On Nonperiodic Euler Flows with Hölder Regularity

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Abstract

In [Ise13], the first author proposed a strengthening of Onsager's conjecture on the failure of energy conservation for incompressible Euler flows with Hölder regularity not exceeding $1/3$. This stronger form of the conjecture implies that anomalous dissipation will fail for a generic Euler flow with regularity below the Onsager critical space $L_t^\infty B_{3,\infty}^{1/3}$ due to low regularity of the energy profile.

This paper is the first and the main paper in a series of two papers, whose results may be viewed as first steps towards establishing the conjectured failure of energy regularity for generic solutions with Hölder exponent less than $1/5$. The main result of the present paper shows that any given smooth Euler flow can be perturbed in $C_{t,x}^{1/5-\epsilon}$ on any pre-compact subset of $\mathbb{R} \times \mathbb{R}^3$ to violate energy conservation. Furthermore, the perturbed solution is no smoother than $C_{t,x}^{1/5-\epsilon}$. As a corollary of this theorem, we show the existence of nonzero $C_{t,x}^{1/5-\epsilon}$ solutions to Euler with compact space-time support, generalizing previous work of the first author [Ise12] to the nonperiodic setting.

1 Introduction

The present work concerns the construction of Hölder continuous solutions to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$

$$\partial_t v^l + \partial_j(v^j v^l) + \partial^l p = 0$$
$$\partial_j v^j = 0 \tag{E}$$

that fail to conserve energy. As we consider solutions with fractional regularity, what we mean by a solution to (E) is a continuous velocity field $v : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ that together satisfy (E) in the sense of distributions. For continuous solutions, this notion of solution may be formulated equivalently in terms of the integral laws of momentum balance and balance of mass, which are commonly used to derive (E) in continuum mechanics; see [DLS14].

A central question concerning weak solutions to (E) is the possibility of dissipation or creation of energy for solutions to Euler in Hölder or Besov type spaces where the known results on energy conservation do not apply. The interest in this question originates from a 1949 note of L. Onsager on statistical turbulence [Ons49], wherein Onsager proposed a mechanism for turbulent energy dissipation driven by frequency cascades that he postulated may exist even among appropriately defined weak solutions to the inviscid equation (E). There Onsager stated that energy is conserved by periodic

solutions in the class $L_t^\infty C_x^\alpha$ if $\alpha > 1/3$, and conjectured that energy conservation may fail for such solutions if $\alpha < 1/3$ (see [DLS14, ES06] for detailed expositions). The conservation of energy stated by Onsager was proven in [Eyi94, CET94], and this result was refined in [CCFS08] to show that energy conservation holds for energy class solutions in the space $L_t^3 B_{3,c_0(\mathbb{N})}^{1/3}$ on either $I \times \mathbb{T}^n$ or $I \times \mathbb{R}^n$ (see also [DR00, IO13] for further proofs). On the other hand, the proof of energy conservation fails for the space $L_t^3 B_{3,\infty}^{1/3}$, and an example in [CCFS08] suggests that anomalous dissipation of energy may be possible in this class. The Besov regularity $\dot{B}_{p,\infty}^{1/3}$ carries a special significance in turbulence theory as it agrees with the $p = 3$ case of the scaling $\langle |v(x + \Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^{\frac{1}{3}} |\Delta x|^{\frac{1}{3}}$ predicted by Kolmogorov's theory [Kol41]. See [Eyi08, Shv10] for further discussion. Recently there has also been a series of advances towards the negative direction of Onsager's conjecture that we will discuss further below [DLS13, DLS12a, Ise12, BDLIS14].

Following the works above, the first author proposed in [Ise13] a stronger form of Onsager's conjecture that will be a main motivation for the present work. The conjecture of [Ise13] states that a generic solution to incompressible Euler with regularity at most $1/3$ will not only fail to conserve energy, but also will possess an energy profile of minimal regularity. For periodic solutions in the class $C_t C_x^\alpha$ with $\alpha < 1/3$, the conjecture may be formulated precisely as follows:

Conjecture 1 (Generic Failure of Energy Regularity). *For any $\alpha < 1/3$, there exists a solution to* (E) *in the class $v \in C_t C_x^\alpha(\mathbb{R} \times \mathbb{T}^n)$ whose energy profile $e(t) = \int_{\mathbb{T}^n} |v|^2(t, x) dx$ fails to have any regularity above the exponent $2\alpha/(1 - \alpha)$, in the sense that[1] $e(t) \notin W^{2\alpha/(1-\alpha)+\epsilon, p}(I)$ for every $\epsilon > 0$, $p \geq 1$ and every open time interval $I \subseteq \mathbb{R}$.*

Furthermore, the set of all such solutions v with the above property is residual (in the sense of category) within the space of all weak solutions to (E) *in the class $v \in C_t C_x^\alpha(\mathbb{R} \times \mathbb{T}^n)$ when the latter space is endowed with the topology from the $C_t C_x^\alpha$ norm.*

Conjecture 1 conveys a sense in which anomalous dissipation should be unstable and nongeneric for weak solutions to Euler with regularity strictly below $1/3$. Assuming Conjecture 1, anomalous dissipation fails to hold for generic solutions to Euler in the class $C_t C_x^\alpha$ when $\alpha < 1/3$, as the energy profile of a typical solution in such a space will fail to be of bounded variation, and hence fail to be monotonic. Instead, the only regularity one can expect for the energy profile of a solution in this class would be provided by the following estimate, proven in [Ise13]:

$$\sup_t \sup_{\Delta t \neq 0} \frac{|e(t + \Delta t) - e(t)|}{|\Delta t|^{\frac{2\alpha}{1-\alpha}}} \leq C_\alpha \|v\|_{C_t \dot{B}_{3,\infty}^\alpha}^3. \tag{1}$$

One expects that the $C^{\frac{2\alpha}{1-\alpha}}$ bound above should be sharp, since the proof of (1) can be viewed as a generalization of the argument used by [CET94] to prove the positive direction of Onsager's conjecture. (The proof of (1) gives more precise information, showing that the fluctuations in the energy profile at time scales of the order τ are governed by contributions from wavenumbers of the order $\tau^{-\frac{1}{1-\alpha}}$.)

The formulation of Conjecture 1 captures part of the intuition that even slight perturbations in a space of solutions with regularity below $1/3$ will typically produce small, rapid oscillations in time for the energy profile of the solution, and the regularity of these oscillations will be governed by the regularity of the perturbation in accordance with the proof of inequality (1). The same intuition offers a picture of what may be expected for solutions in the Onsager critical spaces $L_t^p \dot{B}_{3,\infty}^{1/3}$ for $p \leq \infty$, namely that anomalous dissipation (if possible) would be similarly nongeneric for solutions in $L_t^p \dot{B}_{3,\infty}^{1/3}$ for $p < \infty$, but in contrast would be stable under perturbation for Euler flows in the $L_t^\infty \dot{B}_{3,\infty}^{1/3}$, where having a strictly positive rate of energy dissipation $-\frac{d}{dt} \int \frac{|v|^2}{2}(t, x) dx \geq \varepsilon > 0$ is an open condition.

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[1] Here we use $W^{s,p}$ to denote the Sobolev space with "s" derivatives measured in L^p.

One goal of our work is to give rigorous support to the above intuition in the range of exponents $\alpha < 1/5$ for dimension $n = 3$.

Remark. The aforementioned stability result derives from the following estimate for the difference of the energy profiles e_1, e_2 of two weak solutions to Euler in the class $v_1, v_2 \in L_t^\infty \dot{B}_{3,\infty}^{1/3}$ with domain $I \times \mathbb{T}^n$ or $I \times \mathbb{R}^n$, which was observed in [Ise13, Section 3] by extending the argument of [CET94, CCFS08]:

$$\left\| \frac{d}{dt}(e_2 - e_1) \right\|_{L_t^\infty(I)} \leq C \|v_1 - v_2\|_{L_t^\infty \dot{B}_{3,\infty}^{1/3}} \max \left\{ \|v_1\|_{L_t^\infty \dot{B}_{3,\infty}^{1/3}}, \|v_2\|_{L_t^\infty \dot{B}_{3,\infty}^{1/3}} \right\}^2.$$

Here it is important to consider solutions with uniform in time bounds rather than L_t^p integrability, since the analogous estimate in the class $L_t^p \dot{B}_{3,\infty}^{1/3}$ for $3 < p < \infty$ controls only the $L_t^{p/3}$ norm of $\frac{de}{dt}$ (or the total variation norm of $\frac{de}{dt}$ in the case $p = 3$). For $p < \infty$, one should expect instead that the set of all solutions with nonincreasing energy profiles would be a closed set with empty interior (and hence be nowhere dense) in the space of all $L_t^p \dot{B}_{3,\infty}^{1/3}$ solutions.

In this paper and the companion paper [IO15], we establish two results towards affirming Conjecture 1. The main result of the present paper shows that given any smooth Euler flow, the law of energy conservation can be violated by arbitrarily small localized perturbations in $C_{t,x}^{1/5-\epsilon}$, and the regularity of the perturbed flow is no better than the perturbation.

Theorem 1.1 (Perturbation of smooth Euler flows). *Let $(v_{(0)}, p_{(0)})$ be any smooth solution to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$. Then for any $\epsilon, \delta > 0$ and pre-compact open sets $\Omega_{(0)}$, $\mathcal{U}$ such that $\Omega_{(0)} \neq \emptyset$ and $\overline{\Omega_{(0)}} \subseteq \mathcal{U}$, there exists a weak solution $(v, p) \in C_{t,x}^{1/5-\epsilon} \times C_{t,x}^{2(1/5-\epsilon)}$ to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$ such that the following statements hold:*

1. *The solutions (v, p) and $(v_{(0)}, p_{(0)})$ coincide outside $\mathcal{U}$, i.e.,*

$$(v, p) = (v_{(0)}, p_{(0)}) \text{ on } (\mathbb{R} \times \mathbb{R}^3) \setminus \mathcal{U}. \tag{2}$$

2. *The solutions (v, p) and $(v_{(0)}, p_{(0)})$ differ at most by δ in the $C_{t,x}^{1/5-\epsilon} \times C_{t,x}^{2(1/5-\epsilon)}$ topology, i.e.,*

$$\|v - v_{(0)}\|_{C_{t,x}^{1/5-\epsilon}} + \|p - p_{(0)}\|_{C_{t,x}^{2(1/5-\epsilon)}} < \delta. \tag{3}$$

3. *For every $t \in \mathbb{R}$ and open set $\Omega' \subseteq \mathbb{R}^3$ such that $\{t\} \times \Omega' \subseteq \Omega_{(0)}$, the solution $v(t, x)$ fails to be in the class $v(t, \cdot) \notin C^{1/5}(\Omega')$, and furthermore fails to belong to the Sobolev space $v(t, \cdot) \notin W^{1/5,1}(\Omega')$. As a consequence, v does not coincide with $v_{(0)}$ on any open subset of $\Omega_{(0)}$.*

4. *There exists $t_\star \in \mathbb{R}$ and a smooth, non-negative function $\psi = \psi(x) \geq 0$ with compact support such that*

$$\{x \mid (t_\star, x) \in \mathcal{U}\} \subseteq \{x \mid \psi(x) = 1\}$$

and we have

$$\int_{\mathbb{R}^3} \psi(x) \frac{|v(t_\star, x)|^2}{2} \, dx > \int_{\mathbb{R}^3} \psi(x) \frac{|v_{(0)}(t_\star, x)|^2}{2} \, dx. \tag{4}$$

In particular, the solution v fails to conserve energy if its energy is finite.

We note in passing that Theorem 1.1 provides the first construction of finite energy, continuous solutions failing conserve energy that take place outside the setting of periodic tori. In particular, we obtain failure of energy conservation for $C_{t,x}^{1/5-\epsilon}$ solutions on any bounded domain, and the existence of compactly supported solutions on $\mathbb{R} \times \mathbb{R}^3$ by taking $(v_{(0)}, p_{(0)}) \equiv 0$ and $\Omega_{(0)}$ to be a non-empty pre-compact open subset of a suitable domain $\mathcal{U}$.

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As in previous constructions, the range $\alpha \geq 1/5$ is out of reach of our method. On the periodic torus, the construction of $(1/5 - \varepsilon)$-Hölder solutions that fail to conserve energy was first achieved in [Ise12] improving on initial constructions of $(1/10 - \varepsilon)$-Hölder solutions in [DLS13, DLS12a] (see also [BDLS13, BDLIS14] for a shorter proof closer to the scheme of [DLS13, DLS12a]). We also note the construction of solutions with compact time support in the class $C_{t,x}^0 \cap L_t^1 C_x^{1/3-\varepsilon}$ by [Buc15, BDLS14].

Remark. Theorem 1.1 holds as well for background solutions $(v_{(0)}, p_{(0)})$ which are defined only on some open set $\mathcal{O}$ which contains $\overline{\mathcal{U}}$. Indeed, all our arguments go through essentially verbatim, as all of our techniques are localized. Moreover, in terms of the Cauchy problem, Theorem 1.1 demonstrates that uniqueness and conservation of energy fail for all smooth initial data in the energy class within the class of weak solutions constructed in the Theorem.

In the companion paper [IO15], we prove the existence of solutions to Euler with energy profiles approaching the minimal regularity $2\alpha/(1-\alpha)$ for $0 < \alpha < 1/5$, thus confirming that the $2\alpha/(1-\alpha)$-Hölder estimate (1) is sharp in this range. This result supports the intuition underlying Conjecture 1, as we show moreover that irregularity of the energy profile may arise from a compactly supported perturbation of the 0 solution in $C_{t,x}^\alpha$.

Theorem 1.2 (Euler flows with prescribed energy profile [IO15]). *Let $\alpha < 1/5$, let $I \subseteq \mathbb{R}$ be a bounded open interval, and let $\bar{e}(t) \geq 0$ be any non-negative function with compact support in I which belongs to the class $\bar{e}(t) \in C_t^\gamma$ for some $\gamma > \frac{2\alpha}{1-\alpha}$. Then:*

1. *There exists a weak solution (v, p) to the incompressible Euler equations in the class $v \in C_{t,x}^\alpha(\mathbb{R} \times \mathbb{T}^3)$ with support contained in*

$$\operatorname{supp} v \cup \operatorname{supp} p \subseteq I \times \mathbb{T}^3$$

   such that the energy profile of v is equal to $\int_{\mathbb{T}^3} |v|^2(t, x) dx = \bar{e}(t)$ for all $t \in \mathbb{R}$.

2. *Moreover, one may choose a one parameter family of solutions (v_A, p_A), $0 \leq A \leq 1$, with the above properties such that the energy profile of v_A is equal to $\int_{\mathbb{T}^3} |v_A|^2(t, x) dx = A\bar{e}(t)$ and such that $\|v_A\|_{C_{t,x}^\alpha} \to 0$ as $A \to 0$.*

Theorem 1.2 builds upon work of [DLS13, DLS12a, BDLS13] for prescribing smooth energy profiles the periodic setting and on the organizational framework developed in [Ise12]. We remark that our arguments also allow one to achieve an energy profile that does not have compact support provided the norm $\|e\|_{C_t^\gamma} = \sup_t |e(t)| + \sup_t \sup_{|\Delta t| \neq 0} \frac{|e(t+\Delta t) - e(t)|}{|\Delta t|^\gamma}$ is finite.

We view our proofs of Theorems 1.1 and 1.2 as first steps towards establishing Conjecture 1 in the range of exponents $\alpha < 1/5$. Namely, in the greater scheme of proving Conjecture 1, one could proceed by showing that the set of exceptions to the Conjecture is contained in a countable union of closed subsets of $C_t C_x^\alpha$ having empty interior. Verifying the empty interior condition amounts to proving a perturbation result, which would roughly amount to showing that an arbitrary solution with $v \in C_t C_x^\alpha$ can be perturbed in $C_t C_x^\alpha$ to obtain a solution $\tilde{v} \in C_t C_x^\alpha$ whose energy profile fails to belong to $W^{2\alpha/(1-\alpha)+\epsilon, 1}(I)$ on every open interval $I \subseteq \mathbb{R}$. The second statement of Theorem 1.2 shows that the trivial solution $v = 0$ can be perturbed in $C_t C_x^\alpha$ to achieve any given energy profile $\bar{e}(t)$ which is small in $C^{2\alpha/(1-\alpha)+\epsilon/2}$. Our proof of Theorem 1.1 suggests that a similar perturbation should be possible with the 0 solution replaced by an arbitrary smooth background flow.

An important goal of our work is to emphasize the perspective that Onsager's conjecture is inherently a local problem, where the main issue at hand concerns high frequency oscillations in the velocity field at small spatial scales. Other results that help draw attention to this point of view are the works of [DR00, DLS10, IO13]. This local perspective on the problem is emphasized by the local character of Theorems 1.1 and 1.2, and by the improvements in our construction that allow us to achieve this localization.

In considering the problem of constructing nonperiodic solutions, we are confronted with new issues that are closely connected to the conservation of angular momentum and did not arise in the previous work in the periodic setting. That is, in the setting of the whole space every weak solution to the Euler equations with finite energy and appropriate integrability conserves both linear and angular momentum, and these conservation laws pose further restrictions on the construction of weak solutions that were not present in the periodic setting. Thus, even if one is only interested in constructing solutions with finite energy without requiring the additional property of compact support, there is essentially no way to avoid considerations regarding the conservation of angular momentum. The main difficulty we face in this regard involves the construction of symmetric tensors with a prescribed divergence $\partial_j R^{jl} = U^l$ and good decay. See Sections 1.1.2 and 10 below for further discussion. At the same time, our method of constructing compactly supported solutions by localizing the construction also appears to be the most straightforward approach to obtaining finite energy, continuous solutions on the whole space or on a bounded domain.

In connection with the conservation of angular momentum, we observe that our methods yield a result of h-principle type that is of independent interest. The result we obtain (Theorem A.1 below) states that any smooth incompressible velocity field with compact support that satisfies the conservation of linear and angular momentum can be realized as a limit in $L^\infty_{t,x}$ weak-$*$ of some sequence of compactly supported $C^{1/5-\epsilon}_{t,x}$ Euler flows. This theorem contributes to the growing literature on h-principle type results in fluid equations [DLS12b, Cho13, CS14, IV14]. See Appendix A below for further discussion.

The proof of Theorem 1.1 (as well as Theorem 1.2 in [IO15]) is simplified substantially by the fact that we are able to obtain an exponential growth of frequencies in the iteration, and to truncate a parametrix expansion in the argument after a bounded number of steps. These simplifications are achieved through the use of spatially localized waves, through a family of operators designed to solve the symmetric divergence equation (see Sections 1.1.2 and 1.1.4 below), and through the use of sharp estimates for the regularized velocity field, phase functions and stress that were developed in the work of [Ise12] using an accelerated mollification technique. The same novelties in the proof lead to other features in the construction that are desirable from a physical point of view, including a compatibility with the scaling and Galilean symmetries of the equations, and a self-similarity of the construction. We discuss these further in Sections 1.1.3 and 1.1.4 below. Our proof also features a simple proof of a key property of the mollification along the flow technique introduced in [Ise12], which is included in Section 11.1.

Our overall construction is based on the method of convex integration that has been used to construct Hölder continuous Euler flows in the periodic setting [DLS13, DLS12a, Ise12, BDLIS14]. In particular, we follow rather closely the notation and framework developed in the first author's earlier paper [Ise12]. However, the present construction also involves several modifications compared to [Ise12] that specifically address the issue of angular momentum conservation, and which are used to localize the construction. We have therefore made an effort to give a summary of the new construction that is mostly self-contained, referring to [Ise12] only for some basic results and estimates.

