Long-term behavior of linear and nonlinear waves

Sung-Jin Oh*

Abstract. This article surveys recent progress on the long-term behavior of solutions to linear and nonlinear hyperbolic equations, with a focus on robust physical space techniques. We review classical and modern methods for establishing sharp uniform decay rates and precise global asymptotics for the wave equation. We then describe a general method for determining the leading-order late-time tails for waves in odd spatial dimensions, which is applicable to nonlinear problems on dynamical backgrounds. Finally, we discuss a new physical space approach for establishing the global asymptotics of the variable-coefficient Klein–Gordon equation. Throughout, we emphasize methods that are applicable to problems with variable coefficients and nonlinearities, such as those arising in the study of waves on black hole spacetimes or interacting with solitons.

1 Introduction.

The subject of this article is the precise long-term behavior of solutions to hyperbolic equations. The literature on this fundamental topic is rich and vast, but we will focus on *robust physical space techniques*. These methods lends themselves easily to applications to *nonlinear problems*, are applicable to *variable-coefficient problems*, and yield the precise *global asymptotics* of the solution – not just the upper bound on the decay rate of the uniform norm of the solution. In addition to its inherent interest, the motivation behind this consideration comes from numerous problems involving the interaction of waves with steady states, such as black holes in general relativity and solitons in dispersive PDEs. To wit, variable coefficients naturally arise by linearizing around a nontrivial steady state, and global asymptotics is crucial to understand the precise interaction between the radiation (i.e., the dispersive part of the wave) and the steady state.

This article is structured as follows:

- In **Section 2**, we give a concise review of the case of the wave equation $\Box\phi = 0$ with smooth and decaying initial data, focusing on the ideas of Klainerman, Morawetz, and Dafermos–Rodnianski for establishing sharp uniform decay via a physical space approach.
- In **Section 3**, we describe the recent work [55, 56] of the author with J. Luk, which introduced a general method for determining and rigorously establishing the leading order asymptotics of ϕ along time-like curves (with admissible asymptotics) in the case of odd (but > 1) space dimensions, which is applicable to *nonlinear* problems on *dynamical* backgrounds.
- In **Section 4**, we describe the upcoming work [66] of the author with F. Pasqualotto and N. Tang on a physical space method for establishing the global asymptotics of solutions to variable-coefficient Klein–Gordon equations in 3 spatial dimensions, which are another major class of hyperbolic equations with a rather different long-term behavior.

The following notation and conventions are used throughout the article:

- We use the standard asymptotic notation: $A \lesssim B$ means $A \leq CB$ for some universal constant C, $A \simeq B$ means $A \lesssim B$ and $B \lesssim A$, and $O(\cdot)$, $o(\cdot)$ denote the standard big-O and little-o notation.
- We use Japanese brackets $\langle z \rangle := \sqrt{1 + z^2}$, as well as the notation $z_+ = \max\{z, 0\}$.
- We use (spherical) polar coordinates (t, r, θ), where $r := |x|$ and $\theta := x/r \in \mathbb{S}^{d-1}$. The *retarded time* is $u := t - r$. In Section 4, the following functions are used in the region $\{r < t\}$: the *hyperbolic radius* $\rho := \sqrt{t^2 - r^2}$ and *hyperbolic angle variables* $y := x/\rho$. For level hypersurfaces, we use the following notation: $S_t := \{t = \text{const}\}$ is a time slice and $H_\rho := \{\rho = \text{const}\}$ is a hyperboloid.

*Department of Mathematics, UC Berkeley, Berkeley, CA, USA 94720 and School of Mathematics, KIAS, Seoul, Korea (sjoh@math.berkeley.edu)

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- We use Greek indices $\mu, \nu, \dots$ running from 0 to d (where $x^0 = t$ in rectangular coordinates), and latin indices $1, \dots, d$ running from 1 to d. The Minkowski metric, in rectangular coordinates, is written $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$, and its inverse is written $(\eta^{-1})^{\mu\nu}$; in the matrix form, it is again $\mathrm{diag}(-1, 1, \dots, 1)$. We also use the Einstein summation convention, where repeated upper and lower indices are summed over.

2 A review of physical space approaches for the wave equation. In this section, we provide a review of prior physical space approaches to understand the decay properties of solutions to the (massless) wave equation, and its variable-coefficient and/or nonlinear generalizations. Each method will be demonstrated in the basic case of the initial value problem for the wave equation:

$$\begin{cases} \Box\phi = 0 & \text{in } (0,\infty)_t \times \mathbb{R}^d_x, \\ (\phi, \partial_t\phi) = (g, h) & \text{on } \{t = 0\}, \end{cases} \tag{2.1}$$

where, unless stated otherwise, we assume that (g, h) is smooth and rapidly decaying. Here, $\Box := -\partial_t^2 + \Delta = (\eta^{-1})^{\mu\nu}\partial_\mu\partial_\nu$ is the *d'Alembertian* operator, and $\Delta := \partial^2_{x^1} + \dots + \partial^2_{x^d}$ is the *Laplacian* operator with respect to the spatial variables $x = (x^1, \dots, x^d)$.

2.1 Fundamental solution. As is well-known, there is an explicit integral representation formula for the solution ϕ in terms of the initial data (g, h), using the forward fundamental solution for $\Box$ [32, Section 6.2] (alternatively, see [20, Section 2.4] for a derivation that does not use the fundamental solution). From it, we may observe the following:

- **Finite speed of propagation.** If (g, h) vanishes in a ball $B_R(x_0) \subseteq \mathbb{R}^d$, then ϕ also vanishes in the truncated cone $\{(t, x) : |x - x_0| < R - t, t \in [0, R)\}$. As a consequence, if (g, h) is supported in a ball $B_{R_0}(0)$, then $\phi(t, \cdot)$ is supported in the ball $B_{t+R_0}(0)$ for all $t > 0$.

- **Sharp uniform decay and the radiation field.** The solution ϕ primarily propagates along null rays. For any inertial observer traveling at the speed of light, $\phi(t, x_0 + tv)$ with $|v| = 1$, the solution decays like $t^{-\frac{d-1}{2}}$. More precisely, there exists a *Friedlander radiation field* $\mathring{\Phi}_0 : \mathbb{R} \times \mathbb{S}^{d-1} \to \mathbb{R}$ [21, 22], such that[1]

$$\phi(t, x_0 + tv) = t^{-\frac{d-1}{2}}\mathring{\Phi}_0(-v \cdot x_0, v) + o(t^{-\frac{d-1}{2}}) \quad \text{as } t \to +\infty.$$

  The mapping from the initial data (g, h) to the radiation field $\mathring{\Phi}_0$ is an isomorphism on the natural energy spaces. In particular, $\mathring{\Phi}_0 \equiv 0$ if and only if $\phi \equiv 0$. The rate $t^{-\frac{d-1}{2}}$ is the sharp uniform decay rate, i.e., $\|\phi(t, \cdot)\|_{L^\infty} \lesssim t^{-\frac{d-1}{2}}$, and the right-hand side cannot be improved in view of the discussion so far.

- **Late time tails.** For inertial observers traveling along time-like rays (i.e., with speed $|v| < 1$), the solution decays faster: $\phi(t, x_0 + tv) = o(t^{-\frac{d-1}{2}})$ as $t \to +\infty$. The precise asymptotics in this regime, known as *late time tails*, are important in some problems and will be discussed in more detail in Section 3.

While extremely detailed, an analysis based on explicit fundamental solutions is not robust enough for generalizations to variable-coefficient or nonlinear problems. At the core of the tremendous progress in nonlinear hyperbolic equations over the past several decades has been the development of physical space methods for retrieving these key properties of wave propagation.

2.2 Energy estimates. Energy estimates are the most basic yet most robust a priori estimates for the wave equation. To present these, we introduce the *energy-momentum tensor* for ϕ:

$$Q_{\mu\nu}[\phi] := \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\eta_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi.$$

This tensor can be systematically derived using Nöther's theorem from the coordinate invariance of the wave equation's action functional [30, Section 3.3]. For our purposes, we only need the following properties, which can be checked by direct computation:

[1] The input $(-v \cdot x_0, v)$ appears because $\mathring{\Phi}_0$ is naturally a function of the *retarded time* $u = t - r$ and angle $\theta = x/r$. As $t \to +\infty$, we have $\left(t - |x_0 + tv|, \frac{x_0+tv}{|x_0+tv|}\right) = \left(-v \cdot x_0 + O(t^{-1}|x_0|^2), v + O(t^{-1}|x_0|)\right)$. See Lemma 2.4 and the subsequent discussion.

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LEMMA 2.1 (Properties of $Q_{\mu\nu}$). *For any smooth $\phi : \mathbb{R}^{1+d} \to \mathbb{R}$, the tensor $Q_{\mu\nu}[\phi]$ is symmetric and satisfies:*

1. **Divergence identity.** $\partial^\mu Q_{\mu\nu}[\phi] = (\Box\phi)\partial_\nu\phi$.
2. **Dominant energy condition.** $Q_{\mu\nu}[\phi]X^\mu Y^\nu \geq 0$ *if X and Y are future-pointing causal vectors (i.e., $\eta_{\mu\nu}X^\mu X^\nu, \eta_{\mu\nu}Y^\mu Y^\nu \geq 0$ and $X^0, Y^0 > 0$).*

Contracting $Q_{\mu\nu}[\phi]$ with the time-translation vector field $T := \partial_t$, we obtain the *energy current* ${}^{(T)}J_\mu[\phi] := Q_{\mu\nu}[\phi]T^\nu = Q_{\mu 0}[\phi]$. For a space-like[2] hypersurface $\Sigma \subseteq \mathbb{R}^{1+d}$, we define the *energy flux through* Σ to be

$$E[\phi](\Sigma) := \int_\Sigma {}^{(T)}J_\mu[\phi]\boldsymbol{\nu}^\mu_\Sigma \, \mathrm{d}A_\Sigma, \quad \boldsymbol{\nu}_\Sigma \text{ is the future-pointing unit normal to } \Sigma, \, \mathrm{d}A_\Sigma \text{ is the induced volume form.}$$

For a solution ϕ to (2.1), we have $\partial^\mu({}^{(T)}J_\mu[\phi]) = 0$. By integrating this divergence identity over a spacetime region $\{0 < t < t_0\} \times \mathbb{R}^d$ and applying Stokes' theorem, we arrive at the standard *conservation of energy*: $E[\phi](S_{t_0}) = E[\phi](S_0)$ for all $t_0 > 0$. Since $\boldsymbol{\nu}_{S_t}$ is simply ∂_t, an explicit computation retrieves the standard expression for the energy density: ${}^{(T)}J_0[\phi] = \frac{1}{2}((\partial_t\phi)^2 + |\nabla_x\phi|^2)$.

