a delta distribution on $\mathbb{R}^{1+d}$, is homogeneous of degree $-d-1$. Since $\Box$ lowers the degree of homogeneity by 2, we see that

$$\text{If } E_+ \text{ is homogeneous, then it must be of degree } -d+1. \tag{4.17}$$

A nice feature of assuming E_+ to be homogeneous is that we can focus on E_+ on $\mathbb{R}^{1+d} \setminus \{(0,0)\}$, as homogeneity then allows us to extend E_+ uniquely to $\mathbb{R}^{1+d}$; see Lemma A.5.

Next, recall that $\Box$ is invariant under rotations and Lorentz transformations. Furthermore, as they are linear maps with determinant ± 1 (**Exercise:** Prove this statement!), δ_0 is also invariant under these symmetries. Hence it is natural to look for a solution that is invariant under rotations and Lorentz transforms (recall, e.g., the fundamental solution for the Laplacian). Recall that Lorentz transformations are precisely the linear transformations which leave the scalar quantity $s^2(t,x) := t^2 - |x|^2$ invariant. Note, moreover, that $t^2 - |x|^2$ is homogeneous of degree 2. Combined with the earlier observation (4.17), we see that a reasonable first try would be

$$G(t,x) = \chi(t^2 - |x|^2), \tag{4.18}$$

where χ is a homogeneous distribution of degree $-\frac{d-1}{2}$ on $\mathbb{R}$.

To pin down the homogeneous distribution χ, we now bring up the requirement that E_+ must be supported in the upper half-space $\{t \geq 0\}$. Unfortunately, $G(t,x)$ is symmetric under $t \mapsto -t$ so the the naive guess (4.18) fails to work as it is. However, we may multiply by the Heaviside function $H(t)$ without disturbing homogeneity and Lorentz invariance (i.e., invariance under rotations and Lorentz transforms), as $H(t)$ is also homogeneous of degree 0 and Lorentz invariant. In order to make sense of the distribution $1_{\{t\geq 0\}}G(t,x)$ in $\mathbb{R}^{1+d}\setminus\{(0,0)\}$, we are motivated to find $G(t,x)$ that vanishes in a neighborhood of $\Sigma_0\setminus\{(0,0)\}$. This consideration dictates χ to be a homogeneous distribution on $\mathbb{R}$ (of degree $-\frac{d-1}{2}$) that is supported on $[0,\infty)$, i.e.,

$$\chi = c\chi_+^{-\frac{d-1}{2}} \quad \text{for some constant } c\neq 0.$$

(We refer to Appendix A.4 for a quick recap of the theory of homogeneous distributions.)

Motivated by the preceding considerations, we define $E_+(t,x)$ on $\mathbb{R}^d\setminus\{(0,0)\}$ by the formula

$$E_+(t,x) = c_d\, 1_{\{t\geq 0\}}\chi_+^{-\frac{d-1}{2}}(t^2-|x|^2), \tag{4.19}$$

and extend it to $\mathbb{R}^{1+d}$ by homogeneity (see Lemma A.5). Here, the composition of the distribution $\chi_+^{\frac{d-1}{2}}$ with $t^2-|x|^2$ on $\mathbb{R}^{1+d}\setminus\{(0,0)\}$ is to be interpreted as the following limit in the sense of distributions:

$$h_j(t^2-|x|^2) \rightharpoonup \chi_+^{-\frac{d-1}{2}}(t^2-|x|^2) \quad \text{as } j\to\infty, \tag{4.20}$$

where $h_j\in C_0^\infty$ is a sequence such that $h_j \rightharpoonup \chi_+^{-\frac{d-1}{2}}$. That this procedures is well-defined is justified in Lemma A.4.

We claim that $E_+(t,x)$ is the forward fundamental solution for the d'Alembertian, for an appropriate choice of the constant c_d. We begin with the computation

$$\Box E_+(t,x) = c_d(\partial_t^2-\triangle)(1_{\{t\geq 0\}}\chi_+^{-\frac{d-1}{2}}(t^2-|x|^2)) = c_d 1_{\{t\geq 0\}}(\partial_t^2-\triangle)\chi_+^{-\frac{d-1}{2}}(t^2-|x|^2).$$