1.1 Main Ideas in the Construction

The main new ideas in our construction revolve around the issue of angular momentum conservation and the related problem of localizing the construction to obtain compactly supported solutions. The new ideas we employ result in some new features for the construction that are desirable from a physical point of view, including compatibility with the symmetries of the equations and the exponential growth

of frequencies.

1.1.1 Euler-Reynolds flows and conservation of momentum

The Hölder continuous weak solution to the incompressible Euler equations (E) in Theorem 1.1 is constructed by an iteration scheme, where each step consists of adding a correction to an approximate solution to improve the error while maintaining the desired properties. Beginning with the work of [DLS13], the space of approximate solutions used to build continuous solutions to (E) consists of the solutions to the following underdetermined system known as the "Euler-Reynolds equations":

$$\partial_t v^l + \partial_j(v^j v^l) + \partial^l p = \partial_j R^{jl}$$
$$\partial_j v^j = 0$$

(5)

Here, R^{jl} is a symmetric tensor called the Reynolds stress whose trace-free part[2] measures the error by which (v, p) fail to solve the Euler equations. Solutions to (5) are called Euler-Reynolds flows. A well-known and important property of the equation (5) is that it contains weak limits of solutions to the Euler equations. Namely, the divergence free property of v remains true after taking weak limits, and a weak limit of tensors $v^j v^l$ must be symmetric, even though it may fail to be rank 1.

Under appropriate decay assumptions, the space of Euler-Reynolds flows on $\mathbb{R} \times \mathbb{R}^3$ can also be viewed as the space of incompressible velocity fields which conserve both linear and angular momentum. Namely, the usual laws of conservation of linear and angular momentum

$$\int v^l \, dx = const, \quad \int x^k v^l - x^l v^k \, dx = const$$

(6)

can also be proven for Euler-Reynolds flows under the assumption that $R^{jl} \in L^1_{t,x}$ (which is exactly the integrability one obtains if R^{jl} is obtained from weak limits of Euler flows with uniform bounds on $\|v\|_{L^2_{t,x}}$). Conversely, if v^l is divergence-free and conserves both linear and angular momentum, then (formally) one can represent v as an Euler-Reynolds flow with $p = 0$ by solving the following underdetermined elliptic equation, which we call the *symmetric divergence equation*:

$$\partial_j R^{jl} = U^l$$

(7)

for $U^l = \partial_t v^l + \partial_j(v^j v^l)$. The conservation of linear and angular momentum for v^l ensures that the implied force $U^l = \partial_t v^l + \partial_j(v^j v^l)$ is orthogonal to every element K_l to the kernel of the Killing operator $\partial_j K_l + \partial_l K_j$, which is (up to a sign) the adjoint to the symmetric divergence operator $\partial_j R^{jl}$ on symmetric tensors (with appropriate decay). Formally, this property ensures that U lies in the image of the symmetric divergence operator.

It is a basic principle of convex integration that approximate solutions used in the construction turn out to be weak limits of solutions to the partial differential equation (or inclusion) that is being solved (see, for example, Appendix A below, or [DLS12b, Cho13, Ise12]). As a consequence, we are forced in our construction to work with approximate solutions that likewise satisfy the laws of conservation of linear and angular momentum (6), which are linear and thus survive under weak limits. In particular, our corrections must maintain the conservation of angular momentum (in addition to the divergence-free property and the conservation of linear momentum), which is a new feature compared to the construction on the periodic torus.

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[2]The convention initiated in [DLS13] is slightly different in that the Reynolds stress is represented as $\mathring{R}^{jl}$ and there is an additional requirement that $\mathring{R}^{jl}$ has vanishing trace. Although we will not use this convention, one obtains an equivalent definition of Euler-Reynolds flows since the trace part can be absorbed into the pressure gradient $\partial^l p = \partial_j(p\delta^{jl})$.

1.1.2 Localized solution to the symmetric divergence equation

The main innovation in our construction is a new method for solving equation (7), which enables us to control the support of the solution R^{jl} and to obtain C^0 estimates for R^{jl} and its derivatives compatible with dimensional analysis and with the transport structure of the problem. Our solutions are given by explicit linear operators applied to the data U^l, which retain the property of compact support when the data U^l is compactly supported, and produce solutions to (7) whenever U^l is L^2-orthogonal to constant and rotational vector fields ∂_l and $x^k \partial_l - x^l \partial_k$. As discussed earlier, these vector fields span the kernel of the L^2-adjoint of the symmetric divergence operator. Given a correction to the velocity and pressure fields that is spatially localized and preserves both linear and angular momentum, this method allows us to construct a new Euler-Reynolds stress that is similarly localized in space. This idea is key to our localized convex integration scheme, as we will explain below in Section 1.1.3.

The starting point behind the construction of solution operators to (7) can be illustrated in the context of the strictly simpler problem of finding compactly supported solutions to the divergence equation

$$\partial_l R^l = U \tag{8}$$

where U is a scalar function and R^l is an unknown vector field on $\mathbb{R}^n$. This equation arises often in hydrodynamics, as well as in the foundations of the differential forms approach to degree theory [Tay11, Section 1.19].

Assume now that the scalar field U in (8) has compact support, and satisfies $\int_{\mathbb{R}^n} U \, dx = 0$. These conditions are clearly necessary for a compactly supported solution to (8) to exist. Our starting point for solving (8) is that, when these necessary conditions are satisfied, one can obtain a solution to (8) by Taylor expanding in frequency space

$$\widehat{U}(\xi) = \widehat{U}(0) + \sum_{i=1}^{n} \xi_l \int_0^1 \partial^l \hat{U}(\sigma\xi) d\sigma = \sum_{i=1}^{n} i\xi_l \widehat{R^l}(\xi) \tag{9}$$

$$\widehat{R^l}(\xi) = \frac{1}{i} \int_0^1 \partial^l \hat{U}(\sigma\xi) d\sigma \tag{10}$$

From the physical space expression of (10) one can see that the vector field $\widehat{R}^l$ defined in (10) actually has compact support in a ball of radius ρ about 0 whenever U is supported in the ball of radius ρ. One can see also see from (9) that the vector field R^l defined by (10) solves (8) whenever $\int U dx = \widehat{U}(0) = 0$.

We now view Formula (10) as the frequency space representation of a linear operator applied to the scalar function U. The problem with this operator is that the resulting solution R^l apparently has a singularity at the origin in physical space. Our cure for this problem is to "spread out" the singularity by taking advantage of the translation invariance of Equation (8). Namely, one can construct new solutions to (8) by conjugating the operator defined by (10) with a translation operator, thereby translating the singularity. By taking a smooth average of such conjugates we obtain an operator which is explicit and does not have a singularity while also maintaining control over the support of the solution. The operator obtained in this way turns out to coincide with a known formula introduced by Bogovskii[3] in [Bog80] for solving Equation (8). The novelty here is that we obtain a conceptual derivation of this formula that generalizes to solving the symmetric divergence equation.

In our context, it is also important that the solution $R^{jl}(t, x)$ moves with time along the ambient coarse scale flow of the construction when the data $U^l(t, x)$ travels in the same way. In other words, our operators should commute well with the advective derivative along the coarse scale flow. We achieve good transport properties for our solution operators by taking advantage of the freedom to conjugate

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[3]The authors thank Hao Jia and Peter Constantin for bringing our attention to Bogovskii's formula.

with any smooth family of translations we desire when defining the solution operator at each time slice. By averaging with respect to a family of translations which moves along the ambient coarse scale flow, we are able to achieve solution operators with good commutator properties with respect to the advective derivative. We refer to Section 10 for the full details of the solution to (7).

1.1.3 Localization of the construction

Our construction relies on the use of localized waves that are supported on small length scales which vary inversely with the frequency of the iteration. In contrast, the constructions in the periodic setting use waves supported on length scales of order ≈ 1 independent of the ambient frequency. Thus, in our construction, the number of waves occupying each time slice is very large at high frequencies. The corrections in the construction are modified so that they maintain the balance of angular momentum as well as the divergence-free property.

Due to the use of localized waves and a rearrangement of the error terms in the construction, we always solve (7) with data which satisfies the necessary orthogonality conditions while simultaneously remaining localized to a small length scale ρ. This smallness of support leads to a gain of a factor ρ for the solution to (7), which is an estimate one expects from dimensional analysis

$$\|R\|_{C^0} \lesssim \rho \|U\|_{C^0}.$$

The gain of this smallness parameter ρ allows us to achieve for the first time exponential (rather than double-exponential) growth of frequencies during the iteration. Eliminating the need for double-exponential growth of frequencies in the iteration leads to some technical simplifications in the proof, and also leads to solutions that appear more natural from a physical point of view.

In contrast to the periodic case, where the increment to the energy in each stage of the iteration is a prescribed function of time $e(t)$, we prescribe a local energy increment $e(t, x)$ that is a function of both space and time, allowing for the possibility of compact support in time and space. In order to ensure that our increments satisfy the required bounds on both spatial and advective derivatives, we apply the machinery of mollifying along the flow introduced in [Ise12]. For the Main Lemma of the iteration, we also prove an estimate on the local energy increment; see (25). This estimate applied in our paper to prove the nontriviality of our solutions. We hope that estimates of this type may also be useful in future applications, such as the study of admissibility criteria for the Euler equations as initiated in [DLS10].

Using our bounds on the local energy increments and the other natural estimates of the construction, we prove that the solutions obtained from the iteration fail to belong to $v(t, \cdot) \notin C^{1/5+\delta}$ on essentially every open ball contained in their support. While this lack of regularity is a new result concerning the solutions produced by the iteration, we emphasize that this property actually follows from the construction without any modifications. For instance, the same argument shows that the solutions of [Ise12, BDLS13], which belong to $C_t C_x^{\alpha^*-\epsilon}$ for some $\alpha^* < 1/5$ and all $\epsilon > 0$, actually fail to belong to $v(t, \cdot) \notin C^{\alpha^*+\delta}$ for any $\delta > 0$ on basically their whole support. Our solutions, whose frequencies grow exponentially, necessarily fail to belong to $v(t, \cdot) \notin C^{1/5+\delta}$, and we show that they may also fail to be in $v(t, \cdot) \notin C^{1/5}$ by taking an appropriate choice of frequencies in the iteration.

Our result on the failure of higher regularity confirms that the estimates applied in [Ise12] are sharp, and that any improvement in regularity for the solutions requires modifications in the construction. For example, we see that the solutions of [Buc15], which possess regularity $v(t, \cdot) \in C^{1/3-\epsilon}$ for a.e. t, must be obtained through a nontrivial modification of the construction of [BDLS13]. The proof of this lack of regularity also suggests that such results may be more difficult to obtain without losing control over the energy profile, as we show that failure of spatial regularity follows from the same family of estimates that are used to control the energy profile.

Our techniques for localizing the construction and addressing the issue of angular momentum conservation lead to a framework that accords well with the symmetries of the Euler equations. Thanks

to the combination of our use of localized waves and our new method for solving the divergence equation (7), we have obtained an iteration framework whose bounds are all dimensionally correct with constants that are universal. We have also maintained the property that the estimates of the iteration depend only on estimates for relative velocities as opposed to absolute velocities (for example, $\|\nabla v\|_{C^0}$ as opposed to $\|v\|_{C^0}$). This property is natural in view of the Galilean symmetries of the Euler equations, and the fact that Onsager's conjecture itself is Galilean invariant.

1.1.4   Comparison with ideas in turbulence

Previous constructions of Euler flows by convex integration have led to many features that are regarded as unphysical and sharply contrast the well known description of turbulence in the physics literature. One of the most glaringly unphysical features of previous solutions obtained through convex integration has been the requirement of double-exponentially growing frequencies in the iteration, which result in large gaps in the energy spectrum of the solutions. In contrast, turbulent flows are well-known to exhibit a power law in their energy spectrum, which was first predicted by the foundational theory of Kolmogorov. Another strange feature common to previous constructions of Hölder continuous solutions is the use of waves occupying length scales of size ≈ 1 independent of their frequency. Turbulent flows, on the other hand, have since the seminal ideas of Richardson been described to first approximation as being composed of a self-similar hierarchy of eddies occupying smaller and smaller length scales[4].

The solutions constructed in the present work turn out to have a closer resemblance to the above physical descriptions of turbulent flows. We use waves that are supported on small length scales which allow us to achieve an exponential growth of frequencies for the iteration while obtaining a purely local framework for the construction. This use of localized waves seems to be essentially forced on us by the problem. The resulting solutions exhibit a self-similar structure similar to what was once imagined to be characteristic of turbulence. From the point of view of comparing to turbulence, the most significant, unphysical characteristic that remains for our solutions is the failure to reach the regularity $1/3$ conjectured by Onsager, which is predicted by Kolmogorov's theory and agrees with certain experimental measurements of turbulent flows. In this regard, it is important to remark that the new methods introduced here do not introduce any error terms which would prevent improvements in the Hölder regularity (see Remark 7.5).

A closely related, unphysical property of our solutions is the presence of anomalous time scales in the construction that are inconsistent with the Hölder regularity $1/3$ and also differ from the natural time scales of turbulent flows. The scale of time cutoffs for waves at space scale λ^{-1} in our construction is $\lambda^{-4/5}$, which is far shorter than the scale $\lambda^{-2/3}$ obtained from turbulence theory for the turnover time of eddies at scale λ^{-1}. In view of the time regularity estimate (proved in [Ise13])

$$\|(\partial_t + P_{\leq q} v \cdot \nabla) P_{q+1} v\|_{C^0(I \times \mathbb{R}^n)} \leq C 2^{(1-2\alpha)q} \|v\|_{C_t \dot{C}_x^\alpha (I \times \mathbb{R}^n)}^2,$$

which is saturated in our construction for $\alpha = 1/5$, this feature limits the regularity of the construction to $C_t C_x^{1/5}$. We refer to [Ise13, Section 9] for further discussion.

2   Organization of the Paper

We organize the remainder of the paper as follows. The main body of the paper begins with Section 3, where we state the Main Lemma of the paper, which provides an essentially complete statement of the result of a single iteration of the construction.

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[4]It is important to note, however, that self-similarity in turbulence is known to fail. The phenomenon of intermittency which describes this failure of self-similarity is an active area of research; see, for instance [CS12] for a recent mathematical approach expanding the model of [FS78] and for further references.

After some preliminaries on the geometry of flow maps in Section 4, the construction itself begins in Section 5, which provides a high level summary of the scheme and derives a list of the error terms in the construction. Section 6 finishes the description of the construction up to the choice of some length and time scale parameters that are determined in Section 7. Sections 8 - 9 are devoted to estimating the elements of the construction and the resulting error terms. In Section 10, we derive and prove estimates for the new operators which are applied in Section 9.1 of the construction to solve the symmetric divergence equation (7). In Section 11, we show how that the Main Lemma of the paper implies Theorem 1.1 on the perturbation of smooth Euler flows.

The paper concludes in Appendix A, where we indicate how the Main Lemma of the paper can be used in combination with the operators of Section 10 to yield a sharp h-Principle type result for incompressible Euler on $\mathbb{R} \times \mathbb{R}^3$.

3 The Main Lemma

In this section, we present the Main Lemma which is responsible for the proof of Theorem 1.1. The purpose of this lemma is to describe precisely the result of one step of the convex integration procedure. Theorem 1.1 follows from iteration of this Lemma as we will explain in Section 11.

To state the Main Lemma, we recall the notion of frequency and energy levels for Euler-Reynolds flows introduced in Sections 9 and 10 of [Ise12].

Definition 3.1. Let $L \geq 1$ be a fixed integer. Let $\Xi \geq 2$, and let e_v and e_R be positive numbers with $e_R \leq e_v$. Let (v, p, R) be a solution to the Euler-Reynolds system. We say that the frequency and energy levels of (v, p, R) are below (Ξ, e_v, e_R) (to order L in $C^0 = C^0_{t,x}(\mathbb{R} \times \mathbb{R}^3)$) if the following estimates hold.

$$\|\nabla^k v\|_{C^0} \leq \Xi^k e_v^{1/2} \qquad\qquad k = 1, \ldots, L \qquad (11)$$

$$\|\nabla^k p\|_{C^0} \leq \Xi^k e_v \qquad\qquad k = 1, \ldots, L \qquad (12)$$

$$\|\nabla^k R\|_{C^0} \leq \Xi^k e_R \qquad\qquad k = 0, \ldots, L \qquad (13)$$

$$\|\nabla^k (\partial_t + v \cdot \nabla) R\|_{C^0} \leq \Xi^{k+1} e_v^{1/2} e_R \qquad\qquad k = 0, \ldots, L-1 \qquad (14)$$

Here ∇ refers only to derivatives in the spatial variables.

It is important to note that the bounds in Definition 3.1 are consistent with the dimensional analysis of the Euler equations: namely, the frequency level $\Xi \sim [L]^{-1}$ is an inverse length and the energy levels e_v and e_R have the dimensions of an energy density $e_v, e_R \sim \frac{[L]^2}{[T]^2}$. We refer to Sections 9 and 10 of [Ise12] for the motivation for this definition and further discussion.

Our Main Lemma is based on the Main Lemma in Section 10 of [Ise12] but also keeps track of how the support of the approximate solution enlarges after the addition of a correction. The way the support enlarges is governed by the geometry of the flow map of the velocity field v, so the following definition will be useful for keeping track of this support.

Definition 3.2 (v-adapted Eulerian cylinder). Let $\Phi_s = \Phi_s(t, x)$ be the flow map associated to a vector field v. Given $\tau, \rho > 0$ and a point (t_0, x_0) of the space-time $\mathbb{R} \times \mathbb{R}^3$, we define the v-adapted Eulerian cylinder $\hat{C}_v(\tau, \rho; t_0, x_0)$ centered at (t_0, x_0) with duration 2τ and base radius $\rho > 0$ to be

$$\hat{C}_v(\tau, \rho; t_0, x_0) := \left\{ \Phi_s(t_0, x_0) + (0, h) \ : \ |s| \leq \tau, |h| \leq \rho \right\} \qquad (15)$$

In other words, $\hat{C}_v(\tau, \rho; t_0, x_0)$ is the union of spatial balls of radius ρ about the trajectory of (t_0, x_0) along the flow of v for $t \in [t_0 - \tau, t_0 + \tau]$.

Similarly, if $S \subseteq \mathbb{R} \times \mathbb{R}^3$ is a set, we define

$$\hat{C}_v(\tau, \rho; S) := \bigcup_{(t_0, x_0) \in S} \hat{C}_v(\tau, \rho; t_0, x_0) \tag{16}$$

With these definitions in hand, we can state the Main Lemma.

Lemma 3.1 (The Main Lemma). *Suppose that $L \geq 2$. Let K be the constant in Section 7.3 of [Ise12], and let $M \geq 1$ be a constant. There exist constants $C_0, C > 1$, which depend only on M and L, such that that following holds:*

Let (v, p, R) be any solution of the Euler-Reynolds system whose frequency and energy levels are below (Ξ, e_v, e_R) to order L in C^0.

Define the time-scale $\theta = \Xi^{-1} e_v^{-1/2}$, and let $e(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ be any non-negative function which satisfies the lower bound

$$e(t, x) \geq K e_R \qquad \text{for all } (t, x) \in \hat{C}_v(\theta, \Xi^{-1}; \operatorname{supp} R) \tag{17}$$

(using the notation of Definition 3.2) and whose square root satisfies the estimates

$$||\nabla^k (\partial_t + v \cdot \nabla)^r e^{1/2}||_{C^0} \leq M \Xi^k (\Xi e_v^{1/2})^r e_R^{1/2} \qquad 0 \leq r \leq 1, 0 \leq k + r \leq L \tag{18}$$

Now let N be any positive number obeying the bound

$$N \geq \left(\frac{e_v}{e_R} \right)^{3/2} \tag{19}$$

and define the dimensionless parameter $\mathbf{b} = \left(\dfrac{e_v^{1/2}}{e_R^{1/2} N} \right)^{1/2}$.