The advantage of the energy-momentum formalism is its flexibility. For instance, the finite speed of propagation (as formulated above) can be proven by integrating the divergence of ${}^{(T)}J_\mu[\phi]$ over truncated cones of the form $\{(t,x) : |x - x_0| < R - t, \, 0 < t < t_0\}$ and applying Stokes' theorem. If the initial data vanish on $B_R(x_0)$, the energy at any time $t \in [0, R)$ within the smaller ball $B_{R-t}(x_0)$ must also be zero. This follows because the flux through the null boundary $\{|x - x_0| = R - t\}$ has the sign (due to the dominant energy condition) that prevents energy from flowing into the domain.

2.3 Method of commuting vector fields. We review the classical idea of Klainerman [38], which is to obtain decay bounds for ϕ by deriving energy estimates for weighted derivatives of ϕ. A natural way to obtain such bounds is to use vector fields that commute well with the operator $\Box$. Consider the following vector fields, which are infinitesimal generators of Lorentz boosts, spatial rotations, and scaling transformations on Minkowski space $(\mathbb{R}^{1+d}, \eta)$:

$$\text{Boosts: } L_j := t\partial_{x^j} + x^j\partial_t, \quad \text{Rotations: } \Omega_{jk} := x^j\partial_{x^k} - x^k\partial_{x^j}, \quad \text{Scaling: } S := t\partial_t + x^j\partial_{x^j}.$$

These vector fields have the following commutation properties with $\Box$:

LEMMA 2.2 (Commuting vector fields). *The following identities hold:*

$$[L_j, \Box] = 0, \quad [\Omega_{jk}, \Box] = 0, \quad [S, \Box] = -2\Box.$$

In particular, if $\Box\phi = 0$ and $Z \in \{L_j, \Omega_{jk}, S\}$, then $Z\phi$ also solves the wave equation.

Moreover, the following algebraic identities hold:

$$(t + r)\ell = S + \frac{x^j}{r}L_j, \quad (t - r)\underline{\ell} = S - \frac{x^j}{r}L_j, \quad \frac{t}{r}\Omega_{jk} = \frac{x^j}{r}L_k - \frac{x^k}{r}L_j,$$

where $\ell = \partial_t + \frac{x^j}{r}\partial_{x^j}$ and $\underline{\ell} = \partial_t - \frac{x^j}{r}\partial_{x^j}$ are the outgoing *and* incoming radial null vector fields*, respectively. Schematically, $\underline{\Gamma}\psi = O(Z\psi)$, where $\underline{\Gamma} \in \{\langle t - r\rangle\underline{\ell}, \langle t + r\rangle\ell, \frac{\langle r\rangle}{\langle t+r\rangle}\Omega_{jk}\}$.*

By applying these vector fields repeatedly and using the conservation of energy for each resulting solution, one can control a family of weighted Sobolev norms. Via the schematic relation $\underline{\Gamma}\psi = O(Z\psi)$ and rescaled Sobolev inequalities, these norms can then be related back to the pointwise behavior of the solution. This is precisely the idea of *Klainerman's Sobolev inequality* [38]:

LEMMA 2.3 (Klainerman's Sobolev inequality). *For any sufficiently smooth function $\psi(t,x)$, we have*

$$\langle t + r\rangle^{\frac{d-1}{2}}\langle t - r\rangle^{\frac{1}{2}}|\psi(t,x)| \lesssim \sum_{|I| \leq \lfloor d/2\rfloor + 1} \|Z^I\psi(T, \cdot)\|_{L^2(\mathbb{R}^d)},$$

where Z^I denotes a composition of vector fields from the set $\{L_j, \Omega_{jk}, S, \partial_\mu\}$, I running over suitable multi-indices.

[2]By continuity, we may extend the definition of $E[\phi](\Sigma)$ to incorporate a null hypersurface Σ – in which case $\boldsymbol{\nu}_\Sigma$ and $\mathrm{d}A_\Sigma$ do not make sense separately – so that it still arises as the boundary integral in the application of Stokes' theorem to $\partial^\mu({}^{(T)}J_\mu) = 0$.

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Combining the energy conservation for $Z^I\phi$ with Lemma 2.3, we immediately obtain:

$$|\partial\phi(t,x)| \lesssim \langle t+r\rangle^{-\frac{d-1}{2}} \langle t-r\rangle^{-\frac{1}{2}} \quad \text{for } t>0,$$

which retrieves the sharp uniform decay for $\partial\phi$. Using the rapid decay of the initial data, the finite speed of propagation, and integration of the preceding bound along $\underline{\ell}$, we also obtain

$$|\phi(t,x)| \lesssim \langle t+r\rangle^{-\frac{d-1}{2}} \langle (t-r)_+\rangle^{\frac{1}{2}} \quad \text{for } t>0.$$

This procedure also provides more detailed information about how different derivatives of ϕ decay differently:

$$|\underline{\Gamma}^I\phi(t,x)| \lesssim \langle t+r\rangle^{-\frac{d-1}{2}} \langle (t-r)_+\rangle^{\frac{1}{2}} \quad \text{for } t>0, \tag{2.2}$$

where $\underline{\Gamma}^I$ denotes a composition of vector fields from $\{\langle t-r\rangle\underline{\ell}, \langle t+r\rangle\ell, \frac{\langle r\rangle}{\langle t+r\rangle}\Omega_{jk}\}$. In particular, derivatives tangent to outgoing null cones (i.e., ℓ and angular derivatives) improve the decay rate, a behavior often referred to as *peeling*.

To obtain the existence of $\mathring{\Phi}_0(u,\theta)$, we employ an ODE argument due to Lindblad–Rodnianski [49]. The starting point is the wave equation expressed in (u,r,θ)-coordinates:

LEMMA 2.4. *If $\Box\phi = f$, then in the (u,r,θ)-coordinates, $\Phi := r^{\frac{d-1}{2}}\phi$ solves*

$$-2\partial_u\partial_r\Phi + \partial_r^2\Phi - \frac{(d-1)(d-3)}{4}r^{-2}\Phi + r^{-2}\mathring{\Delta}\Phi = r^{\frac{d-1}{2}}f,$$

where $\mathring{\Delta} := \sum_{j<k}\Omega_{jk}^2$ is the Laplacian with respect to the angular variables.

For a solution ϕ to (2.1), the bound (2.2) implies that all terms on the left-hand side, except the first, are[3] $O(\langle r\rangle^{-2}\langle u_+\rangle^{\frac{1}{2}})$ for large r. This leaves a simple transport equation for $\partial_u\Phi$:

$$\partial_r(\partial_u\Phi) = O(\langle r\rangle^{-2}\langle u_+\rangle^{\frac{1}{2}}) \quad \text{for large } r.$$

Integrating this in r from a fixed large radius to infinity shows that $\partial_u\Phi(u,r,\theta)$ converges to a limit as $r\to\infty$. This limit is precisely $\partial_u\mathring{\Phi}_0(u,\theta)$, the derivative of the radiation field.

2.4 The r^p-weighted energy method and Integrated Local Energy Decay (ILED) The method of commuting vector fields relies on the symmetries of the Minkowski spacetime and is not directly applicable to wave equations on general curved backgrounds, such as black hole spacetimes. The modern approach for such problems, pioneered by Dafermos and Rodnianski [15], combines a generalized vector field method with *integrated local energy decay (ILED)* estimates, also known as *Morawetz estimates* [63]. These estimates capture the core idea that energy, although conserved globally, must radiate away from any compact region of space.

THEOREM 2.5 (ILED, or Morawetz estimate, for $\Box$). *For any $R\geq 1$, there exists a constant $C_R>0$ such that, for any solution ϕ to* (2.1)*, we have*

$$\int_0^\infty \int_{\{|x|\leq R\}} \left((\partial_t\phi)^2 + |\nabla_x\phi|^2 + \frac{\phi^2}{R^2}\right) \mathrm{d}x\mathrm{d}t \leq C_R E[\phi](S_0). \tag{2.3}$$

For the wave equation (2.1), this estimate can be proven using a physical space multiplier based on the radial vector field ∂_r (in the (t,r,θ)-coordinates), which goes back to Morawetz [63]. In more general settings, proving an ILED estimate – which is a weak form of decay for merely finite energy solutions – is closely tied to the geometric and spectral properties of the problem, such as the absence of stable trapped null geodesics and the absence of exponentially growing solutions. There is a vast literature that, unfortunately, cannot be adequately covered in this article. We only mention [17] for the proof of ILED (and uniform boundedness of energy) for all subextremal Kerr exterior backgrounds, [60] for a systematic study in the absence of any trapped null geodesics, and the references therein.

[3]Note that, in the (u,r,θ) coordinates, the partial derivative ∂_u coincides with the time translation vector field, and ∂_r coincides with the outgoing null vector field ℓ.

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Once an ILED estimate is established, the idea of Dafermos–Rodnianski [15] was to combine it with the so-called r^p*-weighted energy estimates*, which are obtained by multiplying the equation in Lemma 2.4 by $r^p \partial_r \Phi$ and integrating by parts (hence the name). Roughly speaking, these estimates quantify the amount of "incoming" energy in the large-r region.

An immediate application of the r^p-weighted energy estimates is a more robust proof of the existence of a radiation field; see [64]. Moreover, defining a new time function τ that smoothly interpolates t for r small and u (plus a constant) for r large, [15] showed that the energy flux $E[\phi](\Sigma_\tau)$ through the level sets $\Sigma_\tau := \{\tau = \text{const}\}$ decays like $O(\tau^{-2})$. Moreover, by commuting the equation with the outgoing radial null vector field ℓ and the spatial rotations Ω_{jk}, one can ultimately derive pointwise decay estimates. This procedure is carried out in detail in Moschidis [64], and one obtains the following with $\alpha = \frac{d}{2}$:

$$\Gamma^I \phi = O_I(\langle\tau\rangle^{-\alpha+\frac{d-1}{2}} \min\{\langle r\rangle^{-\frac{d-1}{2}}, \langle\tau\rangle^{-\frac{d-1}{2}}\}), \tag{2.4}$$

where Γ^I denotes a composition of vector fields from the set $\mathbf{\Gamma} := \{\langle\tau\rangle\partial_\tau, \langle r\rangle\partial_r, \Omega_{jk}\}$ in the (τ, r, θ)-coordinates. We call (2.4) a *vector field bound*. It is weaker than (2.2) in that the derivatives tangent to outgoing null cones only gain $\langle r\rangle^{-1}$, not $\langle t + r\rangle^{-1}$, but it is the robust bound that holds in more general settings[4].