Indeed, $-\triangle$ easily commutes with $1_{\{t\geq 0\}}$, and whenever ∂_t falls on $1_{\{t\geq 0\}}$ the result is zero thanks to the support properties. Using the chain rule, which is easily justified by approximation by C_0^∞ functions, on $\mathbb{R}^{1+d}\setminus(0,0)$ we have

$$\begin{aligned}(\partial_t^2-\triangle)\chi_+^{-\frac{d-1}{2}}(t^2-|x|^2) =& \partial_t(2t\chi_+^{-\frac{d+1}{2}}(t^2-|x|^2)) + \nabla_x\cdot(2x\chi_+^{-\frac{d+1}{2}}(t^2-|x|^2)) \\ =& 4(t^2-|x|^2)\chi_+^{-\frac{d+3}{2}}(t^2-|x|^2) + 2(d+1)\chi_+^{-\frac{d+1}{2}}(t^2-|x|^2).\end{aligned}$$

By the identity $x\chi_+^a(x) = (a+1)\chi_+^{a+1}(x)$, the last line equals

$$4(-\frac{d+1}{2})\chi_+^{-\frac{d+1}{2})}(t^2-|x|^2) + 2(d+1)\chi_+^{-\frac{d+1}{2}}(t^2-|x|^2) = 0.$$

Therefore, we see that $\Box E_+$ is a distribution which is supported only on $\{0\}$; moreover, by construction, $\Box E_+$ is homogeneous of degree $-d-1$. It then follows that (**Exercise:** Verify!)

$$\Box E_+ = c\delta_0 \tag{4.21}$$

for some $c\in\mathbb{R}$.

Checking that E_+ is a fundamental solution now boils down to showing that $c=1$ for an appropriate constant c_d. We claim that the choice

$$c_d = \frac{\pi^{(1-d)/2}}{2} \tag{4.22}$$

leads to $\Box E_+ = \delta_0$; we defer the proof of this claim until Appendix B.

Remark 4.7. Computing the exact constants c and c_d requires explicit computation, but the fact that $c \neq 0$ (and hence some appropriate c_d exists) can be seen by much softer methods. For example, it is sufficient to establish the following uniqueness statement: If $E \in \mathcal{D}'(\mathbb{R}^{1+d})$ is a solution to $\Box E = 0$ with $\mathrm{supp} E \subseteq \{t \geq 0\}$, then $E = 0$. This statement can be proved by, say, Theorem 2.1 (solvability of the initial value problem by Fourier analysis) and duality; we leave this as an exercise for the interested reader.

We now discuss applications of the explicit formula (4.19) for E_+. First, note that for $d \geq 3$ an odd integer, we have

$$E_+(t,x) = c_d 1_{t\geq 0} \chi_+^{-\frac{d-1}{2}}(t^2 - |x|^2) = c_d 1_{t \geq 0} \delta_0^{(\frac{d-3}{2})}(t^2 - |x|^2)$$

which is supported only on the boundary $\{(t,x) : |x| = t\}$ of the cone $\{(t,x) : |x| \leq t\}$. Hence a sharper version of Corollary 4.4 holds in this case. It turns out that this property does *not* hold when $d \geq 2$ is even (see, e.g., the computation of the case $d = 2$ below). This phenomenon is called the *sharp Huygens principle*; we record the precise statement in the following proposition.

Proposition 4.8 (Sharp Huygens principle)**.** *Let $d \geq 3$ be an odd integer. Let $\phi \in C^\infty(\mathbb{R}^{1+d})$ solve the inhomogeneous wave equation $\Box \phi = F$ with initial data $(\phi, \partial_t \phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1)$, and consider a point $(t,x) \in \mathbb{R}^{1+d}$ such that $t > 0$. If*

$$\begin{aligned} F(s,y) =& 0 \quad \text{ in } \{(s,y) : 0 < s < t,\ |y - x| = t - s\}, \\ (\phi_0, \phi_1)(y) =& (0,0) \quad \text{ in } \{y : |y - x| = t\} \end{aligned}$$

then $\phi(t,x) = 0$.

Next, we specialize to the cases $d = 1, 2, 3$ and derive classical representation formulae for the wave equation.

Explicit computation for $d = 1$. We now compute the form of the forward fundamental solution E_+ explicitly in dimension $d = 1$. When $d = 1$, we have

$$E_+(t,x) = c_1 1_{\{t \geq 0\}}\, \chi_+^0(t^2 - |x|^2) = c_1 1_{\{t\geq 0\}} H(t^2 - |x|^2) = c_1 1_{\{(t,x): 0 \leq |x| \leq t\}}.$$

As $c_1 = \frac{1}{2}$, we recover the previous computation.