Then there exists a solution (v_1, p_1, R_1) of the Euler-Reynolds system of the form $v_1 = v + V$, $p_1 = p + P$ whose frequency and energy levels are below

$$(\Xi', e_v', e_R') = (C_0 N \Xi, e_R, \left(\frac{e_v^{1/2}}{e_R^{1/2} N} \right)^{1/2} e_R) \tag{20}$$

$$= (C_0 N \Xi, e_R, \mathbf{b}^{-1} \frac{e_v^{1/2} e_R^{1/2}}{N}) \tag{21}$$

to order L in C^0, and whose stress R_1 is supported in

$$\operatorname{supp} R_1 \subseteq \hat{C}_v(\theta, \Xi^{-1}; \operatorname{supp} e) \tag{22}$$

The correction $V = v_1 - v$ is of the form $V = \nabla \times W$ and can be guaranteed to obey the bounds

$$||V||_{C^0} + (N \Xi)^{-1} ||\nabla V||_{C^0} + (\mathbf{b}^{-1} \Xi e_v^{1/2})^{-1} ||(\partial_t + v^j \partial_j) V||_{C^0} \leq C e_R^{1/2} \tag{23}$$

$$||W||_{C^0} + (N \Xi)^{-1} ||\nabla W||_{C^0} + (\mathbf{b}^{-1} \Xi e_v^{1/2})^{-1} ||(\partial_t + v^j \partial_j) W||_{C^0} \leq C (N \Xi)^{-1} e_R^{1/2} \tag{24}$$

The energy of the correction can be prescribed locally up to errors bounded uniformly in t by

$$\left| \int_{\mathbb{R}^3} |V|^2(t, x) \psi(x) dx - \int_{\mathbb{R}^3} e(t, x) \psi(x) dx \right| \leq C \frac{e_v^{1/2} e_R^{1/2}}{N} \left(||\psi||_{L^1} + \Xi^{-1} ||\nabla \psi||_{L^1} \right) \tag{25}$$

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for any smooth test function $\psi(x) \in C_c^\infty(\mathbb{R}^3)$, where $L^1 = L^1(\mathbb{R}^3)$. The correction to the pressure $P = p_1 - p_0$ satisfies the estimate

$$||P||_{C^0} + (N\Xi)^{-1}||\nabla P||_{C^0} + (\mathbf{b}^{-1}\Xi e_v^{1/2})^{-1}||(\partial_t + v \cdot \nabla)P||_{C^0} \leq Ce_R \tag{26}$$

Finally, the space-time supports of V and P are also contained in

$$\operatorname{supp} V \cup \operatorname{supp} P \subseteq \hat{C}_v(\theta, \Xi^{-1}; \operatorname{supp} e) \tag{27}$$

Lemma 3.1 is very similar to the Main Lemma in [Ise12], but there are a few differences which are important to observe.

Unlike the Main Lemma in [Ise12], Lemma 3.1 is entirely consistent with dimensional analysis and does not impose a restriction on N that would force super-exponential growth of frequencies. Namely, the Main Lemma of [Ise12] imposes an additional condition $N \geq \Xi^\eta$ for some $\eta > 0$, and this condition on the frequency growth parameter forces a double-exponential growth of frequencies when the lemma is iterated to construct solutions to Euler. The condition $N \geq \Xi^\eta$ is also unfavorable for being inconsistent with dimensional analysis, as the parameter Ξ has the dimensions of an inverse length, whereas the parameter N is supposed to be dimensionless. By excluding the requirement $N \geq \Xi^\eta$, Lemma 3.1 is now completely consistent with dimensional analysis, and hence agrees with the scaling symmetries of the Euler-Reynolds equations. Furthermore, Lemma 3.1 allows for an exponential (rather than double-exponential) growth of frequencies in the iteration, which gives our solutions a closer resemblance to the classical picture of turbulent flows.

Another feature of Lemma 3.1 contrasting the Main Lemma of [Ise12] is that Lemma 3.1 keeps track of the enlargement of support of R in terms of the v-compatible Eulerian cylinders in Definition 3.2. Also, the function $e(t, x)$ which determines the increment of energy to the system is a function of both time and space rather than simply a function of time $e(t)$ as in [Ise12]. Thus, the required estimates (18) for $e(t, x)$ are stated in terms of both advective and spatial derivatives, and the lower bound (17) is stated in terms of the v-compatible cylinders. In order to apply Lemma 3.1 to construct solutions, we will have to show that there exist energy profiles which satisfy the necessary conditions (17) and (18) for any given values of Ξ and e_v. We have included an additional parameter M in (18) to ensure that such functions can be constructed.

The estimate (25) for the energy increment also differ from those of [Ise12] for the increment to the total energy. Here our energy increment estimates are localized as they are stated in terms of a test function ψ. This type of estimate allows us to establish local properties of the resulting solutions, including the failure of local $C^{1/5}$ regularity stated in Theorem 1.1. The estimate(25) is also more natural in terms of dimensional analysis. From this point of view, the factor of $(||\psi||_{L^1} + \Xi^{-1}||\nabla\psi||_{L^1})$ has the dimensions of volume when we regard ψ as being dimensionless. The units of volume have been normalized to agree with units of mass $[M]$ in the physical derivation of the Euler equations. Therefore, both sides of (25) have the dimensions of energy $\frac{[M][L]^2}{[T]^2}$.

Finally, we point out that, as in [Ise12], the enlargement of support is expressed in terms of the support of R, rather than the support of v and p. Thus, if the Euler-Reynolds flow (v, p, R) satisfies the Euler equations except on a compact subset $\overline{\Omega} \subseteq \mathbb{R} \times \mathbb{R}^3$ on which R is supported, the corrections to the pressure and velocity and the resulting error R_1 can be made to have compact support in a neighborhood of $\overline{\Omega}$ by the appropriate choice of $e(t, x)$, even if the ambient velocity and pressure (v, p) do not have compact support.

Now we begin the proof of Lemma 3.1. We start in Section 4 with some preliminary lemmas concerning the geometry of the Eulerian cylinders of Definition 3.2 that will play an important role in the proof. We then give a technical outline of the scheme in Section 5 wherein we organize a list of the error terms in the construction. We continue the proof of Lemma 3.1 through Section 10.

4 Preliminaries on Eulerian and Lagrangian Cylinders

Here we collect some basic facts about the geometry of Eulerian cylinders which will be useful during the construction. We will assume throughout this section that we are working with time-dependent vector fields $v(t, x) = (v^1, v^2, v^3)$ defined on $\mathbb{R} \times \mathbb{R}^3$ which are continuous in (t, x) and C^1 in the spatial variables with uniform bounds on $\|\nabla v\|_{C^0}$. We denote by Φ_s the flow map associated to v, which is the one-parameter group of mappings $\Phi_s : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ generated by the space-time vector field $\partial_t + v \cdot \nabla$. If v is defined only on an open subset of $\mathbb{R} \times \mathbb{R}^3$, then likewise $\Phi_s(t, x)$ is defined only on an open subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$.

In addition to Eulerian cylinders, we will also be interested in the concept of a Lagrangian cylinder adapted to a vector field v, which we define as follows.

Definition 4.1 (v-adapted Lagrangian cylinder). Let $v = (v^1, v^2, v^3)$ be as above. Let $\Phi_s = \Phi_s(t, x)$ be the flow map associated to a vector field v. Given $\tau, \rho > 0$ and a point (t_0, x_0) of the space-time $\mathbb{R} \times \mathbb{R}^3$, we define the v-*adapted Lagrangian cylinder* $\hat{\Gamma}_v(\tau, \rho; t_0, x_0)$ centered at (t_0, x_0) with duration 2τ and base radius $\rho > 0$ to be

$$\hat{\Gamma}_v(\tau, \rho; t_0, x_0) := \left\{ \Phi_s(t_0, x_0 + h) \ : \ |s| \leq \tau, |h| \leq \rho \right\} \tag{28}$$

In other words, $\hat{\Gamma}_v(\tau, \rho; t_0, x_0)$ is the union of trajectories for times $t \in [t_0 - \tau, t_0 + \tau]$ emanating from a spatial ball of radius ρ about x_0.

Similarly, if $S \subseteq \mathbb{R} \times \mathbb{R}^3$ is a set, we define

$$\hat{\Gamma}_v(\tau, \rho; S) := \bigcup_{(t_0, x_0) \in S} \hat{\Gamma}_v(\tau, \rho; t_0, x_0) \tag{29}$$

Throughout the proof, we will often make use of the following duality between Eulerian and Lagrangian cylinders:

$$(t', x') \in \hat{C}_v(\tau, \rho; t, x) \iff (t, x) \in \hat{\Gamma}_v(\tau, \rho; t', x') \tag{30}$$

Our first Lemma provides the most basic estimate on the geometry of the flow of v.

Lemma 4.1. Let $v = (v^1, v^2, v^3)$ be as above and let $\Phi_s(t, x) = (t + s, \Phi'_s(t, x))$ be the flow map associated to $\partial_t + v \cdot \nabla$. Then for every $(t_0, x_0), (s, h) \in \mathbb{R} \times \mathbb{R}^3$, we have

$$|h| e^{-s\|\nabla v\|_{C^0}} \leq |\Phi'_s(t_0, x_0) - \Phi'_s(t_0, x_0 + h)| \leq |h| e^{s\|\nabla v\|_{C^0}}. \tag{31}$$

Proof. Define $\mathbf{d}^2(s) := |\Phi'_s(t_0, x_0) - \Phi'_s(t_0, x_0 + h)|^2$. Recalling the definition of Φ'_s, we easily compute

$$\frac{d}{ds} \mathbf{d}^2(s) = 2 \Big(\Phi'_s(t_0, x_0) - \Phi'_s(t_0, x_0 + h) \Big) \cdot \Big(v(t_0 + s, \Phi'_s(t_0, x_0)) - v(t_0 + s, \Phi'_s(t_0, x_0)) \Big).$$

By the mean value theorem, we see that

$$-2\|\nabla v\|_{C^0} \, \mathbf{d}^2(s) \leq \frac{d}{ds} \mathbf{d}^2(s) \leq 2\|\nabla v\|_{C^0} \, \mathbf{d}^2(s).$$

Note that $\mathbf{d}^2(0) = |h|^2$. Thus, applying Gronwall on each side, we obtain

$$|h|^2 e^{-2s\|\nabla v\|_{C^0}} \leq \mathbf{d}^2(s) \leq |h|^2 e^{2s\|\nabla v\|_{C^0}}.$$

Taking the square root, we then arrive at the desired set of inequalities. □

A simple consequence of Lemma 4.1 is the equivalence of Eulerian and Lagrangian cylinders.

Lemma 4.2 (Equivalence of Eulerian and Lagrangian cylinders). *Let v be as above, let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$ and let $\tau, \rho > 0$. Then*

$$\hat{\Gamma}_v(\tau, e^{-\tau \|\nabla v\|_0} \rho; t_0, x_0) \subseteq \hat{C}_v(\tau, \rho; t_0, x_0) \subseteq \hat{\Gamma}_v(\tau, e^{\tau \|\nabla v\|_0} \rho; t_0, x_0) \tag{32}$$

Proof. Let $(t, x) \in \hat{\Gamma}_v(\tau, e^{-\tau \|\nabla v\|_0} \rho; t_0, x_0)$. Then there exist $s \in \mathbb{R}$, $|s| \le \tau$ and $h \in \mathbb{R}^3$ with $|h| \le e^{-\tau \|\nabla v\|_0} \rho$ such that

$$(t, x) = \Phi_s(t_0, x_0 + h) \tag{33}$$
$$= \Phi_s(t_0, x_0) + \tilde{h} \tag{34}$$
$$\tilde{h} = \Phi_s(t_0, x_0 + h) - \Phi_s(t_0, x_0) \tag{35}$$

From Lemma 4.1 we have $|\tilde{h}| \le e^{s \|\nabla v\|_0} |h| \le \rho$. This bound establishes the first containment in (32).

For the second containment, let $(t, x) \in \hat{C}_v(\tau, \rho; t_0, x_0)$. Then there exist $s \in \mathbb{R}$, $|s| \le \tau$ and $h \in \mathbb{R}^3$, $|h| \le \rho$ such that

$$(t, x) = \Phi_s(t_0, x_0) + (0, h) \tag{36}$$
$$= \Phi_s(t_0, x_0 + \tilde{h}) \tag{37}$$
$$(0, \tilde{h}) = \Phi_{-s}(\Phi_s(t_0, x_0) + (0, h)) - \Phi_{-s}(\Phi_s(t_0, x_0)) \tag{38}$$

From Lemma 4.1 we have $|\tilde{h}| \le e^{|s| \|\nabla v\|_0} |h| \le e^{\tau \|\nabla v\|_0} \rho$, which concludes the proof. $\square$

From Lemma 4.1 we can also quickly prove the following containment properties of cylinders

Lemma 4.3. *Let v be as above, let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$ and let $\tau_0, \tau_1, \rho_0, \rho_1$ be positive numbers. Then*

$$\hat{C}_v(\tau_2, \rho_2; \hat{C}_v(\tau_1, \rho_1; t_0, x_0)) \subseteq \hat{C}_v(\tau_1 + \tau_2, \rho_2 + e^{\|\nabla v\|_0 \tau_2} \rho_1; t_0, x_0) \tag{39}$$
$$\hat{C}_v(\tau_2, \rho_2; \hat{\Gamma}_v(\tau_1, \rho_1; t_0, x_0)) \subseteq \hat{C}_v(\tau_1 + \tau_2, \rho_2 + e^{\|\nabla v\|_0 (\tau_1 + \tau_2)} \rho_1; t_0, x_0) \tag{40}$$
$$\hat{\Gamma}_v(\tau_2, \rho_2; \hat{C}_v(\tau_1, \rho_1; t_0, x_0)) \subseteq \hat{C}_v(\tau_1 + \tau_2, e^{\|\nabla v\|_0 \tau_2} (\rho_1 + \rho_2); t_0, x_0) \tag{41}$$

Proof. To see the containment (39), let $(t, x) \in \hat{C}_v(\tau_1, \rho_1; \hat{C}_v(\tau_0, \rho_0; t_0, x_0))$. Then we can write

$$(t, x) = \Phi_{s_2}(\Phi_{s_1}(t_0, x_0) + (0, h_1)) + (0, h_2) \tag{42}$$

with $|s_i| \le \tau_i$ and $|h_i| \le \rho_i$, $i = 1, 2$. We rewrite (42) as

$$(t, x) = \Phi_{s_2 + s_1}(t_0, x_0)) + (0, \tilde{h}_1 + h_2)$$
$$(0, \tilde{h}_1) = \Phi_{s_2}(\Phi_{s_1}(t_0, x_0) + (0, h_1)) - \Phi_{s_2}(\Phi_{s_1}(t_0, x_0))$$

Then $|\tilde{h}_1| \le e^{\|\nabla v\|_0 s_2} |h_1| \le e^{\|\nabla v\|_0 \tau_2} \rho_1$ by Lemma 4.1, and the containment (39) follows by the triangle inequality. The containments (40) and (41) are proven similarly. $\square$

We will sometimes have the need to compare the cylinders of two related velocity fields. To prepare for such a comparison, we start with the following preliminary estimate.

Lemma 4.4. *Suppose that $u(t, x)$ and $v(t, x)$ be vector fields on $\mathbb{R} \times \mathbb{R}^3$ as above. Denote by $^{(v)}\Phi_s(t, x) = (t + s, {}^{(v)}\Phi_s')$ and $^{(u)}\Phi_s(t, x) = (t + s, {}^{(u)}\Phi_s')$ their associated flow maps. Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$. Then we have a comparison estimate*

$$|^{(v)}\Phi_s'(t_0, x_0) - {}^{(u)}\Phi_s'(t_0, x_0)| \le \|v - u\|_{C^0} |s| e^{|s| \|\nabla u\|_0} \tag{43}$$

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Proof. Define
$$\mathbf{d}^2(s) = |^{(v)}\Phi'_s(t_0, x_0) - {}^{(u)}\Phi'_s(t_0, x_0)|^2$$

Then we have

$$\frac{d}{ds}(\mathbf{d}^2) = 2\Big(^{(v)}\Phi'_s(t_0, x_0) - {}^{(u)}\Phi'_s(t_0, x_0)\Big) \cdot \Big(v(^{(v)}\Phi_s(t_0, x_0)) - u(^{(u)}\Phi_s(t_0, x_0))\Big) \tag{44}$$

Writing $v = (v - u) + u$ and applying the mean value theorem, we have

$$|\frac{d}{ds}(\mathbf{d}^2)| \leq 2\|v - u\|_{C^0}\mathbf{d} + 2\|\nabla u\|_{C^0}\mathbf{d}^2 \tag{45}$$

Inequality (43) now follows from (45) and the fact that $\mathbf{d}^2(0) = 0$. $\qquad\square$

The estimate (43) is most useful when the vector field u is the smoother of the two vector fields. Note that the inequality (43) reduces to the trivial bound $|\Phi_s(t_0, x_0) - (t_0 + s, x_0 + s\bar{v})| \leq \|v - \bar{v}\|_{C^0}|s|$ when we take $u = \bar{v}$ to be a constant vector field. We also remark that Lemma 4.4 holds even if v is only continuous, in which case the trajectory $^{(v)}\Phi_s(t, x)$ through any given point may fail to be unique.

From Lemma 4.4, we have the following Cylinder Comparison Lemma

Lemma 4.5 (Cylinder Comparison Lemma). *Let v and u be as in Lemma 4.4, $\tau > 0, \rho > 0$ and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$. Then,*

$$\hat{C}_v(\tau, \rho; t_0, x_0) \subseteq \hat{C}_u(\tau, \rho + \tau\|v - u\|_{C^0}e^{\tau \min\{\|\nabla u\|_0, \|\nabla v\|_0\}}; t_0, x_0) \tag{46}$$

Proof. For $(t, x) \in \hat{C}_v(\tau, \rho; t_0, x_0)$, There exists s and h with $|s| \leq \tau$ and $|h| \leq \rho$ such that

$$\begin{aligned}(t, x) &= {}^{(v)}\Phi_s(t_0, x_0) + (0, h) \\ &= {}^{(u)}\Phi_s(t_0, x_0) + (^{(v)}\Phi_s(t_0, x_0) - {}^{(u)}\Phi_s(t_0, x_0)) + (0, h)\end{aligned}$$

The containment (46) now follows from Lemma 4.4 and the triangle inequality. $\qquad\square$

The following Lemma will be our basic tool in Section 11 for keeping track of the enlargement of support of the approximation solutions during the iteration.