In general, this method provides a robust framework for proving decay such as (2.4) for an *asymptotically flat* (i.e., asymptotic to $\Box$ as $r \to +\infty$ in a suitable sense) linear operator $\mathcal{P}$ that lack symmetries, by reducing the problem to establishing a uniform energy estimate and an ILED. See Tataru [77, 78] for other pioneering work that underscores the role of ILED as the key intermediate decay bound leading to strong decay estimates. We also refer to Oliver–Sterbenz [68] for an alternative approach which is closer to the commuting vector field method.

3 Late time tails for waves in odd space dimensions. The methods in the previous section capture the precise leading-order asymptotics of a wave ϕ along null rays in terms of its radiation field $\mathring{\Phi}_0$. However, those methods do not give sharp information about the behavior of $\phi(t, x_0 + tv)$ as $t \to +\infty$ for inertial observers along time-like rays (i.e., $|v| < 1$). This behavior, which we refer to as *late time tails*, is crucial for studying problems involving the interaction of waves with spatially localized objects, such as black holes in general relativity, solitons in geometric wave equations, and compact obstacles in exterior initial-boundary value problems (IBVP).

For the sake of simplicity, in all discussions of late time tails in this section, **we will concentrate exclusively on the asymptotics of solutions as $t \to +\infty$ while all the other coordinates remain fixed**, where ∂_t generates either the exact or (in the non-stationary case) an asymptotic time translation symmetry for the underlying linear problem. Most methods discussed below – especially the method of [55] – are capable of describing the asymptotics along more general time-like curves.

3.1 Oddity of space dimensions: the strong Huygens' principle and Price's law. An elementary yet remarkable fact about the wave equation $\Box\phi = 0$ is that the parity of the spatial dimension d fundamentally alters its propagation properties.

When d is *even*, the solution exhibits a universal late time tail. From the explicit formula for the solution, one finds the asymptotics $\phi(t, x_0) = \mathfrak{L}[\phi]t^{-(d-1)} + o(t^{-(d-1)})$, where $\mathfrak{L}[\phi]$ is a linear functional of the initial data that is generically nonzero and independent of x_0 (we remark that $o(\cdots)$ depends on ϕ and x_0). The exponent $d - 1$ can be understood from dimensional analysis: the fundamental solution E_+ for the wave equation on $\mathbb{R}^{1+d}$ is homogeneous of degree $-(d - 1)$, and this homogeneity dictates the decay rate. This behavior is expected to persist for a large class of variable-coefficient generalizations.

In contrast, when $d > 1$ is *odd* – including the physical case $d = 3$ – a completely different phenomenon occurs. The wave equation obeys the *strong Huygens' principle*: the solution at a point (t, x) depends only on the initial data on the boundary of the past null cone emanating from (t, x). As a consequence, for compactly supported initial data, the solution $\phi(t, x_0)$ becomes *identically zero* for large enough t. This perfect cancellation is highly unstable. For a large class of variable-coefficient and nonlinear generalizations, the late time tail is restored, but its form is no longer predicted by the Minkowski space wave equation. Determining the precise decay rate becomes a nontrivial problem.

A celebrated example is *Price's Law*, which arose in the study of linear fields on the $(3 + 1)$-dimensional Schwarzschild black hole exterior. Based on numerical and formal arguments, Price [70] predicted the following decay rates for late-time tails:

[4] In fact, results in Section 3 imply that (2.2) cannot be true in general, since the spatial profile φ_0 may not be constant.

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- For the scalar wave equation, the generic decay rate is t^{-3}.
- For the Maxwell field (spin ± 1), the generic decay rate is t^{-5}.
- For linearized gravity (spin ± 2), the generic decay rate is t^{-7}.

Notice that the scalar rate t^{-3} is faster than the t^{-2} predicted by simple dimensional analysis for a $(3+1)$-dimensional problem. This already indicates that Price's law captures a more subtle phenomenon.

All of the above predictions have now been rigorously justified, on not just the Schwarzschild but also the Kerr black hole exterior (stationary axi-symmetric black holes). The first proof of Price's law as an *upper bound* (up to a small loss) was given by [14] in a spherically symmetric, but nonlinear, setting. The sharp upper bound was later proven for the linear wave equation on Schwarzschild without symmetry assumptions [18, 19]. For more general stationary asymptotically flat spacetimes, the Price law upper bound was proven by Tataru [78]; an alternative proof which also applies to non-stationary spacetimes was later given by Metcalfe–Tataru–Tohaneanu [61]. The first *lower bound* result in the spirit of Price's law was achieved in [52] for Reissner–Nordström. (Although the lower bound was only proven on the event horizon in an averaged sense, it was nonetheless sufficient for the instability of the Cauchy horizon. In the recent work of Gautam [23], the pointwise late time tails are justified.) *Sharp pointwise asymptotics* were achieved independently by Angelopoulos–Aretakis–Gajic [2, 4, 3] and by Hintz [31]. For more references, including some of the vast physics literature on the subject, see [55, Section 1.3].

However, it was suggested that deviations from linearity and/or stationarity could drastically alter these tails. This was explored by Bizoń, Chmaj, and Rostworowski in a series of work; see, e.g., [7] and the references therein. A particularly striking observation of Bizoń–Rostworowski [6] involved revisiting of one of the earliest numerical work on Price's law due to Gundlach–Pullin–Price [29]. The latter studied a scalar field ϕ on a *dynamical* spherically symmetric black hole background. Decomposing the solution into spherical harmonics of degree ℓ, $\mathbb{S}_{(\ell)}\phi$, they observed for $\ell = 0, 1, 2$, the tail was $\sim t^{-3}, t^{-4}, t^{-6}$, respectively [29, Fig. 12, 13]. On a (stationary) Schwarzschild background, the rates are $t^{-2\ell-3}$, i.e., t^{-3}, t^{-5}, t^{-7}. The numerical results agree for $\ell = 0$ but show a significantly slower decay for $\ell \geq 1$. Bizoń-Rostworowski [6] argued this was *not* a numerical error, but *a new physical effect caused by the non-stationary background.*

3.2 Beyond linear, stationary setting: Method of [55]. In [55], we introduced a general method for determining and rigorously establishing the universal generic late-time tail of waves on spacetimes with odd spatial dimension ($d > 1$), applicable to *nonlinear problems* on *dynamical backgrounds.* A simplified version of our main theorem reads:

THEOREM 3.1 ([55]). *Let $d > 1$ be odd and let $\mathcal{M} = \mathbb{R}_t \times (\mathbb{R}^d_x \setminus \mathcal{K})$, where $\mathcal{K}$ may be a smooth bounded domain (not necessarily connected) in $\mathbb{R}^d$ in case of the black hole exterior or the exterior IBVP. Consider the equation*

$$\mathcal{P}\phi = \mathcal{N}(\phi) \quad \textit{in } \mathcal{M},$$

where $\mathcal{P}$ is a scalar second-order linear operator on $\mathcal{M}$ and $\mathcal{N}$ is a smooth, possibly quasilinear (but not fully nonlinear) nonlinearity (in the case of the exterior IBVP, assume also the Dirichlet boundary condition $\phi = 0$ on $\mathbb{R}_t \times \partial\mathcal{K}$). Assume the following on $\mathcal{P}$, $\mathcal{N}$, and ϕ:

- **Asymptotic flatness.** *We have $\mathcal{P} \to \Box_{\mathbb{R}^{1+d}}$ as $|x| \to +\infty$ (in a suitable topology for the coefficients in the (τ, x)-coordinates).*
- **Ultimate stationarity.** *We have $\mathcal{P} \to {}^{(\infty)}\mathcal{P}$ as $\tau \to +\infty$ (in a suitable topology for the coefficients in the (τ, x)-coordinates), where ${}^{(\infty)}\mathcal{P}$ is* stationary *in the sense that its coefficients are independent of τ.*
- **Null structure for** $d = 3$. *If $d = 3$, then the quadratic part of the nonlinearity $\mathcal{N}(\phi)$, written in the coordinates (τ, x) in the region $\{r > R_1\}$, does not contain terms of the form ϕ^2, $\phi\partial_\tau\phi$, $(\partial_\tau\phi)^2$, $\phi\partial_\tau^2\phi$, or $\partial_\tau\phi\partial_\tau^2\phi$*
- **Regular threshold.** *The operator ${}^{(\infty)}\mathcal{P}_0$ defined by dropping all terms involving ∂_τ in the (τ, x)-coordinates (supplemented with the Dirichlet boundary condition in the case of the exterior IBVP) is invertible on suitable Sobolev spaces.*

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- **Admissible initial data.** *The initial data for ϕ on $\{t = 0\}$ is smooth and rapidly decaying.*
- **Vector field bounds.** *The solution ϕ obeys a baseline set of decay estimates (i.e., the vector field bounds from Section 2).*

Then the following conclusions hold:

1. **Higher radiation fields.** *For some $J > 0$, we have*

$$\phi(u,r,\theta) = r^{-\frac{d-1}{2}} \left(\mathring{\Phi}_0(u,\theta) + \sum_{j=1}^{J} r^{-j} \mathring{\Phi}_j(u,\theta) + \rho_{J+1} \right) \tag{3.1}$$

for some functions $\mathring{\Phi}_j(u,\theta)$ ($j = 1,\dots,J$) as $r \to +\infty$, and $\rho_{J+1} = o(r^{-J})$.

Moreover, the limits $\mathring{\Phi}_j(\infty,\theta) := \lim_{u\to\infty} \mathring{\Phi}_j(u,\theta)$ exist for each $j = 0,1,\dots,J$ and θ. We have $\mathring{\Phi}_j(\infty,\theta) = 0$ for $j < J$, and if $\mathring{\Phi}_J(\infty,\theta) = 0$, we may continue this expansion with a higher J.