Explicit computation for $d = 2$. Next, we compute the form of the forward fundamental solution E_+ explicitly in dimension $d = 2$. We have

$$E_+(t,x) = c_2 1_{\{t \geq 0\}}\, \chi_+^{-\frac{1}{2}}(t^2 - |x|^2) = c_2' 1_{\{t \geq 0\}} \frac{1}{(t^2 - |x|^2)_+^{\frac{1}{2}}} = c_2' 1_{\{(t,x): 0 \leq |x| \leq t\}} \frac{1}{(t^2 - |x|^2)^{\frac{1}{2}}},$$

outside the origin, and at the origin E_+ is determined by homogeneity. By (4.22) and the definition of $\chi_+^{-\frac{1}{2}}$ (see (A.12) in the appendix), we have $c_2' = \frac{c_2}{\Gamma(\frac{1}{2})} = \frac{c_2}{\sqrt{\pi}} = \frac{1}{2\pi}$. Hence we arrive at the formula

$$E_+(t,x) = \frac{1}{2\pi} 1_{\{(t,x): 0 \leq |x| \leq t\}} \frac{1}{(t^2 - |x|^2)^{\frac{1}{2}}}. \tag{4.23}$$

We may easily compute

$$
\begin{aligned}
E_+ * (\phi_1 \delta_{t=0})(t,x) =& \langle E_+(t-s, x-y) \mid \phi_1(y)\delta_0(s)\rangle_{y,s} \\
=& \frac{1}{2\pi} \int_{\{|x|\leq t\}} \frac{\phi_1(y)}{(t^2 - |x-y|^2)^{\frac{1}{2}}} \, \mathrm{d}y \\
E_+ * (F\, 1_{\{t\geq 0\}})(t,x) =& \langle E_+(t-s, x-y) \mid F(s,y)H(s)\rangle_{y,s} \\
=& \frac{1}{2\pi} \int_0^t \int_{\{|x|\leq s\}} \frac{F(s,y)}{((t-s)^2 - |x-y|^2)^{\frac{1}{2}}} \, \mathrm{d}y \, \mathrm{d}s.
\end{aligned}
$$

Combined with Propositions 4.3 and 4.5, we recover *Poisson's formula*:

Theorem 4.9 (Poisson's formula)**.** *Let ϕ be a solution to the equation $\Box\phi = F$ with $\phi, F \in C^\infty(\mathbb{R}^{1+2})$. Then we have the formula*

$$
\begin{aligned}
\phi(t,x) =& \partial_t \Big(\frac{1}{2\pi} \int_{\{|x|\leq t\}} \frac{\phi_0(y)}{(t^2 - |x-y|^2)^{\frac{1}{2}}} \, \mathrm{d}y \Big) + \frac{1}{2\pi} \int_{\{|x|\leq t\}} \frac{\phi_1(y)}{(t^2 - |x-y|^2)^{\frac{1}{2}}} \, \mathrm{d}y \\
=& + \frac{1}{2\pi} \int_0^t \int_{\{|x|\leq s\}} \frac{F(s,y)}{((t-s)^2 - |x-y|^2)^{\frac{1}{2}}} \, \mathrm{d}y \, \mathrm{d}s.
\end{aligned}
\tag{4.24}
$$

where $(\phi_0, \phi_1) = (\phi, \partial_t \phi) \restriction_{\{t=0\}}$ and $B_{0,t}(x)$ is the ball $\{(0,y) : |x| \leq t\}$.

Conversely, given any initial data $(\phi_1, \phi_2) \in C^\infty(\mathbb{R}^2)$ and $F \in C^\infty(\mathbb{R}^{1+2})$, there exists a unique solution ϕ to the initial value problem (1.3) *defined by the formula* (4.24)*.*

Explicit computation for $d = 3$. Finally, we compute the form of the forward fundamental solution E_+ explicitly in dimension $d = 3$. Recall that $\chi_+^{-1} = \delta_0$; hence

$$
E_+(t,x) = c_3 1_{\{t\geq 0\}} \, \delta_0(t^2 - |x|^2),
$$

outside the origin, and at the origin E_+ is determined by homogeneity. By (4.22), we have $c_3 = \frac{1}{2\pi}$.