Lemma 4.6. *Let v, u be C^1 vector fields on $\mathbb{R} \times \mathbb{R}^3$ such that $v = u$ on $(\mathbb{R} \times \mathbb{R}^3) \setminus Z$, where Z is a closed set. Then for any open set $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ containing Z (i.e., $Z \subseteq \Omega$) and $\tau, \rho > 0$, we have*

$$\hat{\Gamma}_v(\tau, \rho; \Omega) = \hat{\Gamma}_u(\tau, \rho; \Omega).$$

Proof. By symmetry, it suffices to show that $\hat{\Gamma}_u(\tau, \rho; \Omega) \subseteq \hat{\Gamma}_v(\tau, \rho; \Omega)$, or equivalently,

$$(\mathbb{R} \times \mathbb{R}^3) \setminus \hat{\Gamma}_v(\tau, \rho; \Omega) \subset (\mathbb{R} \times \mathbb{R}^3) \setminus \hat{\Gamma}_u(\tau, \rho; \Omega). \tag{47}$$

Let $(t, x) \in (\mathbb{R} \times \mathbb{R}^3) \setminus \hat{\Gamma}_v(\tau, \rho; \Omega)$. By definition, this is equivalent to the statement

$$^{(v)}\Phi_s(t, x) \notin B(\rho; t_0, x_0) \text{ for any } (t_0, x_0) \in \Omega \text{ and } |s| \leq \tau.$$

In particular, $^{(v)}\Phi_s(t, x) \in (\mathbb{R} \times \mathbb{R}^3) \setminus \Omega$ for $|s| \leq \tau$. Notice, however, that $v = u$ in the region $(\mathbb{R} \times \mathbb{R}^3) \setminus \Omega$. Therefore, $(t, x) \in (\mathbb{R} \times \mathbb{R}^3) \setminus \hat{\Gamma}_u(\tau, \rho; \Omega)$, which proves (47). $\qquad\square$

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5 Basic Technical Outline

In this section, we recall the basic technical outline of the scheme and give a list of the error terms which arise.

Let (v, p, R) be a velocity field, pressure and stress tensor which satisfy the Euler-Reynolds equations (5) with frequency energy levels below (Ξ, e_v, e_R). To perform the construction, we add corrections to the velocity and the pressure $v_1 = v + V$, $p_1 = p + P$ where the correction to the velocity is a sum of high frequency, divergence-free waves $V = \sum_I V_I$ which have the form

$$V_I = e^{i\lambda \xi_I}(v_I + \delta v_I) \tag{48}$$

$$= e^{i\lambda \xi_I} \tilde{v}_I \tag{49}$$

The phase function $\xi_I(t, x)$ and amplitude $v_I(t, x)$ are at disposal, but vary slowly in space relative to the large frequency parameter λ. The small corrections δv_I are present to ensure that (48) is divergence-free, and also to make sure that each correction has vanishing linear and angular momentum. Each individual wave has a conjugate wave $V_{\bar{I}} = \bar{V}_I$ which oscillates in the opposite direction $\xi_{\bar{I}} = -\xi_I$ and has amplitude $v_{\bar{I}} = \bar{v}_I$, so that the overall correction is real-valued.

The corrected velocity and pressure now satisfy the system

$$\partial_t v_1^l + \partial_j(v_1^j v_1^l) + \partial^l p_1 = \partial_t V^l + \partial_j(v^j V^l) + \partial_j(V^j v^l) \tag{50}$$

$$+ \sum_{J \neq \bar{I}} \partial_j(V_I^j V_J^l) + \partial_j \left[\sum_I V_I^j \bar{V}_I^l + P\delta^{jl} + R^{jl} \right] \tag{51}$$

$$\partial_j v_1^j = 0$$

Our goal is to represent the terms on the right hand side of (50)-(51) as the divergence $\partial_j R_1^{jl}$ of a symmetric tensor R_1^{jl} which is small and which satisfies appropriate bounds on its spatial and advective derivatives. First it is necessary to define appropriate mollifications v_ϵ and R_ϵ of the given v and R so that the building blocks of the construction will be influenced only by the low frequency part of the given (v, p, R). These mollifications give rise to the following error term:

$$Q_M^{jl} = (v^j - v_\epsilon^j)V^l + V^j(v^l - v_\epsilon^l) + (R^{jl} - R_\epsilon^{jl}). \tag{52}$$

We now gather the remaining terms in (50)-(51). Expanding the first term in (50) using the Ansatz (48) leads us to impose the transport equation

$$\partial_t \xi_I + v_\epsilon^j \partial_j \xi_I = 0 \tag{53}$$

for the phase functions ξ_I. One can interpret equation (53) as an assumption that the high frequency features are carried by the coarse scale flow. It is natural to impose (53) since, as the paper [Ise13] demonstrates, this behavior is forced by the Euler equations in some quantitative sense.

Assuming (53) and using $\partial_j V_I^j = 0$, the remaining error terms in Equation (50) then have the form

$$\partial_j Q_T^{jl} = \partial_t V^l + \partial_j(v_\epsilon^j V^l) + \partial_j(V^j v_\epsilon^l) \tag{54}$$

$$= \sum_I e^{i\lambda \xi_I}(\partial_t + v_\epsilon^j \partial_j)\tilde{v}_I^l \tag{55}$$

$$+ \sum_I e^{i\lambda \xi_I} \tilde{v}_I^j \partial_j v_\epsilon^l \tag{56}$$

The term (55) is referred to as the **transport term** since it involves the advective derivative. In contrast to the work of [Ise12] in the periodic setting, it is necessary to keep the terms (55)-(56)

together for working in the whole space. The reason is that $\partial_j(V^j v_\epsilon^l)$ by itself may fail to be orthogonal to rotational vector fields, even though it is guaranteed to have integral 0 and is therefore orthogonal to constants (i.e. translations). The combination $\partial_j(v_\epsilon^j V^l) + \partial_j(V^j v_\epsilon^l)$ on the other hand is already the divergence of a symmetric tensor, and therefore satisfies the necessary orthogonality conditions to invert the symmetric divergence equation. Therefore, as long as we ensure that the term $\partial_t V$ also satisfies the necessary orthogonality conditions (that is, V conserves both linear and angular momentum), one can hope to solve (54).

We also isolate the **high frequency interference terms** from (51), which we can gather in symmetric pairs so that once again the necessary orthogonality conditions are clearly satisfied

$$\sum_{J \neq \bar{I}} \partial_j(V_I^j V_J^l) = \frac{1}{2} \sum_{J \neq \bar{I}} \partial_j(V_J^j V_I^l + V_I^j V_J^l) \tag{57}$$

$$= \frac{1}{2} \sum_{J \neq \bar{I}} (V_J^j \partial_j V_I^l + V_I^j \partial_j V_J^l) \tag{58}$$

To treat these terms, we draw on the idea introduced in [DLS13] of using Beltrami flows. Following the treatment in [Ise12], this approach involves adding additional correction terms to the pressure

$$P = P_0 + \sum_{J \neq \bar{I}} P_{I,J} \tag{59}$$

where $P_{I,J} = -\frac{1}{2} V_J \cdot V_I$, and imposing the "microlocal Beltrami flow" condition

$$(i\nabla \xi_I) \times v_I = |\nabla \xi_I| v_I$$

so that the waves V_I in (48) serve as curl eigenfunctions to leading order.

After we apply the identity

$$V_I \cdot \nabla V_J + V_J \cdot \nabla V_I = -V_I \times (\nabla \times V_J) - V_J \times (\nabla \times V_I) + \nabla V_I \cdot V_J$$

and add the gradients of the pressure terms $P_{I,J}$, the remainder of the high frequency interference terms (57) can then be written as a main term which is made small after choosing sharp time cutoffs

$$\partial_j Q_H^{jl} = \frac{-1}{2} \sum_{J \neq \bar{I}} \lambda e^{i\lambda(\xi_I + \xi_J)} \left[v_I \times (|\nabla \xi_J| - 1) v_J + v_J \times (|\nabla \xi_I| - 1) v_I \right] \tag{60}$$

plus lower order error terms involving the small corections δv_I, which we express using (49)

$$\partial_j Q_{H'}^{jl} = \frac{-1}{2} \sum_{J \neq \bar{I}} \lambda e^{i\lambda(\xi_I + \xi_J)} \left[\delta v_I \times [(i\nabla \xi_J) \times \tilde{v}_J] + \delta v_J \times [(i\nabla \xi_I) \times \tilde{v}_I] \right]$$

$$- \frac{1}{2} \sum_{J \neq \bar{I}} \lambda e^{i\lambda(\xi_I + \xi_J)} \left[v_I \times [(i\nabla \xi_J) \times \delta v_J] + v_J \times [(i\nabla \xi_I) \times \delta v_I] \right] \tag{61}$$

$$- \frac{1}{2} \sum_{J \neq \bar{I}} e^{i\lambda(\xi_I + \xi_J)} \left[\tilde{v}_I \times (\nabla \times \tilde{v}_J) + \tilde{v}_J \times (\nabla \times \tilde{v}_I) \right]$$

We remark that our estimates for the terms (60) and (61) rely on a nonstationary phase argument, so it is important to check that we have uniform bounds on $\| |\nabla(\xi_I + \xi_J)|^{-1} \|_{C^0}$ for all pairs of indices I, J, $J \neq \bar{I}$ which interact in the construction.

The final term in (51) is called the **stress term** and takes the form

$$Q_S^{jl} = \sum_I (V_I^j \bar{V}_I^l) + P_0 \delta^{jl} + R_\epsilon^{jl} \tag{62}$$

where P_0 is the low frequency part of the correction to the pressure (59). The term Q_S^{jl} is the only error term (including (52)) which is of low frequency. We expand (62) using the Ansatz (48), and to ensure that (62) is small, we choose the amplitudes P_0 and v_I so that the leading order term in (62) cancels. This choice leads to the **stress equation** for the amplitudes:

$$\sum_I v_I^j \bar{v}_I^l = -P_0 \delta^{jl} - R_\epsilon^{jl} \tag{63}$$

The role of the term P_0 in (63) is essentially to ensure that the right hand side of (63) is positive definite, and also to help prescribe the leading order term in the energy increment of the correction as in the estimate (25). Note that equation (63) leads to the estimates $v_I \sim |R|^{1/2}$ and $|P_0| \sim |R|$ for the amplitudes of the corrections indicated in inequalities (23), (26).

The remaining stress term is then given by

$$Q_S^{jl} = \sum_I (\delta v_I^j \bar{v}_I^l + v_I^j \overline{\delta v}_I^l + \delta v_I^j \overline{\delta v}_I^l) \tag{64}$$

Thus, the new stress takes the form

$$R_1^{jl} = Q_M^{jl} + Q_S^{jl} + Q_T^{jl} + Q_H^{jl} + Q_{H'}^{jl} \tag{65}$$

where Q_M^{jl} and Q_S^{jl} are represented by equations (52) and (64) and where Q_T, Q_H and $Q_{H'}$ are obtained by solving the elliptic equations (54), (60), (61).

We now proceed in Section 6 below to describe the correction in more detail. In Section 6.1, we will complete the outline of the scheme by indicating how the error terms in (65) are organized, and how the support of the error terms remains under control during the iteration.

6 The Shape of the Corrections

Our correction has the form of a sum of individual waves

$$V^l = \sum_I V_I^l \tag{66}$$

The individual waves are complex-valued and take the form

$$V_I^l = e^{i\lambda \xi_I} (v_I^l + \delta v_I^l) \tag{67}$$

where the phase function $\xi_I(t, x)$ is allowed to be nonlinear, and the amplitude v_I is complex-valued and required to satisfy $v_I \in \langle \nabla \xi_I \rangle^\perp$ so that the wave (67) is divergence free to leading order. The nonlinear phase functions ξ_I and amplitudes v_I vary slowly in comparison to the large frequency parameter λ. The correction δv_I in (67) is a lower order term defined in Equation (72) below which is present so that each wave V_I is exactly divergence free.

In previous approaches, the divergence-free property was ensured by taking the wave V_I to be the curl of a vector field

$$V_I = \nabla \times W_I \tag{68}$$

as in [Ise12] or by solving a divergence equation to correct the main term as in [DLS13]. Here, we use waves of the form

$$V_I = \nabla \times \nabla \times Y_I \tag{69}$$

where the potential Y_I is given by

$$Y_I = \frac{1}{\lambda^2} e^{i\lambda \xi_I} y_I, \qquad y_I = \frac{1}{|\nabla \xi_I|^2} v_I \tag{70}$$

We impose the double-curl form (69) because our waves are required to be divergence free and also to ensure that the corrections have 0 angular momentum. Thus, the curl form (68) is also achieved, and it will be easy to see that the associated $W_I = \nabla \times Y_I$ obeys all of the same estimates stated in (24) as in [Ise12]. With the Ansatz (69), we have

$$v_I = [(i\nabla \xi_I) \times]^2 \, y_I \tag{71}$$

$$\delta v_I = \frac{1}{\lambda} \nabla \times \left[(i\nabla \xi_I) \times y_I + \frac{\nabla \times y_I}{\lambda} \right] \tag{72}$$

Our amplitudes are required to satisfy the "microlocal Beltrami flow" condition

$$(i\nabla \xi_I) \times v_I = |\nabla \xi_I| v_I \tag{73}$$

so that (67) behaves to leading order like an eigenfunction of the curl operator with eigenvalue $\lambda |\nabla \xi_I|$. Condition (73) allows us to control interference terms between high frequency waves provided we include sharp time cutoffs which keep the phase gradients very close to 1 in absolute value $|\nabla \xi_I| \approx 1$.

To specify the amplitudes v_I more precisely, we must first specify the index set $\mathcal{I}$ for the indices $I \in \mathcal{I}$. The index $I \in \mathcal{I}$ has two parts $I = (k, f)$. The discrete coordinate $k = (k_0, k_1, k_2, k_3) \in \mathbb{Z} \times \mathbb{Z}^3$ indicates the location of the wave V_I in space time. Namely, a wave with location index $k = k(I)$ will be located in a neighborhood of the point $(t(I), x(I)) = (k_0 \tau, k_1 \rho, k_2 \rho, k_3 \rho)$ and more specifically its support will be contained in a Lagrangian cylinder adapted to v_ϵ

$$\mathrm{supp}\, V_I \subseteq \hat{\Gamma}_{v_\epsilon}(\frac{2\tau}{3}, \rho; t(I), x(I)) \tag{74}$$

The waves V_I will be arranged so that every given point and time (t, x) has at most 2^4 location indices k for which the wave $V_{(k,f)}(t, x)$ may be nonzero. The time scale $\tau > 0$ and the space scale $\rho > 0$ are small parameters which will be specified during the construction.

The other part of the index $I = (k, f)$ is the direction coordinate f, which specifies the direction of oscillation of the wave $V_{(k,f)}$. This coordinate $f \in F$ belongs to a finite set F of cardinality $|F| = 12$, which we take as in [Ise12] to be the set of faces of a regular dodecahedron

$$F = \left\{ \pm \frac{(0, 1, \pm\varphi)}{\sqrt{1 + \varphi^2}}, \pm \frac{(1, \pm\varphi, 0)}{\sqrt{1 + \varphi^2}}, \pm \frac{(\pm\varphi, 0, 1)}{\sqrt{1 + \varphi^2}} \right\}$$

with $\varphi = (1 + \sqrt{5})/2$ being the golden ratio. Thus, each location index k supports 12 waves indexed by (k, f) and the number of nonzero waves at a given point (t, x) is bounded by $2^4 \cdot 12$. The reason for the cardinality $|F| = 12$ is that 6 independent directions are necessary in order to span the space of symmetric tensors in equation (63), and each direction f must come with a conjugate direction $-f$ corresponding to the conjugate index $\bar{I} = (k, -f)$.

To explain the amplitude v_I more precisely, we decompose $v_I = a_I + ib_I$ into its real and imaginary parts, which both take values in $a_I, b_I \in \langle \nabla \xi_I \rangle^\perp$ pointwise. The condition (73) is equivalent to the relationship

$$a_I = -\frac{(\nabla \xi_I)}{|\nabla \xi_I|} \times b_I \tag{75}$$

The imaginary part is then represented in the form

$$b_I^l = \tilde{e}^{1/2}(t,x)\eta\left(\frac{t - t(I)}{\tau}\right)\psi_k(t,x)\gamma_I \mathcal{P}_I^\perp(\nabla \xi_{\sigma I})^l \tag{76}$$

Let us explain the terms appearing in equation (76). The factor

$$\eta\left(\frac{t - t(I)}{\tau}\right) = \eta\left(\frac{t - k_0 \tau}{\tau}\right) \tag{77}$$

is an element of a rescaled, quadratic partition of unity

$$\sum_{y \in \mathbb{Z}} \eta^2(t - y) = 1 \tag{78}$$

that is used to glue local solutions of the homogeneous, quadratic equation (63). Hence, the wave $V_{(k,f)}$ is supported in the time interval of size $\frac{2\tau}{3}$ about $t(I) = k_0\tau$ as desired.

Similarly, the factor $\psi_k(t,x)$ is an element of a partition of unity in space

$$\sum_{k_1,k_2,k_3} \psi_{(k_0,k_1,k_2,k_3)}^2(t,x) = 1 \tag{79}$$

which localizes $V_{(k,f)}$ to the cylinder $\hat{\Gamma}_{v_\epsilon}(\frac{2}{3}\tau, \rho; t(I), x(I))$. More specifically, ψ_k solves a transport equation

$$(\partial_t + v_\epsilon^j \partial_j)\psi_k = 0 \tag{80}$$
$$\psi_k(t(I), x) = \psi_k(k_0\tau, x) = \bar{\psi}_k(x) \tag{81}$$

whose initial conditions

$$\bar{\psi}_k(x) = \eta\left(\frac{x_1 - k_1\rho}{\rho}\right)\eta\left(\frac{x_2 - k_2\rho}{\rho}\right)\eta\left(\frac{x_3 - k_3\rho}{\rho}\right) \tag{82}$$

form a rescaled, quadratic partition of unity in space as in (79).

Here we introduce the new element of including a small length scale ρ on which the waves are localized. Having such sharp cutoffs in space is natural in view of the goal of obtaining solutions with compact support. We will also find in Section 10 that these cutoffs play a role in ensuring that our new method of solving the symmetric divergence equation obeys the correct bounds which eliminate the need for super-exponential growth of frequencies. We will see that ρ is chosen to be of size $\sim \Xi^{-1}$, the same length scale on which the building blocks v_I and $\nabla \xi_I$ vary.

The factor

$$\mathcal{P}_I^\perp(\nabla \xi_{\sigma I})^l = \partial^l \xi_{\sigma I} - \frac{(\nabla \xi_{\sigma I} \cdot \nabla \xi_I)}{|\nabla \xi_I|^2}\partial^l \xi_I \tag{83}$$

is a vector field of size ≈ 1 which takes values in the plane $\langle \nabla \xi_I \rangle^\perp$. This vector field was constructed by taking an orthogonal projection of one of the other phase gradients $\nabla \xi_{\sigma I}$, $\sigma I = (k, \sigma f)$ which

occupies the same location indexed by k, but ocillates in a different direction σf. The vector field (83) is essentially the smoothest vector field of size ≈ 1 taking values in $\langle \nabla \xi_I \rangle^{\perp}$ that one can hope to construct. Placing $\mathcal{P}_I^{\perp}(\nabla \xi_{\sigma I})^l$ in (76) ensures that b_I (and hence a_I defined in (75)) takes values in $\langle \nabla \xi_I \rangle^{\perp}$. The index $\sigma I \neq I$ is chosen to satisfy $\overline{\sigma I} = \overline{\sigma} \overline{I}$, which ensures that $b_{\bar{I}} = -b_I$ and hence $a_{\bar{I}} = a_I$, so that $V_{\bar{I}} = \bar{V}_I$ is indeed a conjugate wave.