2. **Leading order late time tails.** *For every $x_0 \in \mathbb{R}^d \setminus \mathcal{K}$, we have*

$$\phi(t,x_0) = c_d \mathfrak{L}\, t^{-\alpha} \varphi_0(x_0) + o(t^{-(J+\frac{d-1}{2})}) \quad \textit{with } \alpha = J + \frac{d-1}{2}, \tag{3.2}$$

where c_d is a nonzero constant depending only on d, $\mathfrak{L} := \int_{\mathbb{S}^{d-1}} \mathring{\Phi}_J(\infty,\theta)\,\mathrm{d}A$, φ_0 is the solution to ${}^{(\infty)}P_0\varphi_0 = 0$ on $\mathbb{R}^d \setminus \mathcal{K}$ with the boundary condition $\varphi_0(x) \to 1$ as $|x| \to +\infty$ (plus the Dirichlet boundary condition in the case of the exterior IBVP), and $o(t^{-(J+\frac{d-1}{2})})$ may depend on x_0.

3. **Strong Huygens correction.** *For $j \le J$, we have*

$$\lim_{u\to+\infty} \mathbb{S}_{(\ell)} \mathring{\Phi}_j(u,\theta) = 0 \quad \textit{for all } j \le \ell + \frac{d-3}{2}, \tag{3.3}$$

where $\mathbb{S}_{(\ell)}$ denotes the $L^2(\mathbb{S}^{d-1})$-projection to the spherical harmonics of degree ℓ.

We start with several remarks concerning the conclusion of this theorem:

- The most important part this theorem is Part 2, which provides *an algorithm for determining the leading order late time tail decay rate $t^{-(J+\frac{d-1}{2})}$*: find the first j such that $\mathring{\Phi}_j(u,\theta)$ does not vanish as $u \to +\infty$. This analysis can be performed at the *formal* level (i.e., without justifying the asymptotic expansion (3.1)), since each $\mathring{\Phi}_j(u,\theta)$ may be determined in terms of the radiation field $\mathring{\Phi}_0(u,\theta)$ by solving a hierarchy of *formal recurrence equations*, which are derived by plugging in the formal asymptotic expansion (i.e., without requiring remainder bounds at any order) $\phi = r^{-\frac{d-1}{2}}(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + \cdots)(u,\theta)$ into the equation[5] $\mathcal{P}\phi = \mathcal{N}(\phi)$. See Subsection 3.3 for more discussions and examples.
- For nonlinear problems, it is a nontrivial task to relate the radiation field $\mathring{\Phi}_0$ to the initial data (note that, in the case of the wave equation, they are famously related to each other by the Radon transform [32, Section 6.2]). Nevertheless, it is often possible to perturb the initial data to vary $\mathring{\Phi}_0$ to show that $\mathfrak{L}$ is *generically* nonzero; see [53, 54], as well as [56]. This idea allows one to establish a universal leading order asymptotics for generic solutions via Theorem 3.1.
- Part 3 captures a subtle but important cancellation due to the rapid decay of the initial data. We emphasize that (3.3) is *false*, even for the wave equation $\Box\phi = 0$, if the initial data do not decay fast enough.

Next, we comment on the theorem's hypotheses:

[5] More precisely, the precise formulation of asymptotic flatness and null structure for $d = 3$ allows for such a derivation. This formal derivation is justified (up to the order that is necessary) through Part 1.

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- Theorem 3.1 is modular: its starting point is the vector field bounds, which is a consequence of the methods in Section 2. This feature allows us to apply this theorem to a large class of examples where global existence and decay is known through previous methods.

- Our formulation of regular threshold does *not* require that ${}^{(\infty)}\mathcal{P}_0$ is elliptic; this feature makes Theorem 3.1 applicable to equations on black hole exteriors [55]. When $\mathcal{P}$ is in product form and is stationary (i.e., $\mathcal{P} = -\partial_t^2 + {}^{(\infty)}\mathcal{P}_0$), the regular threshold assumption is equivalent to the standard assumption of *no eigenvalue or resonance at zero (time) frequency.*

- In the absence of the null structure for $d = 3$, our methods may still be adapted to determine and establish late time tails, but the qualitative features may be drastically different; see Subsection 3.4 below.

The precise formulation and proof of Theorem 3.1 are, unfortunately, too lengthy to present in this survey article. Nevertheless, we can outline the key ideas:

- The overall proof scheme is *iteration* with a finite number of steps, inspired by [14] and [61]. The goal of each step is to improve, at the cost of decreasing the vector-field regularity exponent M (i.e., losing some vector-field regularity), the decay exponent $\alpha \in (J + \frac{d-1}{2}, J + 1 + \frac{d-1}{2})$ in bounds of the form

$$\phi = O^M_{\mathbf{\Gamma}}(\langle\tau\rangle^{-\alpha}) \text{ in } \mathcal{M} \cap \{r \lesssim u\}, \quad \rho_{J+1} = O^M_{\mathbf{\Gamma}}(r^{-(\alpha - \frac{d-1}{2})}) \text{ in } \mathcal{M} \cap \{r \gtrsim u\},$$

  (as well as some bounds on $\mathring{\Phi}_j$, which are suppressed for brevity), where $\psi = O^M_{\mathbf{\Gamma}}(f)$ is a shorthand for $\Gamma^I \psi \lesssim f$, where Γ^I is as in (2.4), for multi-indices I of order $|I| \leq M$.

- When α is sufficiently close to $J+1+\frac{d-1}{2}$, the iteration step extracts a new term $\mathring{\Phi}_{J+1}(u, \theta)$ in the asymptotic expansion. At this point, two scenarios are possible. If $\lim_{u\to+\infty} \mathring{\Phi}_{J+1}(u, \theta) = 0$, then the iteration continues with a larger value of J. If $\lim_{u\to+\infty} \mathring{\Phi}_{J+1}(u, \theta) \neq 0$, then the procedure terminates, and this nonzero limit furnishes the leading order term of the late time tail.

- The basic philosophy in the implementation of an iteration step is to approximate the full equation $\mathcal{P}\phi = \mathcal{N}(\phi)$ by simpler model equations in different regions. More specifically, a single iteration step is divided into three substeps:

  i. **Wave zone,** $\{r \geq u\}$**:** Improve the estimate for ρ_{J+1} by analyzing $\Box\rho_{J+1} =$ (source term);
  ii. **Intermediate zone,** $\{1 \lesssim r \lesssim u\}$**:** Improve the estimate for ϕ in the region where (in particular, $r \simeq u$) by analyzing $\Box\phi =$ (source term);
  iii. **Near-intermediate zone,** $\{r \lesssim u\}$**:** Improve the estimate for ϕ in the region $r \lesssim u$ by analyzing ${}^{(\infty)}\mathcal{P}_0\phi =$ (source term).

  Asymptotic flatness is used in Substeps i and ii, whereas ultimate stationarity is used in Substep iii. The analysis in each substep provides the main contribution for the next, which is why the leading order term of the late time tail is ultimately tied to the expansion in the wave zone.

- A crucial cancellation – which is ultimately responsible for Strong Huygens correction – is that the (unique) forward solution to $\Box\psi = 0$ with characteristic data $\psi|_{\{u=0\}} = \chi_{>1}(r) r^{-J-\frac{d-1}{2}-J} Y_{(\ell)}(\theta)$ *vanishes* in the interior $\{u > 0\}$ (for t large enough) when $\ell \geq j - \frac{d-3}{2}$, where $Y_{(\ell)}$ is a spherical harmonic of degree ℓ. This cancellation eventually implies that the term $r^{-\frac{d-1}{2}-J}\mathbb{S}_{(\ell)}\mathring{\Phi}_J$ in the asymptotic expansion does *not* contribute to the determination of late time tails, and leads to (3.3).

For a more detailed discussion, we refer to the introduction of [55].

3.3 Some consequences of the formal recurrence equations. In this subsection, we delve further into the analysis of formal recurrence equations and derive some consequences of Theorem 3.1. For further reading, we refer the reader to [55, 56].

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3.3.1 Formal recurrence equations. To initiate the discussion, let us start by being more precise about the form of the recurrence equations. Consider the forced wave equation with a right-hand side that admits a formal asymptotic expansion into powers of r^{-1} for r sufficiently large in the (u, r, θ) coordinates:

$$\Box\phi = r^{-\frac{d-1}{2}-1}\left(r^{-1}\mathring{F}_1(u,\theta) + \cdots\right) \tag{3.4}$$

Plugging in the formal asymptotic expansion $\phi = r^{-\frac{d-1}{2}}(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + \cdots)$, and equating the coefficients of each power of r^{-1}, we arrive at the following *formal recurrence equations*:

$$\partial_u \mathring{\Phi}_j(u,\theta) = -\tfrac{1}{2j}\left((j-1)j - \tfrac{(d-1)(d-3)}{4} + \mathring{\Delta}\right)\mathring{\Phi}_{j-1}(u,\theta) + \tfrac{1}{2j}\mathring{F}_j(u,\theta) \quad \text{for } j \geq 1. \tag{3.5}$$

In general, we may put $\mathcal{P}\phi = \mathcal{N}(\phi)$ into the form (3.4) by moving $(\mathcal{P} - \Box)\phi$ to the right-hand side and plugging in the formal asymptotic expansion for ϕ into $\mathcal{N}$. (The precise formulation of the hypotheses in [55] ensures, in particular, that this procedure works.) By the rapid decay of the initial data, we supplement (3.5) with

$$\lim_{u\to-\infty} \mathring{\Phi}_j(u,\theta) = 0. \tag{3.6}$$

Equations (3.5) and (3.6) allow us to determine $\mathring{\Phi}_j$ in terms of $\mathring{\Phi}_0$ in a recursive manner.

3.3.2 Newman–Penrose cancellation and the strong Huygens' principle To understand better the structure of the formal recurrence equations, it is useful to project (3.5) to spherical harmonics and obtain ODEs:

$$\partial_u \mathbb{S}_{(\ell)}\mathring{\Phi}_j = -\tfrac{1}{2j}(j - \tfrac{d-1}{2} - \ell)(j + \tfrac{d-1}{2} + \ell - 1)\mathbb{S}_{(\ell)}\mathring{\Phi}_{j-1} + \tfrac{1}{2j}\mathbb{S}_{(\ell)}\mathring{F}_j \quad \text{for } j \geq 1 \tag{3.7}$$

Observe that the coefficient in front of $\mathbb{S}_{(\ell)}\mathring{\Phi}_{j-1}$ vanishes if $j = \ell + \frac{d-1}{2}$; we refer to this as *Newman–Penrose cancellation.*

In the case of $\Box\phi = 0$, Newman–Penrose cancellation and strong Huygens correction (3.3) explain how the strong Huygens' principle is consistent with Theorem 3.1:

PROPOSITION 3.2. *Let ϕ be a solution to* (2.1) *with smooth and rapidly decaying initial data. Then Theorem* 3.1 *is applicable (with $\mathcal{P} = \Box$ and $\mathcal{N} = 0$), and* (3.2) *holds with $\alpha = +\infty$, i.e., for all $\alpha \geq 0$.*

Of course, this result is true by the strong Huygens' principle, but the point is to understand it in the framework of Theorem 3.1.