Lemma 4.10. *On $\mathbb{R}^{1+3} \setminus \{0\}$, we have the identity*

$$
\delta_0(t^2 - |x|^2) = \frac{1}{2\sqrt{2}t} \mathrm{d}\sigma_{C_0^+}(t,x)
\tag{4.25}
$$

where $C_0^+ := \{(t,x) : t = |x|, t \geq 0\}$ is a forward cone and $\mathrm{d}\sigma_{C_0^+}$ is the induced measure on C_0^+. Moreover, for $t_0 > 0$ we have

$$
\delta_{t_0}(t)\mathrm{d}\sigma_{C_0^+}(t,x) = \sqrt{2}\mathrm{d}\sigma_{S_{t_0}}(t,x)
\tag{4.26}
$$

where S_{t_0} is the sphere $\{(t,x) : t = t_0, |x| = t_0\}$ and $\mathrm{d}\sigma_{S_{t_0}}$ is the induced measure on S_{t_0}.

Proof. Let (r, ω) be the standard polar coordinates on $\mathbb{R}^3 \setminus \{0\}$, i.e.,

$$
(r, \omega) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathbb{S}^2.
$$

We employ the null coordinates (u, v, ω) on $\mathbb{R}^{1+3}$, which is defined by

$$
(u, v) = (t - r, \; t + r).
$$

Then we have $t^2-|x|^2=uv$ and $v=2t=2r$ on C_0^+. Recalling the formula for the induced measure on C_0^+, we see that $\mathrm{d}\sigma_{C_0^+}(u,v,\omega)$ takes the form[4]

$$\int\phi(u,v,\omega)\mathrm{d}\sigma_{C_0^+}(u,v,\omega)=\iint\phi(0,v,\omega)\frac{v^2}{4\sqrt{2}}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega) \tag{4.27}$$

for every $\phi\in C_0^\infty$. Hence we wish to show

$$\langle\delta_0(uv)\mid\phi\rangle=\iint\phi(0,v,\omega)\frac{v}{8}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega).$$

Let $h_j\in C_0^\infty(\mathbb{R})$ be a sequence such that $h_j\rightharpoonup\delta_0$ as $j\to\infty$. Writing out the $\langle h_j(uv)\mid\phi\rangle$ and making a change of variables $\overline{u}=uv$, we obtain

$$\begin{aligned}\langle h_j(uv)\mid\phi(u,v,\omega)\rangle_{u,v,\omega}&=\iiint h_j(uv)\phi(u,v,\omega)\frac{v^2}{8}\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)\\&=\int h_j(\overline{u})\Big(\iint\phi(\frac{\overline{u}}{v},v,\omega)\frac{v}{8}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)\Big)\mathrm{d}\overline{u}\\&\to\iint\phi(0,v,\omega)\frac{v}{8}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)\quad\text{as } j\to\infty,\end{aligned}$$

where we used the fact that ϕ is supported away from $\{v=0\}$, which is simply the origin in $\mathbb{R}^{1+3}$. The proof of (4.25) is complete.

Now we turn to (4.26). Using (4.27), we compute

$$\begin{aligned}&\langle\delta_0(\frac{1}{2}(v+u)-t_0)\mathrm{d}\sigma_{C_0^+}(u,v)\mid\phi(u,v,\omega)\rangle_{u,v,\omega}\\&=\langle h_j(\frac{1}{2}(v+u)-t_0)\mathrm{d}\sigma_{C_0^+}(u,v)\mid\phi(u,v,\omega)\rangle_{u,v,\omega}\\&=\iint h_j(\frac{1}{2}v-t_0)\phi(0,v,\omega)\frac{v^2}{4\sqrt{2}}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)\\&=\iint h_j(\overline{v})\phi(0,2(t_0+\overline{v}),\omega)\sqrt{2}(t_0+\overline{v})^2\mathrm{d}\overline{v}\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)\\&\to\int\phi(0,2t_0,\omega)\sqrt{2}t_0^2\mathrm{d}\sigma_{\mathbb{S}^2}(\omega)=\sqrt{2}\int\phi\,\mathrm{d}\sigma_{S_{t_0}},\end{aligned}$$

which proves (4.26). □

Using Propositions 4.3, 4.5 and Lemma 4.10, now it is not difficult to prove *Kirchhoff's formula*:

Theorem 4.11 (Kirchhoff's formula)**.** *Let ϕ be a solution to the equation $\Box\phi=F$ with $\phi,F\in C^\infty(\mathbb{R}^{1+3})$. Then we have the formula*

$$\begin{aligned}\phi(t,x)=&\partial_t\Big(\frac{1}{2\pi t}\int_{S_{0,t}(x)}\phi_0(y)\,\mathrm{d}\sigma(y)\Big)+\frac{1}{2\pi t}\int_{S_{0,t}(x)}\phi_1(y)\,\mathrm{d}\sigma(y)\\&+\frac{1}{2\pi t}\int_0^t\int_{S_{0,t-s}(x)}F(s,y)\,\mathrm{d}\sigma(y)\end{aligned} \tag{4.28}$$

[4]Strictly speaking, $\mathrm{d}\sigma_{C_0^+}(u,v,\omega)$ is the composition of $\mathrm{d}\sigma_{C_0^+}(t,x)$ with the coordinate map $(u,v,\omega)\mapsto(t,x)$, which is well-defined by Lemma A.1.

where $(\phi_0, \phi_1) = (\phi, \partial_t \phi)\restriction_{\{t=0\}}$ *and* $S_{0,t}(x)$ *is the sphere* $\{(0,y) : |y-x| = t\}$.

Conversely, given any initial data $(\phi_1, \phi_2) \in C^\infty(\mathbb{R}^3)$ *and* $F \in C^\infty(\mathbb{R}^{1+3})$*, there exists a unique solution* ϕ *to the initial value problem* (1.3) *defined by the formula* (4.28)*.*

Remark 4.12. For an alternative approach to derivation of the classical representation formulae, which does not use the theory of distributions, we refer the reader to [2, Chapter 2].

Appendix A. Recap of distribution theory

The purpose of this appendix is to gather some results from distribution theory which are necessary in the lectures.

A.1. **Change of variables for distributions.** In the notes, we often changed coordinates to better suit our needs. The following lemma justifies the procedure of change of coordinates for distributions.

Lemma A.1. *Let* $\Phi : X_1 \to X_2$ *be a diffeomorphism, where* X_1, X_2 *are open subsets of* $\mathbb{R}^d$*. To every distribution* $h \in \mathcal{D}'(X_2)$ *on* X_2*, there exists a way to associate a unique distribution* $h \circ \Phi \in \mathcal{D}'(X_1)$ *on* X_1 *so that* $u \circ \Phi$ *agrees with the usual composition for* $h \in C_0^\infty(X_2) \subseteq \mathcal{D}'(X_2)$ *and the following holds:*

The mapping $\mathcal{D}'(X_2) \to \mathcal{D}'(X_1)$, $h \mapsto h \circ \Phi$ *is linear and continuous in* h*.*

In fact, for $\phi \in C_0^\infty(X_1)$*,* $u \circ \Phi$ *is defined by the formula*

$$\langle u \circ \Phi \mid \phi \rangle = \langle u \mid \frac{1}{|\det \Phi|} \phi \circ \Phi^{-1} \rangle. \tag{A.1}$$

Proof. Uniqueness is clear by density of $C_0^\infty(X_2)$ in $\mathcal{D}'(X_2)$. Let $h_j \rightharpoonup h$ be a sequence of h_j in $C_0^\infty(X_2)$ converging to h in the sense of distributions. Write $\Phi(x) = (y^1(x), \cdots, y^d(x))$ and

$$\frac{\partial(y^1, \cdots, y^d)}{\partial(x^1, \cdots, x^d)} = \det \Phi, \quad \text{and} \quad \frac{\partial(x^1, \cdots, x^d)}{\partial(y^1, \cdots, y^d)} = \det \Phi^{-1}.$$

For any $\phi \in C_0^\infty(X_1)$, we have

$$\begin{aligned} \langle h_j \circ \Phi \mid \phi \rangle &= \int u_n(\Phi(x)) \phi(x) \, \mathrm{d}x \\ &= \int u_n(y) \phi(\Phi^{-1}(y)) \frac{\partial(x^1, \cdots, x^d)}{\partial(y^1, \cdots, y^d)} \, \mathrm{d}y. \end{aligned}$$

Since $\phi(\Phi^{-1}(y)) \frac{\partial(x^1, \cdots, x^d)}{\partial(y^1, \cdots, y^d)}$ is a test function on X_2 (**Exercise:** Verify!), it follows that the last line goes to

$$\to \langle u(y) \mid \phi(\Phi^{-1}(y)) \frac{\partial(x^1, \cdots, x^d)}{\partial(y^1, \cdots, y^d)} \rangle_y = \langle u \mid \frac{1}{|\det \Phi|} \phi \circ \Phi^{-1} \rangle,$$

as desired. ☐

As an immediate corollary, we have the following linear change of variables for formula for the delta distribution.