The factor $\tilde{e}^{1/2}(t, x)$ is a regularized version of the function $e^{1/2}(t, x)$ described in the statement of the Main Lemma (Lemma 3.1). Thus, we will show that

$$\tilde{e}^{1/2}(t, x) \geq K e_R^{1/2} \tag{84}$$

on a neighborhood of the support of R^{jl} which will contain the support of R_ϵ^{jl}. The function $\tilde{e}^{1/2}(t, x)$ satisfies all the bounds stated in (18), and can also be differentiated in space an arbitrary number of times with good bounds. From (76), the amplitude v_I can be written in the form

$$v_I = \tilde{e}^{1/2} \mathring{v}_I \tag{85}$$

The factor $\tilde{e}^{1/2}$ accounts for the size of the amplitudes $|v_I| \leq C e_R^{1/2}$ with C depending on the constant M in (18), while $\mathring{v}_I$ has size of the order $|\mathring{v}_I| \approx 1$. The renormalization (85) leads to a renormalization of the stress equation (63) for the renormalized amplitudes $\mathring{v}_I$. We choose P_0 in (63) to be

$$P_0 = -\frac{1}{3}\tilde{e} + \frac{1}{3}R_\epsilon^{jl}\delta_{jl} = -\frac{1}{3}\tilde{e} + \frac{1}{3}\text{tr } R_\epsilon$$

With this choice, the right hand side of the Stress Equation has a prescribed trace $\tilde{e}(t, x)$

$$\sum_I v_I^j \bar{v}_I^l = \tilde{e}(t, x)\frac{\delta^{jl}}{3} - \mathring{R}_\epsilon^{jl} \tag{86}$$

Here $\mathring{R}_\epsilon^{jl}$ denotes the trace free part of R^{jl}. The function $\tilde{e}$ turns out to be the main term in the increment to the energy (see Section 8.1 below). In terms of the renormalized amplitudes $\mathring{v}_I$, Equation (86) becomes

$$\sum_I \mathring{v}_I^j \bar{\mathring{v}}_I^l = \frac{\delta^{jl}}{3} + \varepsilon^{jl} \tag{87}$$

$$\varepsilon^{jl} = -\frac{\mathring{R}_\epsilon^{jl}}{\tilde{e}} \tag{88}$$

The tensor ε^{jl} in (88) is bounded by $\|\varepsilon^{jl}\|_{C^0} = O(1/K)$ due to the lower bound $e(t, x) \geq K e_R$ assumed for $e(t, x)$ in (84). In Section 7.3 below, we verify that, on the support of R_ϵ, the regularized function $\tilde{e}$ maintains the same lower bound satisfied by e. As long as K is larger than some absolute constant, this bound ensures that the term ε^{jk} in (88) is smaller than the term $\frac{\delta^{jl}}{3}$ in (87), so that the right hand side of (87) is positive definite and solutions $\mathring{v}_I$ to (87) exist.

We can rewrite Equation (87) as a quadratic equation for the unknown coefficients γ_I appearing in (76), which all have size on the order of ≈ 1. It turns out that the coefficients γ_I can be written as

$$\gamma_I = \gamma_f(\nabla \xi_k, \varepsilon^{jl}) \tag{89}$$

for some smooth, real-valued functions γ_f depending only on the gradients of the phase functions occupying the same location $\nabla \xi_I, I \in k \times F$ and the tensor ε^{jl} appearing in (88). In fact, only six different functions γ_f are used for the formula (89), so that one is not worried about seeing an infinite multitude of constants in the construction.

The phase functions themselves are chosen to satisfy the transport equation

$$(\partial_t + v_\epsilon^j \partial_j)\xi_I = 0 \tag{90}$$

$$\xi_I(t(I), x) = \hat{\xi}_I(x) \tag{91}$$

where the initial data $\hat{\xi}_I$ is a linear function whose gradient has absolute value $|\nabla \hat{\xi}_I| = 1$. The direction of the initial data $\hat{\xi}_{(k,f)}$ is obtained by taking the faces $f \in F$ of the dodecahedron, and applying different rotation matrices $O_{[k]}$ to these faces

$$\hat{\xi}_{(k,f)}(x) = f \cdot O_{[k]}(x - x(I)) \tag{92}$$

$$x(I) = (k_1\rho, k_2\rho, k_3\rho) \tag{93}$$

Here we use a family of 2^4 rotations $O_{[k]}$ depending on the equivalence class of $[k] \in (\mathbb{Z}/(2\mathbb{Z}))^4$. These rotations ensure that no two phase functions occupying adjacent location indices k will oscillate in the same direction. More precisely, they satisfy the following Proposition taken from [Ise12, Lemma 7.1].

Proposition 6.1. *There exists a collection of 2^4 rotations $O_{[k]}$ indexed by $[k] \in (\mathbb{Z}/(2\mathbb{Z}))^4$ and a positive number $c > 0$ with the property that*

$$|f \circ O_{[k]} + f' \circ O_{[k']}| \geq c \qquad f, f' \in F \quad [k], [k'] \in (\mathbb{Z}/(2\mathbb{Z}))^4 \tag{94}$$

holds unless $f' = -f$ and $[k] = [k']$.

This arrangement will allow us to have uniform bounds on $|\nabla(\xi_I + \xi_J)|^{-1} \leq A$, so that the phase functions $\xi_I + \xi_J$ appearing in (60) remain uniformly nonstationary (see Proposition 7.1 below).

We refer to Section 7 of [Ise12] for a full derivation of the construction.

6.1 A preliminary bound on the support of the new stress

Having specified the construction in more detail, we can now briefly indicate how the support of the stress R_1^{jl} calculated in (65) will remain under control during the iteration. Here we explain the rationale for including sharp cutoffs in space ψ_k in our definition of the amplitudes v_I.

The support of the terms Q_M and Q_S will be relatively easy to control, and one can see from equations (52), (63) and (64) that these terms will be supported in a neighborhood of the support of R containing the union of the supports of the waves V_I composing V. We therefore focus on the term $Q_O^{jl} = Q_T^{jl} + Q_H^{jl} + Q_{H'}^{jl}$, which is obtained by solving the elliptic equation

$$\partial_j Q_O^{jl} = U^l \tag{95}$$

$$U^l = \partial_t V^l + \partial_j(v_\epsilon^j V^l + V^j v_\epsilon^l) + \sum_{J \neq \bar{I}} \partial_j(V_I^j V_J^l + P_{IJ}\delta^{jl}) \tag{96}$$

We will construct Q_O as a sum of individual parts $Q_O^{jl} = \sum_I Q_{O,I}^{jl}$. Each individual part $Q_{O,I}^{jl}$ accounts for the wave V_I^l and the interaction terms involving V_I^l by solving the equation

$$\partial_j Q_{O,I}^{jl} = U_I^l \tag{97}$$

$$U_I^l = \partial_t V_I^l + \partial_j(v_\epsilon^j V_I^l + V_I^j v_\epsilon^l) \tag{98}$$

$$+ \frac{1}{4} \sum_{J:J \neq \bar{I}} \partial_j(V_I^j V_J^l + V_J^j V_I^l - V_I \cdot V_J \delta^{jl}) \tag{99}$$

where we recall the choice of $P_{I,J}$ in (59).

Note that the force term U_I^l satisfies the orthogonality conditions necessary for (97) to admit a solution, as the individual waves V_I are required to have 0 linear and angular momentum at all times, and because we keep the interactions of line (99) in a symmetric form. Furthermore, observe that the support of U_I is contained in the support of V_I.

Our method of solving the Equation (97) has the property that if the data U_I^l is supported in an Eulerian cylinder $\hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t_0, x_0)$ adapted to v_ϵ and furthermore U_I^l satisfies the orthogonality conditions necessary for the existence of a solution, then the solution $Q_{O,I}^{jl}$ we construct is also supported in the same Eulerian cylinder $\hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t_0, x_0)$. From Lemma 4.2 on the equivalence of Eulerian and Lagrangian cylinders, it follows from (74) that

$$\operatorname{supp} V_I \cup \operatorname{supp} Q_{O,I} \subseteq \hat{C}_{v_\epsilon}\left(\frac{2\tau}{3}, e^{\frac{2}{3}\|\nabla v_\epsilon\|_0 \tau}\rho; t(I), x(I)\right) \tag{100}$$

The containment (100) will play an important role in controlling the support of the overall stress R_1, which is achieved in Section 7.5.

The containment (100) will also be essential for proving that only a limited number of waves V_I and stress terms can be nonzero at any given point. We summarize this basic property of the construction as a Proposition, which we prove in Section 7.6 after the parameters of the construction have been chosen.

Proposition 6.2 (Limited Interactions). *Let $\#(I)$ denote the number of indices I' such that the support of $V_{I'}$ intersects the support of V_I, plus the number of stress terms $Q_{O,I'}$ whose supports intersect the support of $Q_{O,I}$. Then $\#(I)$ is bounded by an absolute constant.*

7 Choosing the Parameters

We now assume that we are given a solution (v, p, R) to the Euler-Reynolds equation with frequency-energy levels below (Ξ, e_v, e_R) to order L in C^0 in the sense of Definition 3.1. In Section 6, we defined a correction of the form

$$V_I = e^{i\lambda\xi_I}(v_I + \delta v_I)$$

up to the choice of several parameters in the construction. These parameters include: the frequency parameter λ; the mollification parameter ϵ_v for v_ϵ; the mollification parameters ϵ_t, ϵ_x for R_ϵ and $\tilde{e}^{1/2}$; the time scale parameter τ; and the length scale parameter ρ.

The purpose of this section is to specify our choices of these parameters. Moreover, we show that the support bounds (22), (27) and Proposition 6.2 hold under our choices, provided that (100) holds. We remark that (100) ultimately follows from our procedure of finding a compactly support solution to the symmetric divergence equation, which will be presented in Sections 9-10.

The large frequency parameter λ has the form

$$\lambda = B_\lambda N \Xi \tag{101}$$

where N is the frequency growth parameter satisfying the conditions of Lemma 3.1, and B_λ is a large constant which will be chosen at the very end of the argument.

7.1 Defining the coarse scale velocity field

To begin the construction, it is necessary to define a suitable regularization v_ϵ of the velocity field v. We define

$$v_\epsilon = \eta_{\epsilon_v} * \eta_{\epsilon_v} * v \tag{102}$$

to be a double mollification of v in the spatial variables at a length scale ϵ_v. Regularity in time for v_ϵ is established from the Euler-Reynolds equations, and having a double mollification is useful for proving the commutator estimate for $(\partial_t + v_\epsilon \cdot \nabla)v_\epsilon$. The most important requirement on the length scale ϵ_v is that ϵ_v^{-1} is smaller than λ, which ensures that the effective frequency of v_ϵ (or the cost of taking a spatial derivative) is small compared to λ.

Associated to the coarse scale velocity field v_ϵ, we also define the coarse scale advective derivatives

$$\frac{\bar{D}}{\partial t} := (\partial_t + v_\epsilon \cdot \nabla), \qquad \frac{\bar{D}^2}{\partial t^2} := (\partial_t + v_\epsilon \cdot \nabla)^2 \tag{103}$$

The regularization in Equation (102) gives rise to an error term of the form

$$(v^j - v_\epsilon^j)V^l + V^j(v^l - v_\epsilon^l)$$

described in Equation (52). The parameter ϵ_v is chosen in order to achieve a good estimate on the leading order part of this error term, which is given by

$$Q_{M,1}^{jl} = \sum_I e^{i\lambda \xi_I}[(v^j - v_\epsilon^j)v_I^l + v_I^j(v^l - v_\epsilon^l)] \tag{104}$$

Strictly speaking, the amplitudes v_I in (104) depend on the choice of v_ϵ. However, the construction of Section 6, in particular Equation (76), guarantees that the amplitudes obey an estimate

$$\| \sum_I |v_I| \|_{C^0} \leq A \| \tilde{e}^{1/2} \|_{C^0} \tag{105}$$

as long as the lower bound $\tilde{e} \geq K e_R$ is satisfied on the support of R_ϵ, and provided the phase gradients $\nabla \xi_I$ remain within a certain distance of their initial values. See Section 7 of [Ise12].

We construct the function $\tilde{e}^{1/2}(t,x)$ in Section 7.2 by regularizing the function $e^{1/2}(t,x)$ given in Lemma 3.1, so we expect to prove a bound of the type

$$\| \tilde{e}^{1/2} \|_{C^0} \trianglelefteq \| e^{1/2} \|_{C^0} \tag{106}$$

Here we recall the notation that the symbol $\trianglelefteq$ denotes an inequality which has not been proven, but will be established later in the construction. (In particular, there is no implied constant.)

Assuming (106), the bound (105) implies an estimate

$$\| \sum_I |v_I| \|_{C^0} \leq AM e_R^{1/2} \tag{107}$$

where M is the constant in Lemma 3.1. Inequality (107) implies that

$$\| Q_{M,1}^{jl} \|_{C^0} \leq AM e_R^{1/2} \|v - v_\epsilon\|_{C^0} \tag{108}$$

We now choose the parameter ϵ_v in (102) to ensure that $Q_{M,1}$ obeys a bound which is consistent with a scheme aimed at the regularity $1/3$ (see Section 13 of [Ise12])

$$\| Q_{M,1}^{jl} \|_{C^0} \trianglelefteq \frac{e_v^{1/2} e_R^{1/2}}{200N} \tag{109}$$

Using well-known estimates for mollifications (see Sections 14 and 15 of [Ise12]), one has that

$$\|v - v_\epsilon\|_{C^0} \leq A \epsilon_v^L \|\nabla^L v\|_{C^0} \tag{110}$$

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provided that the mollifying kernel η_{ϵ_v} satisfies vanishing moment conditions $\int h^a \eta_{\epsilon_v}(h)dh = 0$ for all multi-indices $1 \leq |a| < L$.

We achieve the estimate (109) by taking ϵ_v of the form

$$\epsilon_v = aN^{-1/L}\Xi^{-1} \tag{111}$$

where a is a small constant depending on the A and M in inequalities (105)-(110). Observe that $\epsilon_v^{-1} = N^{1/L}\Xi$ is smaller than $\lambda \approx N\Xi$ since we assume control over at least $L \geq 2$ derivatives in Lemma 3.1. We also note that the choice of ϵ_v here coincides up to a constant with the choice of parameter in Section 15 of [Ise12], which will allow us to quote the estimates from [Ise12].

7.2 Defining the regularized stress and energy increment

In addition to defining the coarse scale velocity field v_ϵ, we also require suitable regularizations of the energy increment $e(t,x)$ and the stress $R^{jl}(t,x)$. These regularizations $\tilde{e}(t,x)$ and $R_\epsilon^{jl}(t,x)$ are used to define the amplitudes in Equations (76) and (89) of Section 6.

Our definition of R_ϵ follows the construction in Section 18 of [Ise12]. We first regularize R in space using a double convolution $R_{\epsilon_x} = \eta_{\epsilon_x} * \eta_{\epsilon_x} * R$, and then regularize in time by averaging along the trajectories of the vector field $(\partial_t + v_\epsilon \cdot \nabla)$ to form

$$R_\epsilon^{jl}(t,x) := \int R_{\epsilon_x}^{jl}(\Phi_s(t,x))\eta_{\epsilon_t}(s)ds \tag{112}$$

The map Φ_s appearing in (112), which we call the **coarse scale flow**, is the one-parameter family of diffeomorphisms of $\mathbb{R} \times \mathbb{R}^3$ generated by the space-time vector field $(\partial_t + v_\epsilon \cdot \nabla)$. Namely, $\Phi_s(t,x)$: $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ is the unique solution to the initial value problem

$$\frac{d}{ds}\Phi_s^0(t,x) = 1, \quad \frac{d}{ds}\Phi_s^i(t,x) = v_\epsilon^i(\Phi_s(t,x)), \quad \Phi_0(t,x) = (t,x)$$

The motivation for averaging along the coarse scale flow comes from the need to estimate the first advective derivative $\frac{\bar{D}}{\partial t}Q_T$ of the transport term Q_T obtained from solving equation (55). In particular, estimating $\frac{\bar{D}}{\partial t}Q_T$ requires estimates on the second advective derivatives of the amplitudes v_I, and therefore requires estimates on $\frac{\bar{D}^2}{\partial t^2}R_\epsilon$ and $\frac{\bar{D}^2}{\partial t^2}\tilde{e}$ by virtue of the construction of Section 6. The key fact which allows for estimates on the second advective derivative is the fact that $\frac{\bar{D}}{\partial t}$ commutes with pullback along the flow, and hence commutes with the averaging in (112)

$$(\partial_t + v_\epsilon^a \partial_a)R_\epsilon^{jl}(t,x) = \int \frac{\bar{D}R_{\epsilon_x}^{jl}}{\partial t}(\Phi_s(t,x))\eta_{\epsilon_t}(s)ds$$

$$= \int \frac{d}{ds}R_{\epsilon_x}^{jl}(\Phi_s(t,x))\eta_{\epsilon_t}(s)ds \tag{113}$$

Integrating by parts in (113) allows one to estimate $\frac{\bar{D}^2}{\partial t^2}R_\epsilon$, whereas estimating spatial derivatives requires preliminary estimates on the coarse scale flow Φ_s. These estimates are established in Section 18 of [Ise12]. There, the double-mollification in space plays a role in the commutator estimates for spatial derivatives $\nabla^k \frac{\bar{D}}{\partial t}R_\epsilon$. We will also give an alternative proof of the identity (113) in Section 11 below.

The parameters ϵ_x and ϵ_t have the form

$$\epsilon_x = cN^{-1/L}\Xi^{-1}, \quad \epsilon_t = cN^{-1}\Xi^{-1}e_R^{-1/2} \tag{114}$$

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where c is a small constant which is chosen to ensure that the error term generated by the mollification satisfies the bound

$$\|R - R_\epsilon\|_{C^0} \leq \frac{e_v^{1/2} e_R^{1/2}}{100N} \tag{115}$$

The important point about the parameter ϵ_t is that ϵ_t is smaller than the natural time scale $\Xi^{-1} e_v^{-1/2}$ within which the flow of v_ϵ remains under control. This upper bound follows from the condition $N \geq \left(\frac{e_v^{1/2}}{e_R^{1/2}} \right)$.

As for the energy increment $e(t,x)$ we define the regularized energy increment $\tilde{e}$ by regularizing the square root of e in essentially the same way. Namely, we define

$$\tilde{e}^{1/2}(t,x) = \int (e^{1/2})_{\epsilon_x}(\Phi_s(t,x))\eta_{\epsilon_t}(s)ds \tag{116}$$

where $(e^{1/2})_{\epsilon_x} = \eta_{\epsilon_x} * \eta_{\epsilon_x} * e^{1/2}$ is a spatial mollification of $e^{1/2}$. With this definition, the inequality (106) follows immediately.

Note that bounds we assume for $e^{1/2}(t,x)$ in (18) are identical to those assumed for R^{jl} in Definition 3.1 up to a factor of $Me_R^{-1/2}$. Therefore, all of the estimates for $\tilde{e}$ follow with the exact same proofs as the estimates for R_ϵ. In particular, we can again choose parameters ϵ_x and ϵ_t of the form (114) depending on the constant M in (18) in such a way that the estimate

$$\|e^{1/2} - \tilde{e}^{1/2}\|_{C^0} \leq \frac{e_v^{1/2}}{100N} \tag{117}$$

is satisfied.

To ensure that $\tilde{e}$ is suitable for the construction, we now must check that the lower bound

$$\tilde{e}(t_0, x_0) \geq Ke_R \tag{118}$$

is satisfied for (t_0, x_0) on the support of R_ϵ, where $\tilde{e} = (\tilde{e}^{1/2})^2$. Inequality (118) is verified in Section 7.3 below, where additional constraints are imposed on the kernels η_{ϵ_x} and η_{ϵ_t}.

7.3 Checking the lower bound on the energy increment

Here we verify that the square root of the regularized energy increment, which takes the form

$$\tilde{e}^{1/2}(t,x) = \int e^{1/2}(\Phi_s(t,x) + (0,h))\eta_{\epsilon_x}(h)\eta_{\epsilon_t}(s)dhds, \tag{119}$$

satisfies the lower bound $\tilde{e}^{1/2}(t,x) \geq K^{1/2} e_R^{1/2}$ for (t,x) in the support of R_ϵ. Here we abuse notation by writing $\eta_{\epsilon_x}(h)$ to abbreviate the expression $\eta_{\epsilon_x} * \eta_{\epsilon_x}(h)$ coming from (116).