Asymptotic flatness, ultimate stationarity, and regular threhold assumptions are true for $\mathcal{P} = \Box$ by design. The vector field bounds hold by, say, any of the methods discussed in Section 2. Hence, it only remains to discuss how the algorithm given by Theorem 3.1 implies Proposition 3.2.

First, note that (3.3) implies $\lim_{u\to+\infty}\mathbb{S}_{(\ell)}\mathring{\Phi}_j = 0$ as long as $j \leq \ell + \frac{d-3}{2}$. For $\Box\phi = 0$, (3.7) holds with $\mathring{F}_j = 0$ for all j. Hence, $\mathbb{S}_{(\ell)}\mathring{\Phi}_j = 0$ for the next j (i.e., $j = \ell + \frac{d-1}{2}$) by Newman–Penrose cancellation. After that, by induction, $\mathbb{S}_{(\ell)}\mathring{\Phi}_j = 0$ for all $j \geq \ell + \frac{d-3}{2}$ by induction. Hence, Theorem 3.1 applies with arbitrary J for $\Box\phi = 0$, which is consistent with the strong Huygens' principle applied to rapidly decaying initial data.

In this argument, one may get a glimpse of how a universal power-type late time tails may arise for a variation of the wave equation: while cancellations such as (3.3) are robust, the analysis of the formal recurrence equations is highly dependent on the extra terms $\mathring{F}_j$ arising from the deviation from the wave equation. This idea will be elucidated in the results and examples discussed below.

3.3.3 General upper bounds. From strong Huygens correction (3.3) and an analysis of the formal recurrence equations (3.5) like Proposition 3.2, we obtain the following general upper bounds:

THEOREM 3.3. *Let $\mathcal{P}$, $\mathcal{N}$, and ϕ satisfy the hypotheses of Theorem* 3.1*. Then the following holds:*

1. *Equation* (3.2) *holds for some $\alpha \geq d - 1$.*
2. *If there exists $J_F \in \mathbb{Z}_{\geq 0}$ such that $\mathring{F}_j = 0$ for all $j < J_F$, then* (3.2) *holds for some $\alpha \geq \frac{d-1}{2} + \max\left\{J_F, \frac{d-1}{2}\right\}$.*

In both cases, it is possible that (3.2) holds with $\alpha = +\infty$ as in Proposition 3.2. This result follows from the same argument that led to Proposition 3.2 (indeed, this theorem generalizes the proposition).

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According to Part 1, the rate $\alpha = d-1$ represents the worst possible case. It is amusing that it coincides with the rate predicted by dimensional analysis. Part 2, on the other hand, explains the reason for the t^{-3}-Price's law for the scalar field in dimension 3: for the linear wave equation on black hole backgrounds, such as Schwarzschild or Kerr, we have $J_F = 2$. However, it turns out that Price's law for $\mathbb{S}_{(\ell)}\phi$ on a Schwarzchild background – which predicts the $t^{-3-2\ell}$ late time tail – is from another cancellation when $\ell > 0$, which we turn to next.

3.3.4 Generalized Price's law: cancellation in the linear, stationary case. In the linear and stationary case, there is always at least one order of cancellation, which generalizes Price's observation:

THEOREM 3.4 (Generalized Price's law [55]). *Let $\mathcal{P}$, $\mathcal{N}$, and ϕ satisfy the hypotheses of Theorem* 3.1. *Assume furthermore that the problem is linear, in the sense that $\mathcal{N} = 0$, and stationary, in the sense that $\mathcal{P} = {}^{(\infty)}\mathcal{P}$. Then* (3.2) *holds for some $\alpha \geq d$ (possibly $\alpha = +\infty$).*

Theorem 3.4 follows from a generalization of the observation of Newman–Penrose [65]: there exists a quantity $I := \mathring{\Phi}_{\frac{d-1}{2}} + \sum_{j=1}^{\frac{d-3}{2}} \mathfrak{c}_j[\mathring{\Phi}_j]$, called the Newman–Penrose constant, that is conserved under (3.5) ($\mathfrak{c}_j$ are linear operators for functions on $\mathbb{S}^{d-1}$). Note that, by strong Huygens correction, each $\mathring{\Phi}_j$ for $j \leq \frac{d-3}{2}$ vanishes as $u \to +\infty$. Combined with the conservation of I, it follows that $\mathring{\Phi}_{\frac{d-1}{2}}$ must also vanish as $u \to +\infty$.

On a spherically symmetric background, the same mechanism shows that there is at least one order of cancellation compared to the baseline decay rate $t^{-(d-1)-2\ell}$; see [55, Section 4]. Price's law for $\mathbb{S}_{(\ell)}\phi$ with $\ell > 0$ (with $\ell = 1$ and 2 being analogues of the Maxwell and the linearized gravity cases, respectively) corresponds to the case when there are no additional cancellations.

3.3.5 Anomalous cancellations for linear, stationary problems. Amusingly, there are many examples of linear and stationary problems where Theorem 3.4 does *not* furnish the sharp upper bound, and exhibit higher order *anomalous cancellations*.

Example 3.5. In this example, we consider the following problem:

$$\phi : \mathbb{R}^{1+3} \to \mathbb{R} \text{ radial}, \quad (\Box + \epsilon V)\phi = 0, \quad V = r^{-3} \text{ in } \{r \gtrsim 1\}.$$

The assumptions in Theorem 3.1 are satisfied if either ϵ is sufficiently small (depending on V), or $-\epsilon V \geq 0$ everywhere; see [55, Section 3.1] and [60].

The formal recurrence equations are:

$$2\partial_u \mathring{\Phi}_1 = 0, \quad 4\partial_u \mathring{\Phi}_2 + 2\mathring{\Phi}_1 + \epsilon\mathring{\Phi}_0 = 0. \tag{3.8}$$

In view of (3.3) (with $\ell = 0$ due to radiality), $\lim_{u\to+\infty} \mathring{\Phi}_0 = 0$, while $\mathring{\Phi}_1 \equiv 0$ by $\partial_u \mathring{\Phi}_1 = 0$ and (3.6). On the other hand, $4\partial_u \mathring{\Phi}_2 + 2\mathring{\Phi}_1 + \epsilon\mathring{\Phi}_0 = 4\partial_u \mathring{\Phi}_2 + \epsilon\mathring{\Phi}_0 = 0$, which implies $\lim_{u\to+\infty} \mathring{\Phi}_2 = -\frac{1}{4}\epsilon \int_{-\infty}^{\infty} \mathring{\Phi}_0(u')\, du'$. The generic late time tail decay rate is given by (3.2) with $J = 2$, or $\alpha = 3$.

Example 3.6. Next, we consider the same problem but posed on $\mathbb{R}^{1+5}$:

$$\phi : \mathbb{R}^{1+5} \to \mathbb{R} \text{ radial}, \quad (\Box + \epsilon V)\phi = 0, \quad V = r^{-3} \text{ in } \{r \gtrsim 1\}.$$

As before, the assumptions of Theorem 3.1 are satisfied if either ϵ is sufficiently small (depending on V), or $-\epsilon V \geq 0$ everywhere.

The formal recurrence equations are:

$$2\partial_u \mathring{\Phi}_1 - 2\mathring{\Phi}_0 = 0, \quad 4\partial_u \mathring{\Phi}_2 + \epsilon\mathring{\Phi}_0 = 0, \quad 6\partial_u \mathring{\Phi}_3 + 4\mathring{\Phi}_2 + \epsilon\mathring{\Phi}_1 = 0, \quad 8\partial_u \mathring{\Phi}_4 + 10\mathring{\Phi}_3 + \epsilon\mathring{\Phi}_2 = 0. \tag{3.9}$$

In view of (3.3) (with $\ell = 0$ due to radiality), $\lim_{u\to+\infty} \mathring{\Phi}_0 = \lim_{u\to+\infty} \mathring{\Phi}_1 = 0$, while $4\partial_u \mathring{\Phi}_2 + \epsilon\mathring{\Phi}_0 = 4\partial_u \mathring{\Phi}_2 + \epsilon\partial_u \mathring{\Phi}_1 = 0$, i.e., $\partial_u I = 0$ where $I := \mathring{\Phi}_2 + \frac{\epsilon}{4}\mathring{\Phi}_1$ (this is the Newman–Penrose constant in this problem). By (3.6), $I \equiv 0$, and thus $\lim_{u\to+\infty} \mathring{\Phi}_2 = 0$ (this is the mechanism behind Theorem 3.4. However, this problem has further cancellations: $6\partial_u \mathring{\Phi}_3 + 4\mathring{\Phi}_2 + \epsilon\mathring{\Phi}_1 = 6\partial_u \mathring{\Phi}_3 + 4I = 0$, which implies $\partial_u \mathring{\Phi}_3 = 0$ and thus $\mathring{\Phi}_3 \equiv 0$. The algorithm finally terminates at the next order: $8\partial_u \mathring{\Phi}_4 + 10\mathring{\Phi}_3 + \epsilon\mathring{\Phi}_2 = 8\partial_u \mathring{\Phi}_4 + \epsilon(-\frac{\epsilon}{4}\mathring{\Phi}_1) = 0$, i.e., $\partial_u \mathring{\Phi}_4 = \frac{\epsilon^2}{4}\mathring{\Phi}_1$. Thus, $\lim_{u\to+\infty} \mathring{\Phi}_4 = \frac{\epsilon^2}{4} \int_{-\infty}^{\infty} \mathring{\Phi}_1(u')\, du'$. In this case, the generic late time tail decay rate is given by (3.2) with $J = 4$, i.e., $\alpha = 6$ (instead of the upper bound exponent 5 in Theorem 3.4).

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A similar phenomenon, but at many more orders, occurs for the linear wave equation on the (5+1)-dimensional Schwarzschild and Reissner–Nordström black hole exteriors:

THEOREM 3.7 ([55, 56]). *Solutions to the wave equation on the* $(5+1)$*-dimensional Schwarzschild spacetime arising from compactly supported initial data satisfy* (3.2) *with* $\alpha = 10$.