Corollary A.2. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a invertible linear transformation. Then we have*

$$\delta_0 \circ \Phi = \frac{1}{|\det \Phi|} \delta_0.$$

A.2. **Time-dependent distributions.** We recall the following lemma from [3] concerning time regularity of solutions to evolutionary PDEs. We use the coordinates (t, x) on $\mathbb{R}^{1+d}$.

Lemma A.3. *Let $I \times X \subseteq \mathbb{R}^{1+d}$, where I is an open interval in $\mathbb{R}$ and X is an open set in $\mathbb{R}^d$. Suppose that $h \in \mathcal{D}'(I \times X)$ satisfies a PDE of the form*

$$\partial_t^m h + a_{m-1} \partial_t^{m-1} h + \cdots + a_0 h = F$$

where a_k is a differential operator in x with coefficients in $C^\infty(I \times X)$ and $F \in C(I; \mathcal{D}'(X))$. Then it follows that $h \in C^m(I; \mathcal{D}'(X))$.

Moreover, if h extends to a distribution in $\mathcal{D}'(J \times X)$, where J is an open interval containing $\overline{I}$, and $F \in C(\overline{I}; \mathcal{D}'(X))$, then $h \in C^m(\overline{I}; \mathcal{D}'(X))$.

For a proof, see [3, Theorem 4.4.8].

A.3. **Composition of a distribution with a map.** In our notes, we considered composition of a distribution h on $\mathbb{R}$ (e.g., χ_+^a) with a function $A : \mathbb{R}^d \to \mathbb{R}$ (e.g., $f = t^2 - |x|^2$). To compute this, we used a smooth approximation $h_j \rightharpoonup h$, performed computation for h_j and then passed to the limit. For completeness, we state a lemma which says that this procedure is well-defined.

Lemma A.4 (Composition of a distribution with a map)**.** *Let $A : X_1 \to X_2$ be a* submersion *(i.e., a map whose differential* $\mathrm{d}A$ *is surjective everywhere), where $X_k \subseteq \mathbb{R}^{d_k}$ (for $k = 1, 2$) is an open subset. To every distribution $h \in \mathcal{D}'(X_2)$ on X_2, there exists a way to associate a unique distribution $h \circ A \in \mathcal{D}'(X_1)$ on X_1 so that $h \circ \Phi$ agrees with the usual composition for $u \in C_0^\infty(X_2) \subseteq \mathcal{D}'(X_2)$ and the mapping $\mathcal{D}'(X_2) \to \mathcal{D}'(X_1)$, $h \mapsto h \circ \Phi$ is linear and continuous.*

This lemma can be proved similarly as Lemma A.1, using the implicit function theorem. See [3, Theorem 6.1.2] for a proof.

A.4. **Homogeneous distributions.** Next, we recap the theory of homogeneous distributions. A distribution $h \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ is said to be *homogeneous of degree a* if for all $\lambda > 0$ and $\phi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, we have

$$\langle h \mid \phi \rangle = \lambda^a \langle h \mid \phi_\lambda \rangle \quad \text{where } \phi_\lambda(x) := \lambda^d \phi(\lambda x). \tag{A.2}$$

We begin by reviewing the theory on $\mathbb{R}$. Consider the function

$$x_+^a := 1_{\{x \geq 0\}} \, x^a,$$

which is locally integrable (and hence a distribution) when $\mathrm{Re}\, a > -1$. To analytically extend x_+^a, note the functional equation

$$\frac{\mathrm{d}}{\mathrm{d}x} x_+^a = a x_+^{a-1}, \tag{A.3}$$

which holds for $\operatorname{Re} a > 0$. This identity can be used to analytically continue x_+^a (that is, $a \mapsto \langle x_+^a \mid \phi \rangle$ for any $\phi \in C_0^\infty(\mathbb{R})$) to $a \in \mathbb{C} \setminus \{-1, -2, \ldots\}$; the problem is that we cannot determine x_+^{-1} from the identity. To cancel the problematic factor of a, we define