What we are given in the Main Lemma is that the function $e^{1/2}(t,x)$ being averaged in (119) already satisfies the lower bound $e^{1/2} \geq K^{1/2} e_R^{1/2}$ on any v-adapted Eulerian cylinder $C_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; t_0, x_0)$ centered at a point (t_0, x_0) in the support of R.

To ensure that the function $\tilde{e}^{1/2}$ inherits the necessary lower bound from $e^{1/2}$, we impose an additional assumption that both kernels in (119) are non-negative

$$\eta_{\epsilon_t}, \eta_{\epsilon_x} \geq 0 \tag{120}$$

This assumption prohibits us from imposing the vanishing moment condition $\int h^a \eta_{\epsilon_x}(h)dh = 0$ for moments of second order $|a| = 2$, as it will be necessary for $\int |h|^2 \eta_{\epsilon_x}(h)dh > 0$. As a consequence, we

are forced to take $L = 2$ for our choice of ϵ_x in the choice of parameters (114) for $\tilde{e}^{1/2}$. This choice of parameter results in slightly worse bounds for derivatives of $\tilde{e}^{1/2}$ compared to what would be achieved by a larger value of L, but these slightly weaker estimates do not affect the proof. The key properties we maintain are the fact that $\epsilon_x^{-1} \approx N^{1/2}\Xi$ is smaller than the frequency $\lambda \approx N\Xi$ by a factor of $N^{1/2}$, and the factors of ϵ_x^{-1} do not appear in the estimates until more than two derivatives of $\tilde{e}^{1/2}$ are taken.

Assuming the conditions (120), we can now check that the lower bound $\tilde{e}^{1/2}(t_0, x_0) \geq K^{1/2}e_R^{1/2}$ holds for (t_0, x_0) in the support of R_ϵ provided the constants in ϵ_t and ϵ_x are chosen appropriately small. First we make a simple observation that the support of R_ϵ is contained in a Lagrangian cylindrical neighborhood of the support of R

$$\operatorname{supp} R_\epsilon \subseteq \hat{\Gamma}_{v_\epsilon}(\epsilon_t, \epsilon_x; \operatorname{supp} R) \tag{121}$$

The containment (121) follows immediately from the Definition (112) of R_ϵ and the Definition 4.1 of a Lagrangian cylinder.

From the Definition (119) and the condition that η_{ϵ_t} and η_{ϵ_x} are non-negative with $\int \eta_{\epsilon_t}(s)ds = \int \eta_{\epsilon_x}(h)dh = 1$, we know that the lower bound (118) is satisfied at a point (t_1, x_1) provided that $e^{1/2}(t, x) \geq K^{1/2}e_R^{1/2}$ on the Eulerian cylinder $(t, x) \in \hat{C}_{v_\epsilon}(\epsilon_t, \epsilon_x; t_1, x_1)$. Combining this observation with (121) and the assumed lower bound (17) on $e^{1/2}$, we obtain the desired lower bound (118) as a corollary of the following Lemma.

Lemma 7.1. *If the constant c in (114) is chosen sufficiently small, then*

$$\hat{C}_{v_\epsilon}(\epsilon_t, \epsilon_x; \hat{\Gamma}_{v_\epsilon}(\epsilon_t, \epsilon_x; \operatorname{supp} R)) \subseteq \hat{C}_v(\Xi^{-1}e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} R) \tag{122}$$

Proof. According to Lemma 4.3, we have

$$\hat{C}_{v_\epsilon}(\epsilon_t, \epsilon_x; \hat{\Gamma}_{v_\epsilon}(\epsilon_t, \epsilon_x; \operatorname{supp} R)) \subseteq \hat{C}_{v_\epsilon}(2\epsilon_t, \epsilon_x(1 + e^{\|\nabla v_\epsilon\|_0 \epsilon_t}); \operatorname{supp} R)$$
$$\subseteq \hat{C}_{v_\epsilon}(2\epsilon_t, 3\epsilon_x; \operatorname{supp} R)$$

for the appropriate choice of c in (114). According to the Cylinder Comparison Lemma 4.5, we have

$$\hat{C}_{v_\epsilon}(2\epsilon_t, 3\epsilon_x; \operatorname{supp} R) \subseteq \hat{C}_v(2\epsilon_t, 3\epsilon_x + 2\|v - v_\epsilon\|_{C^0}\epsilon_t e^{\|\nabla v_\epsilon\|_0 \epsilon_t}; \operatorname{supp} R)$$

Substituting the choice (114) and applying the estimates $\|v - v_\epsilon\|_{C^0} \leq e_v^{1/2}$ and $\left(\frac{e_v^{1/2}}{e_R^{1/2}N}\right) \leq 1$, we have

$$\hat{C}_{v_\epsilon}(2\epsilon_t, 3\epsilon_x; \operatorname{supp} R) \subseteq \hat{C}_v(2c\Xi^{-1}e_v^{-1/2}, 6c\Xi^{-1}; \operatorname{supp} R)$$

This establishes Lemma 7.1, and consequently the lower bound (118), when c is chosen to be a sufficiently small constant. $\qquad\square$

7.4 Choosing the time scale of the construction

Having chosen the parameters for mollifying the velocity, energy increment and stress, we have now completely specified the building blocks in the construction up to the choice of three parameters. The three parameters which remain are: the time scale τ, which determines the lifespan of the time cutoffs $\eta\left(\frac{t - t(I)}{\tau}\right)$ of Equation (77) which enter into the amplitudes, the space scale ρ, which determines the size of the support in space for the initial data $\bar{\psi}_k(t(I), x)$ of the spatial cutoffs $\psi_k(t, x)$ in equation (82), and the constant B_λ in the definition of the frequency parameter $\lambda = B_\lambda N\Xi$. Among these three, the first parameter we specify is the lifespan parameter τ.

The parameter τ takes the form

$$\tau = b\Xi^{-1}e_v^{-1/2} \tag{123}$$

where $b \leq 1$ is a small, dimensionless parameter which we will now specify.

The choice of the lifespan parameter τ is restricted by several aspects of the construction. First of all, τ cannot be larger than a multiple of $\Xi e_v^{1/2}$ as the elements of the construction which are transported by v_ϵ cannot be controlled with good bounds for times larger than $\|\nabla v_\epsilon\|_{C^0}^{-1}$. Secondly, it is necessary for the gradients of the phase functions to remain within a certain, finite distance c_0 of their initial values in order to ensure the construction is well-defined

$$\|\nabla\xi_I - \nabla\hat{\xi}_I\|_{C^0} \leq c_0 \tag{124}$$

When the requirement (124) is satisfied for a sufficiently small constant c_0, we may guarantee that the phase functions in the construction remain nonstationary, which is necessary for gaining cancellations while solving the equation $\partial_j Q^{jl} = e^{i\lambda\xi_I}u_I^l$ with oscillatory data. Namely, we have the following Proposition

Proposition 7.1 (Nonstationary Phase). *There exists $b_0 > 0$ and an absolute constant $A > 0$ such that for τ of the form (123) with $b < b_0$ we have*

$$\| \, |\nabla\xi_I|^{-1}\|_{C^0} + \| \, |\nabla(\xi_I + \xi_J)|^{-1}\|_{C^0} \leq A \tag{125}$$

for all indices I and all pairs of indices I, J with $J \neq \bar{I}$ whose supports intersect.

By Proposition 6.1, the construction is arranged so that (125) is satisfied for by the initial data for the phase gradients $\nabla\hat{\xi}_I$ and $\nabla(\hat{\xi}_I + \hat{\xi}_J)$.The bound (125) remains satisfied (with a larger constant) provided (124) holds. It is also necessary to impose (124) with a possibly smaller constant c_0 to ensure that equation (87) admits solutions in $\mathring{v}_I \in \langle\nabla\xi_I\rangle^\perp$ with uniform bounds. See Lemma 7.5 and Proposition 7.2 of [Ise12].

Assuming $\tau \leq \Xi^{-1}e_v^{-1/2}$, the estimate we obtain from the transport equation for $\nabla\xi_I$ is

$$\|\nabla\xi_I - \nabla\hat{\xi}_I\|_{C^0} \leq A\Xi e_v^{1/2}\tau = Ab \tag{126}$$

Therefore, all the aforementioned requirements on the phase gradients $\nabla\xi_I$ of the construction can be guaranteed by choosing τ of the form (123), where $b \leq b_0$ is an appropriately small constant such that the desired bound (124) holds.

Choosing $b = b_0$ to be a small constant (or something close) would in principle be necessary to obtain the conjectured $1/3$ regularity of solutions for the type of convex integration scheme we consider. However, the smallness of the parameter τ plays a crucial role in controlling the High Frequency Interference Terms, and for this reason we are forced to choose b much smaller than a constant, ultimately leading to solutions with lesser regularity $1/5$. This obstruction to higher regularity was studied in [Ise12], where it was observed that the Transport Term of Equation (55), which obeys the bound (see Section 19 of [Ise12])

$$\|Q_T\|_{C^0} \leq C\lambda^{-1}\tau^{-1}e_R^{1/2} + \text{ Lower order terms }, \tag{127}$$

$$\leq Cb^{-1}\frac{e_v^{1/2}e_R^{1/2}}{B_\lambda N} + \text{ Lower order terms} \tag{128}$$

can only be guaranteed to have the size $\frac{e_v^{1/2}e_R^{1/2}}{N}$ desired for the $1/3$ regularity if the b chosen in (123) is taken to be a constant. On the other hand, the High Frequency Interference Terms, which obey the bound

$$\|Q_H\|_{C^0} \leq C \, be_R + \text{ Lower order terms} \tag{129}$$

require b to be significantly smaller than a constant in order for an improvement in the error to be observed. The estimate (129) arises from Equation (60), which shows that Q_H will only be small provided the terms $\||\nabla\xi_I| - 1\|_{C^0}$ are small. Optimizing between (128) and (129) leads to the choice

$$b = b_0 B_\lambda^{-1/2} \left(\frac{e_v^{1/2}}{e_R^{1/2} N} \right)^{1/2} \tag{130}$$

Now the only parameters which remain to be chosen are the length scale ρ and the large parameter B_λ.

7.5   Choosing the length scale and controlling the support of R_1

A new feature of our construction is the presence of a small length scale parameter ρ which determines the size of the region on which the spatial cutoffs $\psi_k(t,x)$ are supported. The purpose of these sharp cutoffs is to control the supports of the corrections V^l, P and the new stress R_1^{jl} obtained at the end of each stage of the iteration, which are required to stay within a neighborhood of the support of the energy increment $e^{1/2}(t,x)$ according to Lemma 3.1, i.e.,

$$\operatorname{supp} V \subseteq \hat{C}_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} e), \tag{131}$$

$$\operatorname{supp} P \subseteq \hat{C}_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} e), \tag{132}$$

$$\operatorname{supp} R_1 \subseteq \hat{C}_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} e). \tag{133}$$

We will take the parameter ρ to have the form

$$\rho = c_\rho \Xi^{-1} \tag{134}$$

where c_ρ is a small constant associated to ρ which we choose here so that the containments (131)-(133) can be guaranteed. Note that these containments are identical to (22) and (27) in the Main Lemma.

Remark. Before we proceed to choose ρ, it is important to point out that length scales significantly smaller than (134) would be forbidden for a construction aimed at proving the conjectured 1/3 regularity. Namely, the presence of sharp space cutoffs at scale ρ gives rise to a term of size

$$\|Q_S\|_{C^0} \le \rho^{-1}(N\Xi)^{-1} e_R + \dots \tag{135}$$

within the stress term Q_S defined in (64). That is, Q_S is schematically of size

$$|Q_S| \sim \sum_I \frac{|\nabla v_I| \cdot |v_I|}{\lambda},$$

and terms of size (135) appear when the derivative hits the spatial cutoff[5]. Ideally, the bound (135) should be of size $\frac{e_v^{1/2} e_R^{1/2}}{N}$ to obtain 1/3-Hölder solutions (see Section 13 of [Ise12]), and this requirement gives restrictions on the use of length scales smaller than (134). $\square$

We now proceed to estimate the support of R_1 in terms of the parameter ρ. As discussed in Section 6.1, the term composing R_1 with the largest support is the term $Q_O^{jl} = Q_T^{jl} + Q_H^{jl} + Q_{H'}^{jl}$, which is obtained as a solution to the elliptic equation $\partial_j Q_O^{jl} = U^l$. According to the containment (100), the term $Q_O^{jl} = \sum_I Q_{O,I}^{jl}$ is obtained as a sum of localized pieces, with

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_{v_\epsilon}(\tau, e^{\|\nabla v_\epsilon\|_0 \tau} \rho; t(I), x(I)) \tag{136}$$

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[5]The same estimate also arises from the term (61).

The term $Q_{O,I}$ is nonzero only when the wave V_I is nonzero, so we now study the conditions under which V_I is nonzero.

By construction, the support of each wave V_I is contained in the support of its spatial cutoff ψ_k and its corresponding time cutoff $\eta_{k_0}(t)$, which together are supported on some Lagrangian cylinder

$$\operatorname{supp} V_I \subseteq \hat{\Gamma}_{v_\epsilon}(\frac{2\tau}{3}, \rho; t(I), x(I)) \tag{137}$$

A wave V_I can only be nonzero if the cylinder supporting V_I intersects the support of $\tilde{e}^{1/2}$, implying that the terms V_I and $Q_{O,I}$ are nontrivial only when

$$(t(I), x(I)) \in \hat{C}_{v_\epsilon}(\tau, \rho; \operatorname{supp} \tilde{e})$$

by the duality (30) between Eulerian and Lagrangian cylinders. Thus from (136), we have

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_{v_\epsilon}(\tau, A\rho; \hat{C}_{v_\epsilon}(\tau, \rho; \operatorname{supp} \tilde{e})) \tag{138}$$

Here the constant A is an absolute constant which changes from line to line, and we have used the fact that $e^{\|\nabla v\|_0 \tau} \le e^1 \le A$ is bounded. By Lemma 4.3, the right-hand side is bounded by

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_{v_\epsilon}(2\tau, A\rho; \operatorname{supp} \tilde{e}) \tag{139}$$

From the definition of $\tilde{e}$, we have

$$\operatorname{supp} \tilde{e} \subseteq \hat{\Gamma}_{v_\epsilon}(\epsilon_t, \epsilon_x; \operatorname{supp} e),$$

and it follows from Lemma 4.3 that

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_{v_\epsilon}(2\tau + \epsilon_t, A\rho + A\epsilon_x; \operatorname{supp} e) \tag{140}$$

where A is an absolute constant coming from the bound $\|\nabla v\|_0 \epsilon_t \le \Xi e_v^{1/2} \epsilon_t \le 1$. From the cylinder comparison Lemma 4.5, we obtain

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_v(2\tau + \epsilon_t, A\rho + A\epsilon_x + A\|v - v_\epsilon\|_{C^0}(\tau + \epsilon_t); \operatorname{supp} e) \tag{141}$$

Using the estimate $\|v - v_\epsilon\|_{C^0} \le e_v^{1/2}$ guaranteed in line (111), we can therefore guarantee the bound

$$\operatorname{supp} Q_{O,I} \subseteq \hat{C}_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} e) \tag{142}$$

after possibly choosing smaller constants c, c_ρ and b_0 in the definitions (114), (134), (123) and (130) for the parameters ϵ_x, ϵ_t, ρ and τ. We also see that the sum $Q_O = \sum_I Q_{O,I}$ has the same bound on its support from $\operatorname{supp} Q_O \subseteq \bigcup_I Q_{O,I}$. Finally, it is clear that the other terms Q_M and Q_S contributing to R_1 in (52) and (63) have even smaller support. Therefore the containment (133) has been guaranteed.

By construction, these choices also guarantee that

$$\operatorname{supp} \tilde{e} \subseteq \hat{C}_v(\Xi^{-1} e_v^{-1/2}, \Xi^{-1}; \operatorname{supp} e),$$

which implies the desired containments (131)-(132) for V and P.

7.6 Bounding the number of interaction terms

Having chosen the time and length scales of the construction, we can now verify Proposition 6.2, which states that each wave V_I and stress term $Q_{O,I}$ shares support with a bounded number of distinct indices.

First, for a given index I, let $\#(I)$ denote the number of indices I' such that the support of $V_{I'}$ intersects the support of V_I. Recall from (137) that each wave is contained in a cylinder $\operatorname{supp} V_I \subseteq \hat{\Gamma}_{v_\epsilon}(\frac{2\tau}{3}, \rho; t(I), x(I))$. Therefore, if V_J is a wave whose support intersects the support of V_I, the cylinders corresponding to the two waves intersect, and by (30) we have

$$(t(J), x(J)) \subseteq \hat{C}_{v_\epsilon}\left(\frac{2\tau}{3}, \rho; \hat{\Gamma}_{v_\epsilon}\left(\frac{2\tau}{3}, \rho; t(I), x(I)\right)\right)$$

By Lemma 4.3 and the bound $\|\nabla v_\epsilon\|_0 \tau \leq 1$, we have

$$(t(J), x(J)) \subseteq \hat{C}_{v_\epsilon}\left(\frac{4\tau}{3}, 10\rho; t(I), x(I)\right) \tag{143}$$

The number of lattice points $(t(J), x(J)) = (k_0\tau, k_1\rho, k_2\rho, k_3\rho)$ with $k_i \in \mathbb{Z}$ which can belong to a cylinder (143) is clearly bounded, and so is the number of indices $J = (k_0, k_1, k_2, k_3, f) \in \mathbb{Z}^4 \times F$ which occupy such locations, since at most a finite number $|F|$ indices J share a given location index k. Thus, the number of waves $\#(I)$ which interact with V_I is bounded by an absolute constant.

To finish the proof of Proposition 6.2, it suffices to bound the number of stress terms $Q_{O,I}$ occupying a given point. This number is bounded by following the same line of reasoning, but considering the Eulerian cylinders in (136) containing the support of $Q_{O,I}$, and applying the corresponding bound (41) in Lemma 4.3.

8 Estimates for the Corrections

In this section, we verify the estimates stated in the Main Lemma (Lemma 3.1) concerning the corrections V and P. More precisely, we establish the estimates

$$\|\nabla^k v_1\|_{C^0} \lesssim (\Xi')^k (e_v')^{1/2} \qquad\qquad k = 1, \ldots, L \tag{144}$$

$$\|\nabla^k p_1\|_{C^0} \lesssim (\Xi')^k e_v' \qquad\qquad k = 1, \ldots, L \tag{145}$$

concerning the frequency and energy levels of $v_1 = v + V$ and $p_1 = p + P$, with $(\Xi', e_v') = (C_0 N\Xi, e_R)$. We also prove the bounds (23) and (26) for the corrections V and P, respectively, and the local energy increment bound (25). The estimates considered in this Section will also prepare us for estimating the resulting stress R_1 in the next two Sections.

First we state the bounds satisfied by the elements of the construction obtained from solving a transport equation. We recall the following estimates were established for the phase gradients $\nabla\xi_I$ in the construction of [Ise12]. To state the estimates, it will be convenient to use the notation $y_+ := \max\{y, 0\}$.