Moreover, solutions to the wave equation on any $(5+1)$*-dimensional subextremal Reissner–Nordström spacetime (i.e.,* $|Q| < M$*, where* Q *is the charge and* M *is the mass parameters) arising from compactly supported initial data satisfy* (3.2) *with* $\alpha = 10$ except *when* $\frac{Q^2}{M^2} = \frac{6}{11}$*. In the latter case,* (3.2) *holds*[6] *with* $\alpha = 12$.

The result on the Schwarzschild spacetime confirms the heuristics and numerics in [5, 8]. To the best of our knowledge, the anomalous cancellation in the Reissner–Nordström case does not seem to have been previously observed, even in the heuristic literature.

3.3.6 Emergence of a slow tail for nonstationary problems. In contrast, time-dependence in the equation can disrupt these cancellations and lead to slower decay.

Example 3.8. We consider the same problem as Example 3.6, but with ϵ time-dependent:

$$\phi : \mathbb{R}^{1+5} \to \mathbb{R} \text{ radial}, \quad (\Box + \epsilon(\tau)V)\phi = 0, \quad V = r^{-3} \text{ in } \{r \gtrsim 1\},$$

where we assume that $\epsilon(\tau)$ settles down as $\tau \to +\infty$ to a constant ϵ that has been considered in Example 3.6.

The formal recurrence equations are again (3.9), but with $\epsilon = \epsilon(u)$ (which is a light abuse of notation, since τ is simply a translate of u for r large). As before, $\lim_{u\to+\infty} \mathring{\Phi}_0 = \lim_{u\to+\infty} \mathring{\Phi}_1 = 0$ by (3.3), but a different phenomenon occurs at the next order: $4\partial_u \mathring{\Phi}_2 + \epsilon\mathring{\Phi}_0 = 4\partial_u \mathring{\Phi}_2 + \epsilon\partial_u \mathring{\Phi}_1 = 0$ implies $\partial_u(\mathring{\Phi}_2 + \frac{\epsilon}{4}\mathring{\Phi}_1) = -\partial_u\epsilon\mathring{\Phi}_1$. Hence, $\lim_{u\to+\infty} \mathring{\Phi}_2 = -\int_{-\infty}^{\infty} \partial_u\epsilon(u')\mathring{\Phi}_1(u')\,\mathrm{d}u'$, which can be seen to be generically non-zero. In this case, the generic late time tail decay rate is given by (3.2) with $J = 2$, i.e., $\alpha = 4$. Note that this slow decay exponent is consistent with the upper bound exponent in Theorem 3.3.

Remark 3.9. The same analysis applied to the problem on $\mathbb{R}^{1+3}$ yields (3.2) with $\alpha = 3$, which is also dictated by Theorem 3.3 in view of the decay of V. This is the same phenomenon as the exponent 3 in the $\ell = 0$ case for Gundlach–Price–Pullin, and is also consistent with the generic lower bound justified in [52, 53, 54] on (dynamic) Reissner–Nordström black hole exteriors.

This mechanism rigorously explains the numerical observations of Gundlach, Pullin, and Price, as well as the formal analysis of Bizoń and Rostworowski. We state this result informally as follows:

THEOREM 3.10 ([55, 56]). *The numerical results of Gundlach–Pullin–Price [29] and the formal analysis of Bizoń–Rostworowski [6] are correct: The slower decay for* $\ell \geq 1$ *modes is true, due to the dynamical background.*

It also suggests that the t^{-4} late time tail upper bound for the Maxwell equation on non-stationary backgrounds due to Metcalfe–Tataru–Tohaneanu [62] is sharp. We leave its verification, which involves generalizing Theorem 3.1 to the Maxwell equation, as an open problem.

3.3.7 Nonlinear problems. Theorem 3.1 also applies nonlinear problems. We discuss one example, where the geometric properties of the nonlinearity significantly alters the generic late time tail.

Consider the *wave maps* equation, which is a geometric wave equation for Riemannian-manifold-valued maps. For $\boldsymbol{\phi} : \mathbb{R}^{1+3} \to (\mathcal{N}^2, h)$ the equation reads, in local coordinates $(\mathbf{X}^I)$ near a point $p \in \mathcal{N}^2$ in the target,

$$\Box\boldsymbol{\phi}^I = -\boldsymbol{\Gamma}^I_{JK}(\boldsymbol{\phi})\partial_\mu\boldsymbol{\phi}^J\partial^\mu\boldsymbol{\phi}^K,$$

where $\boldsymbol{\Gamma}^I_{JK}(\boldsymbol{\phi})$ are the Christoffel symbols in $(\mathbf{X}^I)$. By the vector field method in Section 2.3, there exists a global solution $\boldsymbol{\phi}^I$ satisfying vector field bounds provided that the initial data are sufficiently small (i.e., image is localized near p), smooth, and rapidly decaying [39]. Denote by $K(p')$ the *Gauss curvature* of $\mathcal{N}^2$ at $p' \in \mathcal{N}^2$.

THEOREM 3.11 (Late time tails for wave maps [55, 56]). *The global solution* $\boldsymbol{\phi}^I$ *to the wave maps equation satisfying vector field bounds obey* (3.2) *for:* $\alpha = 3$ *if* $K(p) \neq 0$*, some* $\alpha \geq 4$ *if* $K(p) = 0$*, and* $\alpha = +\infty$ *if* $K \equiv 0$ *near* p.

[6] In this case, the leading order term is not given by the $\ell = 0$ spherical harmonic, but rather by $\ell = 1$.

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We remark that the last case is geometrically obvious, since $K \equiv 0$ near p implies that $\mathcal{N}^2$ is flat there, and hence there exist local coordinates that make $\mathbf{\Gamma}^I_{JK}$ zero. It is natural to conjecture that α depends on the degree of vanishing of K at p; we leave this as an open problem.

3.4 Further developments and outlook.

3.4.1 Einstein vacuum equation in wave coordinates. For the Einstein vacuum equation in wave coordinates, the nonlinear terms do not possess null structure (hence, Theorem 3.1 is not directly applicable), but have "weak null structure." This feature was heavily exploited in the proof of the small data global existence and decay for this system by Lindblad–Rodnianski [49], which provided an alternative proof of the global nonlinear stability of Minkowski spacetime.

Lindblad [48] delved further into the question of global asymptotics for this system, and identified a slow t^{-1} leading-order term for the metric components that is nonzero for all non-flat solutions. Mao [58] revisited this global asymptotics by extending our framework. More interestingly, [58] demonstrates that the slow t^{-1} tail is a coordinate (or gauge) effect, in the sense that it can be removed by a coordinate transformation to another "wave-like" coordinate.

3.4.2 Quasilinear equations with weak null structure. For quasilinear equations with weak null structure, such as those studied by Lindblad [46, 47], Alinhac [1] and Yu [79, 80, 81], a curious phenomenon occurs: the late time tail is not described by a single power law, but instead, it is a superposition over a continuum of decay rates [57]. In fact, [57] determines the coefficients in this superposition in terms of the radiation field (defined by Yu [79, 80]). This information, combined with the scattering theory established by Yu [79, 80], leads to remarkably strong *rigidity results*, which precludes nontrivial solutions with small, smooth, localized initial data from having a too fast late time tail. A sample statement is as follows: *Consider the initial-value problem for $\Box\phi = \phi\Delta\phi$ in $(0, \infty) \times \mathbb{R}^3$ with $(\phi, \partial_t\phi) = \epsilon(g, h)$ on $\{t = 0\}$, where $(g, h) \in C^\infty_c(\mathbb{R}^3)$. If the global solution ϕ (whose existence was proved in [1, 47]) satisfies $|\phi(t, x_0)| \leq Ct^{-1-\delta}$ for some $C, \delta > 0$, and ϵ is sufficiently small (compared to δ), then $\phi \equiv 0$.* This extends a theorem of [81] in the radial case.

3.4.3 Nonlinear stability problem for steady states. The ideas for establishing late time tail upper bounds have been instrumental in recent proofs of the nonlinear stability of solitons and other steady states for quasilinear wave equations in low dimensions; see [67, 76], which extended the codimension-1 nonlinear stability of the catenoid under the hyperbolic vanishing mean curvature equation (or the time-like minimal surface equation) in high dimensional spacetimes [51] to dimensions $d = 3$ and 4, respectively.

3.4.4 Strong cosmic censorship for perturbations of Kerr black hole. Penrose's *Strong Cosmic Censorship* (SCC) conjecture is a major theoretical problem in general relativity, motivated by a paradox at the heart of most important solutions. The *subextremal Kerr black holes* – explicit solutions to the Einstein vacuum equations describing rotating black holes – appear to violate the principle of determinism! More precisely, inside the black hole in each of these solutions lies a smooth boundary called the *Cauchy horizon*, beyond which the future evolution of the spacetime (and hence the fate of any physical system therein) is not uniquely determined by the initial data, even though they are complete. Such a bizarre behavior inside these black holes is troubling because the subextremal Kerr black holes are believed to faithfully describe the exterior of rotating black holes in our universe.

The SCC conjecture attempts to settle this problem by boldly asserting that such failures of determinism are *non-generic*. Specifically, for solutions corresponding to generic initial data close to that of a subextremal Kerr spacetime, SCC predicts that the smooth Cauchy horizon becomes a *singularity*, so that there is no way to extend the spacetime as a solution to the Einstein equation. Hence, where determinism is possibly violated would be outside of general relativity's purview!

The strongest mathematical evidence supporting SCC comes from the study of a spherically symmetric toy model: the spherically symmetric Einstein–Maxwell–(real) scalar field system. For maximal solutions –or more precisely, the maximal globally hyperbolic developments (MGHD) [30, Section 7.6] – arising from admissible two-ended[7] asymptotically flat initial data with nontrivial Maxwell fields, Dafermos [9, 10] and Dafermos–Rodnianski [14] showed that a nonempty Cauchy horizon exists (i.e., $\mathcal{CH}^+ \neq \emptyset$) and, remarkably, the C^0 extendibility of the

[7]This condition is needed to allow for nontrivial Maxwell fields; note that a spherically symmetric Maxwell field is necessarily trivial on a *one*-ended asymptotically flat spacetime.