$$\chi_+^a := \frac{x_+^a}{\Gamma(a+1)}. \tag{A.4}$$

where Γ is the *Gamma function*, defined by the formula

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, \mathrm{d}t.$$

for $\operatorname{Re} a > 0$ and analytically continued to $\mathbb{C} \setminus \{0, -1, \cdots\}$ by the identity

$$\Gamma(a+1) = a\Gamma(a). \tag{A.5}$$

By (A.3) and (A.5), we have the identity

$$\frac{\mathrm{d}}{\mathrm{d}x} \chi_+^a(x) = \chi_+^{a-1} \tag{A.6}$$

for $\operatorname{Re} a > 0$, as $\frac{a x_+^{a-1}}{a\Gamma(a)} = \chi_+^{a-1}$. Now we may analytically continue χ_+^a for $a \in \mathbb{C}$, i.e.,

(A.7) $a \mapsto \langle \chi_+^a \mid \phi \rangle$ can be analytically continued to $a \in \mathbb{C}$ for any $\phi \in C_0^\infty(\mathbb{R})$.

Note that $\chi_+^{-1}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \chi_+^0(x) = \frac{\mathrm{d}}{\mathrm{d}x} H(x) = \delta_0(x)$. Then by the preceding identity, we see that

$$\chi_+^{-k}(x) = \delta_0^{(k-1)}(x). \tag{A.8}$$

We recall some well-known functional equations for the Gamma function $\Gamma(a)$:

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \tag{A.9}$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 s^{a-1}(1-s)^{b-1} \, \mathrm{d}s. \tag{A.10}$$

$$\Gamma(a)\Gamma(a+\frac{1}{2}) = 2^{1-2a}\sqrt{\pi}\Gamma(2a). \tag{A.11}$$

The first formula is called *Euler's reflection formula*; for a proof, see [5, Chapter 6]. The function defined by the second formula is called the *Beta function* $B(a,b)$; it can be easily proved by writing out $\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a-1} t^{b-1} \, \mathrm{d}s\mathrm{d}t$ and making the change of variables $s = uv$, $t = u(1-v)$. The third formula, called *Legendre's duplication formula*, can be derived by using the second formula twice, with an appropriate change of variables (**Exercise:** Prove these formulae!).

We also record the following formulae concerning the homogeneous distribution χ_+^a for convenience:

$$\chi_+^{-\frac{1}{2}-k}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} \chi_+^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \Big(\frac{1}{x_+^{1/2}} \Big) \tag{A.12}$$

$$\chi_+^a * \chi_+^b = \chi_+^{a+b+1} \tag{A.13}$$

For the first identity, we used $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which follows from (A.9). The second identity is in fact equivalent to (A.10).

Another example of a homogeneous distribution is the limit

$$(x \pm i0)^a = \lim_{\epsilon \to 0+} (x \pm i\epsilon)^a,$$

where the limit is taken in the sense of distributions. Using the logarithm $\log z = \log|z| + i\mathrm{Arg}(z)$, where Arg measures the angle from the x-axis, we see that

$$(x \pm i0)^a = x_+^a + e^{\pm ia\pi} x_-^a \,,$$

at least for $\mathrm{Re}\, a > 0$.

Now we turn to the theory on $\mathbb{R}^d$.

Lemma A.5 (Homogeneous extension to the origin). *If $h \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ is homogeneous of degree a, and a is not an integer less than or equal to $-d$, then h has a unique extension to a homogeneous distribution $\dot{h} \in \mathcal{D}'(\mathbb{R}^d)$ of degree a, so that the map $u \mapsto \dot{u}$ is continuous.*

See [3, Theorem 3.2.3] for a proof.

Finally, by the scaling properties of the Fourier transform, it follows that the Fourier transform of a homogeneous distribution is also a homogeneous distribution. More precisely, we have the following lemma.

Lemma A.6 (Fourier transform of homogeneous distributions). *Let $u \in \mathcal{S}'(\mathbb{R}^d)$ be a homogeneous distribution of degree a. Then $\widehat{u}$ is homogeneous of degree $-a-n$.*

Appendix B. Computation of precise constant for E_+: Proof of (4.22)

Here we give a proof of the formula (4.22) for the constant in the forward fundamental solution for the d'Alembertian. We recall the formula here for the convenience of the reader:

$$c_d = \frac{\pi^{(1-d)/2}}{2}. \tag{4.22}$$

This formula can be read off from [3, Theorem 6.2.1], which in fact applies to more general constant coefficient second order differential operators. We present another argument[5] here, which is based on the use of the null coordinates (u, v, ω).