Proposition 8.1 (Transport Estimates). *Let $L \geq 2$ be as in Lemma 3.1. There exist constants C_a such that for all $a \geq 1$ and $0 \leq r \leq 2$, the bounds*

$$\Xi^{-1}\left\|\nabla^a\left(\frac{\bar{D}}{\partial t}\right)^r \nabla\psi_k\right\|_{C^0} + \left\|\nabla^a\left(\frac{\bar{D}}{\partial t}\right)^r \nabla\xi_I\right\|_{C^0} \leq C_a \Xi^a (\Xi e_v^{1/2})^r N^{(a+(r-1)_+ +1-L)_+/L} \tag{146}$$

are satisfied. Here ∇^a denotes any spatial derivative of order a. Moreover, if $D^{(a,r)}$ denotes any derivative of the form

$$D^{(a,r)} = \nabla^{a_1}(\partial_t + v_\epsilon \cdot \nabla)^{r_1} \nabla^{a_2}(\partial_t + v_\epsilon \cdot \nabla)^{r_1}\nabla^{a_3} \tag{147}$$

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with $a = a_1 + a_2 + a_3, r = r_1 + r_2, a_i, r_i \geq 0$ and $r \leq 2$, we also have the bound

$$\Xi^{-1}\|D^{(a,r)}\nabla\psi_k\|_{C^0} + \|D^{(a,r)}\nabla\xi_I\|_{C^0} \leq C_a\Xi^a(\Xi e_v^{1/2})^r N^{(a+(r-1)_++1-L)_+/L} \tag{148}$$

According to Proposition 8.1, every spatial derivative costs at most $|\nabla| \leq N^{1/L}\Xi$ in the estimate, and each coarse scale advective derivative costs at most $|\frac{\bar{D}}{\partial t}| \leq \Xi e_v^{1/2}$. In particular, as $L \geq 2$, the cost of a derivative $|\nabla|$ is smaller than the frequency parameter $\lambda \approx N\Xi$ by a factor of $N^{-(1-1/L)} \leq N^{-1/2}$, which means that the terms ψ_k and ξ_I can be regarded as having frequency less than λ. Also, since we have imposed that $L \geq 2$, it is important to note that the factors $N^{1/L}$ do not appear in the estimate until at least two derivatives have been taken.

Proposition 8.1 was established for the phase gradients $\nabla\xi_I$ in Section 17 of [Ise12], relying on the transport equation

$$(\partial_t + v_\epsilon^j\partial_j)\partial^l\xi_I = -\partial^l v_\epsilon^j\partial_j\xi_I \tag{149}$$

satisfied by the phase gradients. The estimates for the cutoff gradients $\nabla\psi_k$ can be proved similarly, as they obey the identical transport equation as the phase gradients. We remark that the estimates for second advective derivatives $\left(\frac{\bar{D}}{\partial t}\right)^2$ of $\nabla\xi_I$ and $\nabla\psi_k$ are more subtle to prove than the rest, and require the following estimates for $(\partial_t + v_\epsilon \cdot \nabla)v_\epsilon$ and its spatial derivatives:

Proposition 8.2 (Coarse Scale Velocity Estimates). *Let $L \geq 2$ be as in Lemma 3.1. The vector field v_ϵ defined in (102) satisfies the bounds*

$$\|\nabla^a v_\epsilon\|_{C^0} \leq C_a\Xi^a e_v^{1/2} N^{(a-L)_+/L}, \quad a \geq 1 \tag{150}$$

$$\|\nabla^a(\partial_t + v_\epsilon \cdot \nabla)v_\epsilon\|_{C^0} \leq C_a\Xi^{1+a} e_v N^{(1+a-L)_+/L}, \qquad a \geq 0 \tag{151}$$

These estimates are obtained by commuting the mollifier $\eta_{\epsilon+\epsilon} = \eta_{\epsilon_v}$ with the Euler-Reynolds equations, and using a commutator estimate akin to [CET94]. We refer to Section 16 of [Ise12] for the proof.

We also state the bounds satisfied for the terms $\tilde{e}$ and R_ϵ which were defined in Section 7.2 using a mollification along the flow of v_ϵ. For compactness, we use the notation of line (147) and also

$$(r \geq b) = \begin{cases} 1 & \text{if } r \geq b \\ 0 & \text{if } r < b \end{cases}$$

Proposition 8.3 (Stress and Energy Increment estimates). *Let $L = 2$. Then for every $a \geq 0$ and $0 \leq r \leq 2$, there is a constant C_a such that*

$$e_R^{1/2}\|D^{(a,r)}\tilde{e}^{1/2}\|_{C^0} + \|D^{(a,r)}R_\epsilon\|_{C^0} \leq C_a\Xi^a e_R(\Xi e_v^{1/2})^{(r\geq1)}(N\Xi e_R^{1/2})^{(r\geq2)}N^{(a+1-L)_+/L} \tag{152}$$

Proposition 8.3 was established in Section 18 of [Ise12] for the term R_ϵ. A large part of the work goes into estimating the coarse scale flow Φ_s associated to v_ϵ, and into establishing basic properties of mollification along the flow. Since the function $e^{1/2}$ that was regularized to form $\tilde{e}^{1/2}$ obeys the same estimates as those assumed for $e_R^{-1/2}R$, the same estimates follow for $\tilde{e}^{1/2}$. The restriction to $L = 2$ (which was not present in [Ise12]) arises from the considerations in Section 7.3.

From Propositions 8.1 and 8.3, we obtain estimates for the basic building blocks of the construction

Proposition 8.4 (Amplitude estimates). *For $L = 2$, the amplitudes v_I satisfy the bounds*

$$\|D^{(a,r)}v_I\|_{C^0} \leq C_a\Xi^a e_R^{1/2}\tau^{-r}N^{(a+1-L)_+/L} \tag{153}$$

$$\|D^{(a,r)}\delta v_I\|_{C^0} \leq C_a B_\lambda^{-1}N^{-1}\Xi^a e_R^{1/2}\tau^{-r}N^{(a+2-L)_+/L} \tag{154}$$

for $a \geq 0$ and $0 \leq r \leq 2$.

The estimates for v_I follow from Propositions 8.1 and 8.3 after repeated applications of the chain and product rule using the expressions (76) and (75) for the real and imaginary parts of v_I. The estimates (154) for the small correction terms δv_I then follow from the estimates (153) for v_I and the estimates for $\nabla \xi_I$ of Proposition 8.1 using the expression (72) for δv_I. The details are carried out in Sections 20 and 21 of [Ise12], although there the correction δv_I has a slightly different form. The main point is that, schematically, δv_I has the form

$$\delta v_I \sim \frac{1}{\lambda} \nabla v_I + \frac{1}{\lambda^2} \nabla^2 v_I$$

up to some factors involving phase gradients. The first derivative ∇ hitting v_I costs a factor of $|\nabla| \leq \Xi$ compared to the bound $\|v_I\|_{C^0} \leq e_R^{1/2}$, whereas the factor $\frac{1}{\lambda}$ gains a factor of $(B_\lambda N \Xi)^{-1}$ in the estimate, and the additional term involving $\frac{\nabla^2}{\lambda^2}$ is lower order. The additional restriction to $L = 2$ in the estimates arises from the considerations in Section 7.3 as in Proposition 8.3. This restriction does not affect the final conclusion of the Main Lemma.

The bounds (23) and (26) stated in Lemma 3.1 for the corrections V and P to the velocity and pressure, are straightforward applications of Propositions 8.1-8.4. Furthermore, the frequency and energy level bounds (144) and (145) for $v_1 = v + V$ and $p_1 = p + P$ (with $C_0 > 1$ sufficiently large) follow in a similar manner. A key point in this implication is Proposition 6.2 which states that only a bounded number of waves can interact at any point. The relevant arguments are carried out in Section 22 of [Ise12]. The estimate (27) on the support of V and P has been established during the proof of the containment (22) in Section 7.5, as we have that $\operatorname{supp} V \cup \operatorname{supp} P \subseteq \operatorname{supp} \tilde{e}$. The estimate (24) for the potential $W = \sum_I \nabla \times Y_I$ defined in line (69) is also a straightforward application of the same estimates, even though our terms $W_I = \nabla \times Y_I$ have a slightly different form than the corresponding terms in [Ise12].

Regarding the corrections, the only parts of Lemma 3.1 which do not follow from the proof of the Main Lemma of [Ise12] is the estimate (25) concerning the local energy increments. We now turn to the proof of this estimate.

8.1 Local estimates on the energy increment

Here we verify the estimate (25) on the energy increment of the solution.

Let $\psi(x)$ be a smooth test function on $\mathbb{R}^3$ with compact support and let $t \in \mathbb{R}$. We wish to estimate the error in prescribing the energy estimate. The main point is that, if we expand $V = \sum_I V_I$ into individual waves, the main interactions come from conjugate waves $I, \bar{I}$

$$\int |V|^2(t,x)\psi(x)dx = \sum_{I,J} \int V_I \cdot V_J \psi(x)dx =: E_1 + E_2 + E_3, \tag{155}$$

$$E_1 = \sum_I \int |v_I|^2(t,x)\psi(x)dx \tag{156}$$

$$E_2 = \sum_I \int (v_I^j \overline{\delta v}_I^l + \delta v_I^j \bar{v}_I^l + \delta v_I^j \overline{\delta v}_I^l)\delta_{jl}\psi(x)dx \tag{157}$$

$$E_3 = \sum_{J \neq \bar{I}} \int e^{i\lambda(\xi_I + \xi_J)} \tilde{v}_I \cdot \tilde{v}_J \psi(x)dx \tag{158}$$

Taking the trace of (86), we see that the main term (156) is equal to

$$E_1 = \int \tilde{e}(t,x)\psi(x)dx \tag{159}$$

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The term E_1 gives rise to the main term in (25), with an error bounded by

$$\left| \int e(t,x)\psi(x)dx - \int \tilde{e}(t,x)\psi(x)dx \right| \leq \int |e^{1/2}(t,x) - \tilde{e}^{1/2}(t,x)|e^{1/2}(t,x)|\psi(x)|dx$$

$$+ \int |e^{1/2}(t,x) - \tilde{e}^{1/2}(t,x)|\tilde{e}^{1/2}(t,x)|\psi(x)|dx$$

$$\leq C \frac{e_R^{1/2} e_v^{1/2}}{N} \|\psi\|_{L^1} \tag{160}$$

from (117). The term E_2 is bounded by

$$|E_2(t)| \leq C \frac{e_R}{B_\lambda N} \|\psi\|_{L^1}$$

from Proposition 8.4, and finally E_3 is estimated by integration by parts

$$E_3 = \frac{1}{i\lambda} \sum_{J \neq \bar{I}} \int \frac{\partial^a(\xi_I + \xi_J)}{|\nabla(\xi_I + \xi_J)|^2} \partial_a e^{i\lambda(\xi_I + + \xi_J)} \tilde{v}_I \cdot \tilde{v}_J \psi(x)dx \tag{161}$$

$$= \frac{-1}{i\lambda} \sum_{J \neq \bar{I}} \int e^{i\lambda(\xi_I + \xi_J)} \partial_a \left[\frac{\partial^a(\xi_I + \xi_J)}{|\nabla(\xi_I + \xi_J)|^2} \tilde{v}_I \cdot \tilde{v}_J \psi(x) \right] dx \tag{162}$$

$$|E_3| \leq C \frac{1}{B_\lambda N \Xi} (\Xi e_R \|\psi\|_{L^1} + e_R \|\nabla \psi\|_{L^1}) \tag{163}$$

Here we use Proposition 6.2 to bound the number of interacting waves, and also take advantage of the uniform bounds on $\| |\nabla(\xi_I + \xi_J)|^{-1}\|_{C^0}$ for nonconjugate interacting waves I, J in Proposition 7.1. Estimate (163) concludes the proof of (25).

9 Estimates for the new Stress

To complete the proof of the Main Lemma (Lemma 3.1), we must calculate the new stress R_1 and establish the following estimates

$$\|\nabla^k R_1\|_{C^0} \trianglelefteq (\Xi')^k e_R', \qquad\qquad k = 0, \dots, L \tag{164}$$

$$\|\nabla^k(\partial_t + v_1 \cdot \nabla)R_1\|_{C^0} \trianglelefteq (\Xi')^k (\Xi'(e_v')^{1/2})e_R', \qquad\qquad k = 0, \dots, L-1 \tag{165}$$

$$(\Xi', e_v', e_R') = \left(C_0 N\Xi, e_R, \frac{e_v^{1/4} e_R^{3/4}}{N^{1/2}} \right) \tag{166}$$

Recall from Section 5 that the new stress is composed of several terms

$$R_1^{jl} = Q_M^{jl} + Q_S^{jl} + Q_T^{jl} + Q_H^{jl} + Q_{H'}^{jl} \tag{167}$$

For the terms Q_M and Q_S, we can appeal to [Ise12] Section 25, where the estimates (164)-(166) are verified for essentially identical terms. The only difference in our case is the presence of sharper cutoffs ψ_k and a regularized energy increment $\tilde{e}^{1/2}$ which do not affect the estimates. We are therefore left with the terms Q_T, Q_H and $Q_{H'}$ calculated in (54), (60), (61).

As outlined in Section 6.1, these terms are calculated by solving the symmetric divergence equation

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with high frequency data

$$\partial_j Q_{O,I}^{jl} = U_I^l \tag{168}$$

$$U_I^l = U_{T,I}^l + \sum_{J:J\neq\bar{I}} U_{H,IJ}^l \tag{169}$$

$$U_{T,I}^l = \partial_t V_I^l + \partial_j(v_\epsilon^j V_I^l + V_I^j v_\epsilon^l) \tag{170}$$

$$U_{H,IJ}^l = \frac{1}{4}\partial_j(V_I^j V_J^l + V_J^j V_I^l - 2V_I \cdot V_J \delta^{jl}) \tag{171}$$

The terms $U_{T,I}^l$, $U_{H,IJ}^l$ consist of the individual terms in the summations (54), (60)-(61).

A key point in solving the equation (168) is that we expect to gain a factor λ^{-1} in the estimate $\|Q_{O,I}\|_{C^0} \leq \lambda^{-1}\|U_I\|_{C^0}$ up to lower order terms, because the data on the right hand side has high frequency λ. For example, the transport term (170) has the form

$$U_{T,I}^l = e^{i\lambda\xi_I} u_I^l \tag{172}$$

$$u_I^l = (\partial_t + v_\epsilon^j \partial_j)\tilde{v}_I^l + \tilde{v}_I^j \partial_j v_\epsilon^l \tag{173}$$

Furthermore, we desire a solution $Q_{O,I}$ to (168) which also has compact support around the support of U_I. Concerning the support of the waves, note that the terms in (169) have the common feature that they are supported in the cylinder

$$\operatorname{supp} U_{T,I} \cup \operatorname{supp} U_{H,IJ} \subseteq \operatorname{supp} V_I \subseteq \hat{C}_{v_\epsilon}(\frac{2}{3}\tau, A\rho; t(I), x(I))$$

with $A = e^{\frac{2}{3}\|\nabla v_\epsilon\|_0 \tau}$ as discussed in Section 7.5. Our solution $Q_{O,I}$ will have support in the same cylinder, from which (100) follows.

Before we can find a compactly supported solution $Q_{O,I}$ to (168), it is necessary to check that the terms U_I^l satisfy the orthogonality conditions necessary to solve (168). For the terms $U_{H,IJ}^l$ and the term $\partial_j(v_\epsilon^j V_I^l + V_I^j v_\epsilon^l)$ in $U_{T,I}^l$, the orthogonality conditions are obvious as both terms have already been represented as the divergence of a symmetric tensor with compact support. For the term $\partial_t V_I^l$, the orthogonality conditions follow from our technique of taking V_I of double curl form. Namely, if K^l is any solution to the equation $\partial_j K_l + \partial_l K_j = 0$ on $\mathbb{R}^n$, then K^l is a linear combination of translational and rotational vector fields, and in particular its second derivative $\nabla^2 K$ vanishes. It follows that

$$\int \partial_t V_I \cdot K \, dx = \frac{d}{dt}\int \nabla \times \nabla \times Y_I \cdot K \, dx \tag{174}$$

$$= \frac{d}{dt}\int Y_I \cdot \nabla \times \nabla \times K \, dx = 0 \tag{175}$$

Thus, there is no immediate obstruction to obtaining a compactly supported solution $Q_{O,I}$ to (168).

It now remains to construct a solution to equation (168) and to establish the oscillatory estimate $\|Q_{O,I}\|_{C^0} \leq \lambda^{-1}\|U_I\|_{C^0}$ up to lower order terms. These tasks are taken up in Sections 9.1 and 10 below.

9.1 Applying the parametrix

Here we consider the general problem of finding compactly supported solutions to the symmetric divergence equation

$$\partial_j Q^{jl} = e^{i\lambda\xi} u^l \tag{176}$$

where the right hand side is supported on a cylinder $\hat{C}_{v_\epsilon}(\frac{2}{3}\tau, A\rho; t(I), x(I))$ and satisfies the necessary orthogonality conditions for a solution to exist. In our applications, the phase function ξ is either one of the phase functions ξ_I or the sum $\xi_I + \xi_J$ of two interacting, nonconjugate phase functions. In every case, the amplitude u^l turns out to satisfy the estimates

$$\|\nabla^k u^l\|_{C^0} + \tau\|\nabla^k \frac{\bar{D}}{\partial t} u^l\|_{C^0} \le C_k B_\lambda^{-1/2}\lambda(N^{1/2}\Xi)^k e'_R, \qquad k \ge 0 \tag{177}$$

where $e'_R = \frac{e_v^{1/4}e_R^{3/4}}{N^{1/2}}$ is the target size of the new stress R_1 expressed in (166). The amplitudes u^l are also supported in a cylinder

$$\operatorname{supp} u^l \subseteq \hat{C}_{v_\epsilon}(\frac{2\tau}{3}, A\rho; t(I), x(I))$$

of size $\rho \sim c_\rho \Xi^{-1}$, where c_ρ is the constant chosen in Section 7.5. See Section 26 of [Ise12] for details, particularly Section 26.2. Here the factors of $N^{1/2}$ come from the factors of $N^{1/2}$ in the estimates of Section 8.

In solving the first order, elliptic equation (176), we expect the solution Q to gain a factor λ^{-1} in the estimate $\|Q\|_{C^0} \le \lambda^{-1}\|u\|_{C^0}$ modulo lower order terms. In [DLS13], [DLS12a], De Lellis and Székelyhidi gave an approach to obtaining this cancellation based on the method of nonstationary phase. The approach we take here follows the approach in [Ise12], which is a slight adaptation of the method in [DLS13, DLS12a] generalized to nonlinear phase functions. The main distinction is that the approach we take does not involve proving that the operators $R^{jl}[U]$ we construct for solving the equation (176) exhibit cancellation when the input U has the form $U^l = e^{i\lambda\xi}u^l$. Instead, we obtain the necessary cancellation through a parametrix expansion of the solution. We also avoid the use of Schauder estimates, which would impose a super-exponential growth of frequencies in the iteration by requiring C^α rather than C^0 control of the data.

To begin, we write down a first order approximate solution to (176) of the form

$$Q_{(1)}^{jl} = \frac{1}{\lambda}e^{i\lambda\xi}q_{(1)}^{jl} \tag{178}$$

where the amplitude $q_{(1)}^{jl}$ is a symmetric tensor solving the underdetermined linear equation

$$i\partial_j\xi q_{(1)}^{jl} = u^l \tag{179}$$

pointwise. Following [Ise12], we begin constructing a solution to (179) by first decomposing u^l into

$$u^l = u_\perp^l + \frac{(u \cdot \nabla\xi)}{|\nabla\xi|^2}\partial^l\xi = u_\perp^l + u_\parallel^l,$$

so that $u_\perp \in \langle\nabla\xi\rangle^\perp$ and $u_\parallel \in \langle\nabla\xi\rangle$ pointwise. We then define

$$q_{(1)}^{jl} = -i(q_\perp^{jl} + q_\parallel^{jl}) = q^{jl}(\nabla\xi)[u], \tag{180}$$

where the tensors

$$q_\perp^{jl} = \frac{1}{|\nabla\xi|^2}(\partial^j\xi u_\perp^l + \partial^l\xi u_\perp^j), \quad q_\parallel^{jl} = \frac{(u \cdot \nabla\xi)}{|\nabla\xi|^2}\delta^{jl}$$

solve $\partial_j\xi q_\perp^{jl} = u_\perp^l$ and $\partial_j\xi q_\parallel^{jl} = u_\parallel^l$ pointwise.