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solution beyond $\mathcal{CH}^+$, which may seem to be at odds with SCC. Nevertheless, J. Luk and the author [53, 54] proved SCC by showing that for generic (in the sense of open and dense with respect to suitable topologies) admissible two-ended data, the MGHD exhibits a blow-up near every point on $\mathcal{CH}^+$, i.e., the first derivatives of the scalar field fail to be in $L^2(\mathcal{O})$ for any neighborhood $\mathcal{O}$ of $p \in \mathcal{CH}^+$. In sum, these results, along with the more recent results [23], confirm the *mass inflation scenario* (see [10, 11] and the references therein).

Beyond spherical symmetry, one immense difficulty of SCC is that it is nonperturbative. A natural starting point, which retains the core of the problem, is to consider perturbations of Kerr spacetimes. Thanks to recent work of Dafermos–Luk [13] on the C^0-stability of the Kerr interior, the conjecture can be formulated as follows:

CONJECTURE 3.12 (Strong Cosmic Censorship conjecture for perturbations of Kerr). *The MGHD to the Einstein vacuum equation corresponding to a generic small regular and localized perturbations of the initial data for a subextremal Kerr spacetime has complete null infinity $\mathcal{I}^+$ and a bifurcate null future boundary $\mathcal{CH}^+$, through which the spacetime extends as a Lorentzian manifold with C^0 metric, but not as one with the derivative of the metric locally in L^2.*

In the slowly rotating case, in light of the spectacular progress made on the nonlinear stability of the Kerr black hole exterior [41, 12, 44, 42, 43, 72, 27, 28], the key remaining issue is the question of generic inextendibility past $\mathcal{CH}^+$. Conjecture 3.12 is still wide open even in this subcase, but [53, 54] provides a two-step road map:

- **Step 1.** Show that generic initial data lead to a solution with a universal nontrivial long-term asymptotics of gravitational waves on the boundary of the black hole (i.e., the event horizon $\mathcal{H}^+$).
- **Step 2.** Show that the universal long-term asymptotics on $\mathcal{H}^+$ from Step 1 indeed turns the smooth Cauchy horizon $\mathcal{CH}^+$ into an inextendible bifurcate null boundary (weak null singularity) inside the black hole.

We believe that Theorem 3.1 constitutes a major step toward addressing Step 1, which was the most difficult component of the analysis in [53, 54]. In this direction, the major next goal is *to extend Theorem* 3.1 *to solutions of the Einstein vacuum equations that asymptote to subextremal Kerr black hole exteriors* (cf. the aforementioned work of Mao [58] for solutions that asymptote to the Minkowski spacetime).

4 Klein–Gordon equation.

In this final section, we discuss the long-term behavior of solutions to the Klein–Gordon equation, which exhibits a markedly different asymptotic structure from the wave equation.

4.1 Constant-coefficient Klein–Gordon equation.

We begin with the initial value problem for the constant-coefficient Klein–Gordon equation with unit mass:

$$\begin{cases} (\Box - 1)\phi = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^d_x, \\ (\phi, \partial_t \phi) = (g, h) & \text{on } \{t = 0\}, \end{cases} \tag{4.1}$$

where, unless stated otherwise, we assume that (g, h) is smooth and rapidly decaying.

4.1.1 The method of stationary phase.

For the Klein–Gordon equation, the forward fundamental solution does not take a simple form as in the wave equation. An effective way to understand the solution to (4.1) is to instead use the representation via the Fourier transform, which takes form $\phi(t, x) = \sum_{\pm} \int_{\mathbb{R}^d} e^{i(\pm t\langle\xi\rangle + x\cdot\xi)} a_{\pm}(\xi)\,\mathrm{d}\xi$, where $\langle\xi\rangle := \sqrt{1 + |\xi|^2}$ (with 1 coming from the unit mass) and the amplitudes $a_{\pm}(\xi)$ are determined by the initial data. The long-term asymptotics can be extracted using the method of stationary phase.

For an observer at (t, tv) moving with velocity v (spatial translation by x_0 does not alter the analysis in a significant way), the critical points of the phase $t(\pm\langle\xi\rangle + v \cdot \xi)$ occur at $\xi = \mp\xi_v$, where $\xi_v := \frac{v}{\sqrt{1-|v|^2}}$. These points only exist for sub-luminal speeds $|v| < 1$, or equivalently, in the interior of the light cone $|x| < t$. According to the method of stationary phase [75, Section 8.2], we obtain the asymptotics

$$\phi(t, tv) = c_d \sum_{\pm} e^{\pm it\sqrt{1-|v|^2}} t^{-\frac{d}{2}} \left(\frac{1}{\sqrt{1-|v|^2}}\right)^{\frac{d}{2}+1} a_{\pm}(\mp\xi_v) + o(\rho^{-\frac{d}{2}}) = c_d \sum_{\pm} e^{\pm i\rho} \rho^{-\frac{d}{2}} \langle\tfrac{x}{\rho}\rangle a_{\pm}(\mp\tfrac{x}{\rho}) + o(\rho^{-\frac{d}{2}}), \tag{4.2}$$

as $t \to +\infty$ for $|v| < 1$ (or $|x| < t$), where we used $x = tv$ and $\rho = \sqrt{t^2 - |x|^2}$ in the last expression. With a little more work, one can show that the rate $t^{-\frac{d}{2}}$ is the sharp uniform decay rate: $\|\phi(t, \cdot)\|_{L^\infty} \lesssim t^{-\frac{d}{2}}$. Clearly, the right-hand side cannot be improved in view of the appearance of $a_{\pm}$ in the asymptotics.

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4.1.2 The vector field method in hyperboloidal coordinates. A physical space approach can recover the uniform decay rate and these asymptotics. The key is to adapt the vector field method to the geometry of the hyperboloid $H_{\rho_0} = \{\rho = \rho_0\}$.

We now sketch the physical space proof of the uniform decay rate due to Klainerman [37], but using a different set of coordinates that simplifies the argument. For brevity, we only focus on the interior of the null cone $\{r < t\}$, which is covered by the *hyperboloidal coordinates* (ρ, y), where $y = \frac{x}{\rho}$ (see [40] for the analysis near and outside the null cone). In these coordinates, the scaling and Lorentz boost vector fields take particularly simple forms: $S = \rho\partial_\rho$ and $L_j = \langle y\rangle \partial_{y^j}$.

For (4.1), the *energy-momentum tensor* for ϕ takes the form

$$Q_{\mu\nu}[\phi] := \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\eta_{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi + \phi^2).$$

This tensor obeys similar properties as Lemma 2.1, with $\Box$ replaced by $\Box - 1$. Define the energy flux $E[\phi](\Sigma)$ through Σ exactly as in Section 2.3; in this case, the energy flux through H_ρ plays an important role. By a direct computation, thanks to the extra term ϕ^2 in $Q_{\mu\nu}[\phi]$, it follows that

$$E[\phi](H_\rho) \gtrsim \int_{H_\rho} \phi^2 \rho^d \, \mathrm{d}y. \tag{4.3}$$

By Lemma 2.2, if $(\Box - 1)\phi = 0$, then $(\Box - 1)L_j\phi = 0$. Similarly, the rotation vector field Ω_{jk} commutes with $\Box - 1$, but the scaling vector field S does *not* commute well with $(\Box - 1)$, which is an important distinction from the wave equation.

By applying the Lorentz boost vector fields $L_j = \langle y\rangle\partial_{y^j}$ repeatedly and using the conservation of energy for each resulting solution, one can control $E[L^I\phi](H_\rho)$ for any I. Combining (4.3) with the Sobolev embedding with radial weights gives *Klainerman's Sobolev inequality*:

$$|\phi(\rho, y)| \lesssim \rho^{-\frac{d}{2}} \langle y\rangle^{-\frac{d}{2}} \sum_{I:|I| \le \lfloor \frac{d}{2} \rfloor + 1} \left(\int_{H_\rho} |(\langle y\rangle\nabla_y)^I \phi|^2 \rho^d \, \mathrm{d}y \right)^{\frac{1}{2}} \lesssim t^{-\frac{d}{2}} \sum_{I:|I| \le \lfloor \frac{d}{2} \rfloor + 1} E[L^I\phi](H_\rho)^{\frac{1}{2}}, \tag{4.4}$$

which proves the uniform decay rate in $\{r < t\}$.

To obtain the asymptotics in $\{r < t\}$, we employ an ODE argument due to Lindblad–Soffer [50]. In analogy with Subsection 2.3, the starting point is the Klein–Gordon equation expressed in the (ρ, y)-coordinates:

LEMMA 4.1. *If $(\Box - 1)\phi = f$, then in the (ρ, y)-coordinates, $\Phi := \rho^{\frac{d}{2}}\phi$ solves*

$$-(\partial_\rho^2 + 1)\Phi + \rho^{-2}\Delta_{\mathbb{H}^d}\Phi = \rho^{\frac{d}{2}} f, \tag{4.5}$$

where $\Delta_{\mathbb{H}^d}$ is the Laplacian on the hyperbolic space in the y coordinates. We also have $\Delta_{\mathbb{H}^d} = \sum_j L_j^2 - \mathring{\Delta}$.

For a solution ϕ to (4.1), the bound (4.4) implies that all terms on the left-hand side, except the first, are $O(\rho^{-2})$ (since L_j and Ω_{jk} commute with $\Box - 1$). This leaves the harmonic oscillator ODE for Φ:

$$(\partial_\rho^2 + 1)\Phi = O(\rho^{-2})$$

Integrating this ODE shows that $\Phi(\rho, y)$ converges to a linear combination of $e^{\pm i\rho}$ as $\rho \to \infty$, recovering (4.2).

4.2 Physical space approach for the variable-coefficient Klein–Gordon equation. In [66], we introduce a physical space method for establishing sharp uniform decay and global asymptotics for variable-coefficient Klein–Gordon equations in $d = 3$. The main result, in a simplified form, is:

THEOREM 4.2. *Let $\mathcal{M} = \mathbb{R}_t \times (\mathbb{R}_x^3 \setminus \mathcal{K})$, where $\mathcal{K}$ may be a smooth bounded domain (not necessarily connected) in $\mathbb{R}^3$ in case of the exterior IBVP. Consider the equation*

$$(-\partial_t^2 + \mathcal{P}_0 - 1)\phi = 0, \tag{4.6}$$

where ϕ is a real-valued function on $\mathcal{M}$ and $\mathcal{P}_0$ is a scalar second-order linear operator on $\mathbb{R}^3 \setminus \mathcal{K}$ (in the case of the exterior IBVP, assume also the Dirichlet boundary condition $\phi = 0$ on $\mathbb{R}_t \times \partial\mathcal{K}$).