Proof of (4.22). First, given $g, h \in C_0^\infty$, note that we have the simple formula (by integration by parts)

$$\langle \Box g \mid h \rangle = \int \partial_t g \partial_t h \, \mathrm{d}t\mathrm{d}x - \int \nabla_x g \cdot \nabla_x h \, \mathrm{d}t\mathrm{d}x$$

Suppose that g, h is rotationally invariant. Then in the polar coordinates (t, r, ω), we see that

$$\langle \Box g \mid h \rangle = \omega_{d-1} \iint (\partial_t g \partial_t h - \int \partial_r g \cdot \partial_r h) r^{d-1} \, \mathrm{d}t\mathrm{d}r$$

where $\omega_{d-1} = \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma$ is the $d-1$-dimensional volume of the unit sphere $\mathbb{S}^{d-1}$. Making another change of variables to the null coordinates $(u, v, \omega) = (t - r, t + r, \omega)$, we then have the formula

$$\langle \Box g \mid h \rangle = \omega_{d-1} \iint (\partial_v f \partial_u g + \partial_u f \partial_v g) \Big(\frac{v-u}{2}\Big)_+^{d-1} \mathrm{d}u\mathrm{d}v. \tag{B.1}$$

[5] We thank P. Isett for communicating this proof.

Now recall that E_+ is a function of $t^2 - |x|^2$, which equals uv in the null coordinates. Using $g = \Box E_+ = \delta_0$ and $h = H(v) = 1_{\{t+|x|\leq 1\}}$, the identity (B.1) can then be used to deduce

$$1 = \langle \Box E_+ \mid 1_{\{|x|+t\leq 1\}} \rangle = \omega_{d-1} \iint \partial_u E_+(uv) \partial_v 1_{\{v\leq 1\}} \Big(\frac{v-u}{2}\Big)_+^{d-1} \, \mathrm{d}u\mathrm{d}v \tag{B.2}$$

where the integral is interpreted suitably. (**Exercise:** Using the support properties of E_+ and $1_{\{v\leq 1\}}$, show that (B.2) makes sense. Indeed, show that the right-hand side is the limit

$$\omega_d \iint 1_{v+u\geq 0} \partial_u g_j(uv) \partial_v h_j(1-v) (\frac{v-u}{2})_+^{d-1} \, \mathrm{d}u\mathrm{d}v \qquad \text{as } j \to \infty,$$

where $g_j, h_j \in C_0^\infty(\mathbb{R})$, $g_j(x) \rightharpoonup c_d \chi_+^{-\frac{d-1}{2}}(x)$ and $h_j(x) \rightharpoonup 1_{\{x\geq 0\}}$.)

Now note that

$$\begin{aligned} \partial_u E_+ =& 1_{\{v+u\geq 0\}} c_d v \chi_+^{-\frac{d+1}{2}}(uv), \\ \Big(\frac{v-u}{2}\Big)_+^{d-1} =& 2^{-d+1}(d-1)! \chi_+^{d-1}(v-u), \\ \partial_v 1_{v\leq 1} =& -\delta(1-v). \end{aligned}$$

Substituting these identities into (B.2), it follows that

$$\begin{aligned} c_d^{-1} =& 2^{-d+1}(d-1)! \, \omega_{d-1} \iint \chi_+^{-\frac{d+1}{2}}(u) \chi_+^{d-1}(v-u) \, \mathrm{d}u \\ =& 2^{-d+1}(d-1)! \, \omega_{d-1} \chi_+^{-\frac{d+1}{2}} * \chi_+^{d-1}(1). \end{aligned}$$

Using the identities (see Appendix A.4)

$$\begin{aligned} \chi_+^a * \chi_+^b =& \chi_+^{a+b+1} \\ \chi_+^a(1) =& \frac{1}{\Gamma(a+1)} \end{aligned}$$

and the formula $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$, we see that

$$c_d^{-1} = 2^{-d+1}(d-1)! \, \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \, \frac{1}{\Gamma(\frac{d+1}{2})}. \tag{B.3}$$

By Legendre's duplication formula (A.11) with $a = \frac{d}{2}$, we have

$$\Gamma(\frac{d}{2})\Gamma(\frac{d+1}{2}) = 2^{-d+1}\sqrt{\pi}(d-1)!$$

Substituting the preceding computation into (B.3), we obtain (4.22). □

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