Assume the following on $\mathcal{P}_0$ and ϕ:

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- **Asymptotic flatness (in product form).** *We have* $\mathcal{P}_0 \to \Delta_{\mathbb{R}^3}$ *as* $|x| \to +\infty$ *(in a suitable fashion).*
- **Regular threshold.** *The operator* ${}^{(t)}\mathcal{P}_0$ *defined by freezing the coefficients at time* t *(supplemented with the Dirichlet boundary condition in the case of the exterior IBVP) is invertible on suitable Sobolev spaces.*
- **(Dual) integrated local energy decay.** *There exist* $\beta_c^* > 0$ *and* $s_c^* \geq 0$ *such that, for every* ψ *solving* $(-\partial_t^2 + \mathcal{P}_0 - 1)\psi = f$ *with zero initial data (supplemented with the Dirichlet boundary condition in the case of the exterior IBVP) and all* $N \geq 0$*, we have* $\sup_{t>0} E[\partial_{t,x}^{(\leq N)}\psi](S_t) \lesssim_N \int_0^\infty \int (\langle r\rangle^{1+\beta_c^*}\partial_{t,x}^{(\leq N+s_c^*)} f)^2 \,\mathrm{d}x\mathrm{d}t.$
- **Admissible initial data.** *The initial data for* ϕ *on* $\{t=0\}$ *is smooth and rapidly decaying.*
- **A-priori energy estimate.** *We have* $\sup_{t>0} E[\partial_{t,x}^{(\leq M_0)}\psi](S_t) < +\infty$ *for some* M_0 *that is sufficiently large.*

Then we have

$$\phi(t,x) = \begin{cases} \sum_\pm \rho^{-\frac{3}{2}} e^{\pm i\rho}\gamma_\pm(\frac{x}{\rho})\varphi_0(t,x) + o(t^{-\frac{3}{2}}) & \text{in } |x| < t, \\ o(t^{-\frac{3}{2}}) & \text{in } |x| > t, \end{cases} \tag{4.7}$$

where $\gamma_\pm$ *is a bounded continuous function,* $\varphi_0(t,\cdot)$ *is the solution to* ${}^{(t)}\mathcal{P}_0\varphi_0(t,\cdot) = 0$ *on* $\mathbb{R}^d \setminus \mathcal{K}$ *with the boundary condition* $\varphi_0 \to 1$ *as* $|x| \to +\infty$ *(plus the Dirichlet boundary condition in the case of the exterior IBVP), and* $o(t^{-\frac{3}{2}})$ *holds* uniformly *in* x*. (Similar global asymptotics also hold for higher derivatives of* ϕ*, which we suppress.)*

This result is analogous to the r^p-weighted energy method and Theorem 3.1, as it derives sharp asymptotics from a similar set of abstract assumptions: asymptotic flatness (in product form), (dual) integrated local energy decay, regular threshold, and a weak bound on the solution (i.e., a-priori energy estimate). Here, the asymptotic flatness assumption is stated for the product form, since the oscillatory factor $e^{i\pm\rho}$ in (4.7) makes ∂_t acts in a very different way compared to the wave equation case. Also, we have chosen to state the dual form of integrated local energy decay, since this is the form that is used directly in the argument (see below). Finally, we note that an energy estimate for (4.6) – which impiles the a-priori energy estimate – holds for a reasonable class of $\mathcal{P}_0$, but we have chosen not to take it as an assumption to keep the statement simpler and more general.

To demonstrate the main difficulty in the proof and how it is overcome, consider the simple model case

$$\phi : \mathbb{R}^{1+3} \to \mathbb{R}, \quad (\Box - 1 + \epsilon V)\phi = 0, \quad \epsilon \in \mathbb{R}, \quad \operatorname{supp} V \subseteq \{r \lesssim 1\} \text{ and } V \text{ is real-valued everywhere.}$$

As before, the assumptions are satisfied if either $|\epsilon|$ is sufficiently small (depending on V), or $-\epsilon V \geq 0$ everywhere.

A naive first idea might be to try to directly apply the vector field method for $\Box - 1$. Then we immediately face the following problematic commutator:

$$[L_j, \Box - 1 + \epsilon V] = \epsilon[L_j, V] = \epsilon t \partial_j V, \quad \text{hence } (\Box - 1 + \epsilon V)L_j\phi = \epsilon t\partial_j V.$$

The last equation is not very useful, since the right-hand side has the growing weight t, whereas the weights in L_j also grow at most like t in time; this is not enough for handling the growth on the right-hand side. Moreover, *a-posteriori*, the asymptotics (4.7) show that $E[L_jL_k\phi](S_t)$ should *not* be bounded in t like the constant-coefficient case. The expected global asymptotics (4.7), however, motivates our *good commutator*:

$$\mathcal{L} := \sum_j L_j^2 + \epsilon\rho^2 V.$$

This second-order operator has the following desirable features over Lorentz boosts L_j:

- In the region $\{r \gtrsim t\}$, $\mathcal{L} \approx \sum_{j=1}^3 L_j^2$, which may control L_jL_k through elliptic estimates.
- In the (previously problematic) region $\{r \lesssim 1\}$, $\mathcal{L} \approx t^2(\Delta + \epsilon V)$, which precisely annihilates the spatial profile φ_0! Algebraically, the cancellation property manifests through an improved commutator identity (valid everywhere in $\{r < t\}$): $[\mathcal{L}, -\partial_t^2 + \mathcal{P}_0 - 1] = -2\epsilon(SV)\rho\partial_\rho + \text{(better)}$. Here, the worst term on the right-hand side has the growing weight ρ ($\simeq t$ on $\operatorname{supp} SV \subseteq \operatorname{supp} V$), whereas the weights in $\mathcal{L}$ grow like ρ^2 in time, which is stronger.

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Still, this commutator identity is not useful if we naively use the energy estimate to handle the right-hand side, since it will involve an integration in time, thereby losing a full power of t (which is barely not enough). The idea is to use instead the *dual integrated local energy decay estimate*, which only involves L^2-integral in time (and thus loses only $t^{\frac{1}{2}}$) at the cost of requiring some space decay of the right-hand side, which we have thanks to asymptotic flatness. This gain serves as the basis for an iteration scheme (with finitely many steps) to establish (4.7). In each step of the iteration, the goal is to improve the growth rate of $E[\mathcal{L}\phi](H_\rho)$ in powers of ρ, as well as pointwise decay rates for ϕ and its derivatives (cf. [55], but also the earlier approaches of [16] and [71]).

The price we pay for using the good commutator $\mathcal{L}$ is that the coercivity of $E[\mathcal{L}\phi](H_\rho)$ is unclear (even after adding a lower order control $KE[\phi](H_\rho)$ with a sufficiently large $K > 0$). To handle this difficulty, we introduce *two-scale elliptic analysis* of $\mathcal{L}$: the basic philosophy is to approximate $\mathcal{L}\phi = \sum_j L_j^2\phi +$ (better) in $\{r \gtrsim t\}$ and $\mathcal{L}\phi = t^2\mathcal{P}_0\phi +$ (better) in $\{r \lesssim t\}$. In implementing the latter, the regular threshold assumption plays a key role.

Finally, we recover the sharp uniform decay and global asymptotics by adapting the method of testing by wave packets, which is a powerful idea due to Ifrim–Tataru [33, 34] that requires much less control of ϕ compared to the ODE integration method.

Remark 4.3. Theorem 3.1 does *not* apply to a Schwarzschild black hole exterior, one of the main reasons being the slow r^{-1} decay rate of the $(g_{Sch}^{-1})^{tt} - (\eta^{-1})^{tt}$ (where g_{Sch} is a Schwarzschild metric). In fact, the asymptotics are *completely* different as $t \to +\infty$ for x fixed; see the remarkable recent work [73, 69] on this problem.

Remark 4.4. Another interesting work in the multi-dimensional setting is Léger–Pusateri [45], who used distorted Fourier transform to analyze the problem $(\Box - 1 + V)\phi = \phi^2$ and established global existence and decay for small, smooth, and localized data. Here, $\Delta + V$ was also allowed to have a bound state satisfying the Fermi golden rule, in which case global existence and decay exploit radiation damping [74].

4.2.1 Applications.

The key advantage of the physical space approach (especially compared to prior non-physical-space approaches) is that it is easily adaptable to nonlinear (even quasilinear) problems.

An interesting example is $\mathcal{P}_0 = \Delta$ and $\mathcal{K}$ is a finite union of smooth convex domains (i.e., finitely many convex obstacles), and $\mathcal{N}$ a combination of quadratic nonlinearities of the form $\phi\partial_\mu\partial_\nu\phi$, $\partial_\mu\phi\partial_\nu\phi$, $\phi\partial_\nu\phi$, and ϕ^2, such that $\mathcal{N}(\phi) = 0$ on $\mathbb{R}_t \times \partial\mathcal{K}$ for ϕ satisfying the Dirichlet boundary condition. In this case, trapped geodesics exist, but are all unstable. By the work of Ikawa [36, 35], integrated local energy decay (with a derivative loss) holds, and our approach is applicable. This case extends the seminal result of Metcalfe–Sogge [59] for quasilinear wave equations (with null structure) outside obstacles to quasilinear Klein-Gordon equations.

4.3 Outlook.

We end this section with a list of follow-up questions.

- **Soliton stability.** A major application of this work is to the nonlinear stability of solitons for relativistic field theories with mass, such as the Abelian–Higgs or Yang–Mills–Higgs models. *Adapting our physical space approach to these multi-dimensional*[8] *soliton stability problems* is a key future goal.

- **Incorporation of space-time resonances.** The method of *space-time resonances*, developed by Germain–Masmoudi–Shatah [25, 24, 26], is a powerful Fourier-based framework for the analysis of nonlinear dispersive equations with a constant-coefficient linear part. *Integrating the delicate nonlinear cancellations captured by this method into a robust physical space framework* (like ours) would be a significant advance.

- **Generalization to other space dimensions.** *Extending our results to other dimensions* – especially the physical cases $d = 1, 2$ – is a natural next step. This presents new challenges, as the regular threshold assumption fails in these cases due to a zero-energy resonance.

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[8]In the one-dimensional case, there has been much progress on the soliton stability problem based on *distorted Fourier transform*.

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