The important properties of the map defined in[6] (180) are that $q^{jl}(\nabla\xi)[u]$ is linear in u, homogeneous of degree -1 in $\nabla\xi$, and smooth outside of $\nabla\xi = 0$. Thus, the main term $Q_{(0)}$ in (178) obeys the

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[6]Another example of a satisfactory map q^{jl} can be read off from the symbol of the operator in Definition 4.2 of [DLS13]. Our construction of (180) can likewise be regarded as giving the symbol of an order -1 operator which solves the symmetric divergence equation.

desired estimate $\|Q_{(1)}\|_{C^0} \leq C\lambda^{-1}\|u\|_{C^0}$, using the uniform bounds on $\| |\nabla\xi|^{-1} \|_{C^0}$ which are satisfied by all the phase functions involved in the construction. We can then construct an exact solution to (176) of the form $Q^{jl} = Q_{(1)} + \widetilde{Q}^{jl}_{(1)}$ by letting $\widetilde{Q}^{jl}_{(1)}$ solve the equation $\partial_j\widetilde{Q}^{jl}_{(1)} = -\frac{1}{\lambda}e^{i\lambda\xi}\partial_j q^{jl}_{(1)}$, noting that the right hand side now has a smaller amplitude than before thanks to the factor of $1/\lambda$.

To improve on the first order expansion (178), we build the solution to (176) as an approximate solution plus an error

$$Q^{jl} = Q^{jl}_{(D)} + \widetilde{Q}^{jl}_{(D)} \tag{181}$$

$$Q^{jl}_{(D)} = \sum_{k=1}^{D} \frac{1}{\lambda} e^{i\lambda\xi} q^{jl}_{(k)} \tag{182}$$

The amplitude $q^{jl}_{(k)}$ of the k'th term is obtained by solving the linear equation

$$i\partial_j\xi q^{jl}_{(k)} = u^l_{(k)}, \quad u^l_{(1)} = u^l, \qquad u^l_{(k+1)} = \frac{-1}{\lambda}\partial_j q^{jl}_{(k)} \tag{183}$$

using the function $q^{jl}_{(k)} = q^{jl}(\nabla\xi)[u_{(k)}]$ defined in (180). For Q^{jl} to be a solution of (176), the remainder term in (181) must be chosen to solve the equation

$$\partial_j\widetilde{Q}^{jl}_{(D)} = e^{i\lambda\xi}u^l_{(D+1)} \tag{184}$$

Thanks to the estimate (177), the bounds on the amplitude $u_{(k)}$ become smaller with each iteration of the parametrix by a factor of

$$\frac{|\nabla|}{\lambda} \leq CB_\lambda^{-1}\frac{N^{1/2}\Xi}{N\Xi} \leq CB_\lambda^{-1}N^{-1/2} \tag{185}$$

After taking D terms in the expansion, the bounds for $u^l_{(D)}$ have the form

$$\|\nabla^k u^l_{(D)}\|_{C^0} + \tau\|\nabla^k\frac{\bar{D}}{\partial t}u^l_{(D)}\|_{C^0} \leq C_k B_\lambda^{-D}N^{-D/2}B_\lambda^{-1/2}\lambda(N^{1/2}\Xi)^k e'_R \tag{186}$$

In particular, since $\lambda = B_\lambda N\Xi$, for $D \geq 2$ we have

$$\|u^l_{(D)}\|_{C^0} + \tau\|\frac{\bar{D}}{\partial t}u_{(D)}\|_{C^0} \leq C_k B_\lambda^{-1}N^{-D/2+1}\Xi e'_R \tag{187}$$

Our goal is to make sure the solution $\widetilde{Q}^{jl}_{(D)}$ to (184) has C^0 norm bounded by a multiple of e'_R. In previous constructions of Hölder continuous solutions on the torus, it has been necessary to assume a super-exponential growth of frequencies (i.e. $N \geq \Xi^\eta$ for some $\eta > 0$), so that the estimate (187) gains a power of $N^{-D/2+1} \leq N^{-1}\Xi^{-1} \approx \lambda^{-1}$ once D is chosen sufficiently large. In our case, however, we will gain a smallness factor of $\rho \sim \Xi^{-1}$ from our new method of solving Equation (184), thus eliminating the apparent need for super-exponential growth of frequencies.

We take $D = 3$, which leaves us with the following estimate for the amplitude in (184)

$$\|u^l_{(D+1)}\|_{C^0} + \tau\|\frac{\bar{D}}{\partial t}u_{(D+1)}\|_{C^0} \leq CB_\lambda^{-1}\Xi e'_R \tag{188}$$

This choice of D leads also to the estimates

$$\|\nabla^k u^l_{(D+1)}\|_{C^0} + \tau\|\nabla^k\frac{\bar{D}}{\partial t}u^l_{(D+1)}\|_{C^0} \leq C_k B_\lambda^{-1}(N^{1/2}\Xi)^k\Xi e'_R \tag{189}$$

The data $U^l_{(D+1)} = e^{i\lambda\xi}u^l_{(D+1)}$ on the right hand side of (184) now obeys the estimates

$$\|\nabla^k U^l_{(D+1)}\|_{C^0} + \tau\|\nabla^k \frac{\bar{D}}{\partial t}U^l_{(D+1)}\|_{C^0} \leq C_k B_\lambda^{-1}(B_\lambda N\Xi)^k \Xi e'_R \tag{190}$$

According to Theorem 10.1, there is a solution $\widetilde{Q}^{jl}_{(D)}$ to the equation (184) with support in the same Eulerian cylinder

$$\mathrm{supp}\, \widetilde{Q}^{jl}_{(D)} \subseteq \hat{C}_{v_\epsilon}(\frac{2\tau}{3}, A\rho; t(I), x(I))$$

such that $\widetilde{Q}^{jl}_{(D)}$ obeys the estimates

$$\|\nabla^k \widetilde{Q}^{jl}_{(D)}\|_{C^0} + \tau\|\nabla^k \frac{\bar{D}}{\partial t}\widetilde{Q}^{jl}_{(D)}\|_{C^0} \leq C_k B_\lambda^{-1}(B_\lambda N\Xi)^k e'_R \tag{191}$$

We emphasize in particular that the estimate for the solution of Theorem 10.1 gains a factor of $A\rho \sim \Xi^{-1}$, which is consistent with dimensional analysis of the equation.

If B_λ is sufficiently large, then we can guarantee that each term $\widetilde{Q}^{jl}_{(D)}$ has size bounded by $\|\widetilde{Q}^{jl}_{(D)}\|_{C^0} \leq \frac{1}{B}e'_R$ where B can be any large constant. In particular, by Proposition 6.2 on limited interactions, we can guarantee that the sum of all stress terms $\widetilde{Q}^{jl}_{(D)}$ obtained by this procedure is bounded uniformly by $\frac{1}{500}e'_R$, and that the bound (191) is also satisfied for the sum of these terms.

On the other hand, the parametrix term (182) also satisfies the same estimate (191), with the main term contribution coming from the first term $Q^{jl}_{(1)}$, and the number of such $Q^{jl}_{(D)}$ which are nonzero at any given point is likewise bounded. Choosing B_λ sufficiently large, we can therefore guarantee that the entire contribution $Q^{jl}_O = \sum_I Q^{jl}_{O,I}$ to the stress R_1 obeys the estimates

$$\|Q_O\|_{C^0} \leq \frac{1}{40}e'_R \tag{192}$$

At last, we choose B_λ so that the bound (192) is satisfied, which implies the desired bound for

$$\|R_1\|_{C^0} \leq \|Q_M\|_{C^0} + \|Q_S\|_{C^0} + \|Q_O\|_{C^0} \leq e'_R \tag{193}$$

With the construction fully determined, it now remains to check that the spatial and advective derivatives of R_1 obey the bounds demanded by the Main Lemma (Lemma 3.1).

With the above choice of B_λ, we obtain

$$\|\nabla^k Q^{jl}_O\|_{C^0} + \tau\|\nabla^k \frac{\bar{D}}{\partial t}Q^{jl}_O\|_{C^0} \leq C_k(N\Xi)^k e'_R \tag{194}$$

The bound (194) is clearly enough to conclude that the new frequency-energy levels are satisfied for the spatial derivatives of Q_O, as the cost of a spatial derivative is at most $|\nabla| \leq C_0 N\Xi$. Also, the cost of taking an advective derivative is bounded by

$$\left|\frac{\bar{D}}{\partial t}\right| \leq \tau^{-1} = C\left(\frac{e_v^{1/2}}{e_R^{1/2}N}\right)^{-1/2} \Xi e_v^{1/2} \tag{195}$$

which is no larger than the required estimate

$$\left|\frac{\bar{D}}{\partial t}\right| \unlhd \Xi'(e'_v)^{1/2} = C_0 N\Xi e_R^{1/2} \tag{196}$$

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thanks to the condition $N \geq \left(\frac{e_v^{1/2}}{e_R^{1/2}} \right)$. From (196), it is straightforward to conclude the necessary bounds on

$$(\partial_t + v_1 \cdot \nabla)Q_O = \frac{\bar{D}}{\partial t}Q_O + (v - v_\epsilon) \cdot \nabla Q_O + V \cdot \nabla Q_O$$

using the spatial derivative bounds (194). Namely, the derivative $(\partial_t + v_1 \cdot \nabla)$ costs at most

$$|(\partial_t + v_1 \cdot \nabla)| \leq \left| \frac{\bar{D}}{\partial t} \right| + |(v - v_\epsilon) \cdot \nabla| + |V \cdot \nabla| \tag{197}$$

$$\leq \left| \frac{\bar{D}}{\partial t} \right| + C\frac{e_v^{1/2}}{N}(N\Xi) + Ce_R^{1/2}(N\Xi) \tag{198}$$

$$\leq CN\Xi e_R^{1/2} \tag{199}$$

as desired. One can then take spatial derivatives up to order $L - 1$ for each term at a cost of at most $|\nabla| \leq C_0 N\Xi$ per derivative as desired, which is carried out in detail in Sections 24 - 26 of [Ise12]. Combining the above estimates with the bounds for the terms Q_M and Q_S already estimated in [Ise12], we conclude our proof of the Main Lemma (Lemma 3.1).

10 Solving the Symmetric Divergence Equation

We now present our method of solving the underdetermined elliptic equation from which we recover the new stress in the construction. The analysis in this Section is independent of the earlier part of this paper, and in particular holds on $\mathbb{R} \times \mathbb{R}^d$ for any $d \geq 2$.

For a symmetric tensor $R^{jl} = R^{lj}$ and vector field U^l on $\mathbb{R}^d$, consider the divergence equation

$$\partial_j R^{jl} = U^l. \tag{200}$$

In what follows, (200) will be referred to as the *symmetric divergence equation*.

10.1 Main result for the symmetric divergence equation

The following is our main result regarding compactly supported solutions to the symmetric divergence equation (200).

Theorem 10.1 (Compactly supported solutions to the symmetric divergence equation)**.** *Let* A, N, Ξ, e_v *be positive numbers,* $L \geq 1$ *be a positive integer and* $v_\epsilon = (v_\epsilon^1, \ldots, v_\epsilon^d)$ *be a vector field on* $\mathbb{R} \times \mathbb{R}^d$ *such that*

$$\|\nabla^\beta v_\epsilon\|_{C^0_{t,x}} \leq A\Xi^{|\beta|}e_v^{1/2}, \qquad 1 \leq |\beta| \leq L. \tag{201}$$

Furthermore, let U^l *be a vector field with zero linear and angular momenta, i.e.,*

$$\int U^l(t,x)\,dx = 0, \quad \int (x^j U^l - x^l U^j)(t,x)\,dx = 0 \tag{202}$$

for all t*, and such that*

$$\operatorname{supp} U \subseteq \hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t(I), x(I)), \tag{203}$$

for some $(t(I), x(I)) \in \mathbb{R} \times \mathbb{R}^d$ *and* $0 < \bar{\tau} \leq \Xi^{-1}e_v^{-1/2}$*. Assume also that for*

$$\Lambda > 0, \qquad 0 < \hat{\tau} \leq \Xi^{-1}e_v^{-1/2},$$

the vector field U obeys the estimates

$$
\begin{aligned}
\|\nabla^\beta U\|_{C^0_{t,x}} &\leq A\Lambda^{|\beta|} & |\beta| = 0,\ldots,L, \\
\|\nabla^\beta(\partial_t + v_\epsilon \cdot \nabla)U\|_{C^0_{t,x}} &\leq A\hat{\tau}^{-1}\Lambda^{|\beta|} & |\beta| = 0,\ldots,L-1.
\end{aligned}
\tag{204}
$$

Then there exists a solution $R^{jl}[U]$ to the symmetric divergence equation (200), depending linearly on U^l, with the following properties:

1. The support of $R^{jl}[U]$ stays in the cylinder $\hat{C}_{v_\epsilon}(\bar{\tau},\bar{\rho};t(I),x(I))$, i.e.,

$$
\operatorname{supp} R^{jl}[U] \subseteq \hat{C}_{v_\epsilon}(\bar{\tau},\bar{\rho};t(I),x(I)).
\tag{205}
$$

2. There exists $C > 0$ such that for $|\beta| = 0,\ldots,L,$

$$
\|\nabla^\beta R^{jl}[U]\|_{C^0_{t,x}} \leq CA\bar{\rho} \sum_{m=0}^{|\beta|} \bar{\rho}^{-(|\beta|-m)}\Lambda^m
\tag{206}
$$

3. There exists $C > 0$ such that for $|\beta| = 0,\ldots,L-1,$

$$
\|\nabla^\beta(\partial_t + v_\epsilon \cdot \nabla)R^{jl}[U]\|_{C^0_{t,x}} \leq CA\hat{\tau}^{-1}\bar{\rho} \sum_{m_0+m_1+m_2=|\beta|} \Xi^{m_0}\bar{\rho}^{-m_1}\Lambda^{m_2}
\tag{207}
$$

where the sum is over all triplets of non-negative integers (m_0,m_1,m_2) such that $m_0+m_1+m_2 = |\beta|$.

Remark. This theorem should be compared with Theorem 27.1 in [Ise12], which was proved by solving a transport equation[7] using a Helmholtz-type solution operator. The key difference is, of course, that the present theorem preserves the support property (203) whereas Theorem 27.1 in [Ise12] does not. Furthermore, note that Theorem 27.1 in [Ise12] gives estimates in L^p_x with $1 < p < \infty$ (more specifically, $p = 4$), whereas the present theorem operates directly in $C^0_{t,x}$. In accordance with scaling, the $C^0_{t,x}$ estimate gain a factor of $\bar{\rho}$ (i.e., the spatial scale of U^l), which is crucial for removing the super-exponential growth assumption $N \geq \Xi^\eta$ in the Main Lemma.

Finally, we remark that by computing the kernel of the integral operator more carefully, it can be shown that $R^{jl}[U]$ is a classical pseudodifferential operator of order -1. In particular, by Calderón-Zygmund theory $R^{jl}[U]$ gains one derivative in L^p_x for $1 < p < \infty$, as in Theorem 27.1 in [Ise12]. We have elected not to give a detailed proof, as this statement is not used in the present paper. See [OT15] for the analysis of the case of the divergence equation $\partial_l R^l = U$.

10.2 Derivation of the solution operator

The purpose of this subsection is to give a derivation of the solution operator $R^{jl}[U]$ for (200) in Theorem 10.1.

For the moment, we shall omit the time variable and work entirely on $\mathbb{R}^d$. Let $U^l = U^l(x)$ be a vector field supported on some ball $B(\bar{\rho};x_0)$. For simplicity, we will furthermore assume that U is smooth and $x_0 = 0$.

---

[7]We remark that the method used in [Ise12] seems to be very special to the torus. A key ingredient in this approach is that the transport by a divergence free vector field preserves the integral zero property, which is the only necessary condition to solve the symmetric divergence equation on $\mathbb{T}^3$. On the other hand, the orthogonality conditions from angular momentum conservation seem to prevent such an approach from applying to the whole space.

Our first idea is to use the Fourier transform and Taylor expand $\widehat{U}^l(\xi)$ about the frequency origin $\xi = 0$ in the Fourier space. We will then try to write the terms of the Taylor expansion as the divergence of a symmetric tensor, up to some terms evaluated at $\xi = 0$. Translating the resulting formula to the physical space, we shall obtain a solution operator which possess the desired (physical space) support property, albeit with a mild singularity at 0.

Indeed, we first compute

$$
\begin{aligned}
\widehat{U}^l(\xi) =& \widehat{U}^l(0) + \Big(\int_0^1 \partial^k \widehat{U}^l(\sigma\xi)\, d\sigma \Big) \xi_k \\
=& \widehat{U}^l(0) + \frac{1}{2}\Big(\int_0^1 (\partial^k \widehat{U}^l + \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_k + \frac{1}{2}\Big(\int_0^1 (\partial^k \widehat{U}^l - \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_k \\
=& \widehat{U}^l(0) + \frac{1}{2}(\partial^k \widehat{U}^l - \partial^l \widehat{U}^k)(0)\xi_k + \frac{1}{2}\Big(\int_0^1 (\partial^k \widehat{U}^l + \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_k \\
& + \frac{1}{2}\Big(\int_0^1 (1-\sigma)(\partial^j \partial^k \widehat{U}^l - \partial^j \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_j\xi_k.
\end{aligned}
$$

Note that the third term on the right-hand side is (formally) the Fourier transform of a divergence of a symmetric tensor. The last term may also be written as a sum of two terms of such type by the following calculation:

$$
\begin{aligned}
\frac{1}{2}\Big(\int_0^1 (1-\sigma)(\partial^j \partial^k \widehat{U}^l - \partial^j \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_j\xi_k =& \frac{1}{2}\Big(\int_0^1 (1-\sigma)\xi_k(\partial^j \partial^k \widehat{U}^l + \partial^l \partial^k \widehat{U}^j)(\sigma\xi)\, d\sigma \Big)\xi_j \\
& - \Big(\int_0^1 (1-\sigma)\xi_k(\partial^l \partial^j \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_j
\end{aligned}
$$

Therefore, assuming

$$
\widehat{U}^l(0) = \int U^l(x)\, dx = 0, \qquad \Big(\frac{1}{i}\partial^k \widehat{U}^l - \frac{1}{i}\partial^l \widehat{U}^k\Big)(0) = \int (x^l U^k - x^k U^l)(x)\, dx = 0, \tag{208}
$$

which is equivalent to the assumption (202) on U, the following formula for $\widehat{U}^l(\xi)$ holds:

$$
\begin{aligned}
\widehat{U}^l(\xi) =& \frac{1}{2}\Big(\int_0^1 (\partial^j \widehat{U}^l + \partial^l \widehat{U}^j)(\sigma\xi)\, d\sigma \Big)\xi_j \\
& + \frac{1}{2}\Big(\int_0^1 (1-\sigma)\xi_k(\partial^j \partial^k \widehat{U}^l + \partial^l \partial^k \widehat{U}^j)(\sigma\xi)\, d\sigma \Big)\xi_j \\
& - \Big(\int_0^1 (1-\sigma)\xi_k(\partial^l \partial^j \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\xi_j
\end{aligned}
$$

Let us define $r^{jl}[U] := r_0^{jl}[U] + r_1^{jl}[U] + r_2^{jl}[U]$, where

$$
r_0^{jl}[U] = \mathcal{F}^{-1}\Big[\frac{1}{2}\Big(\int_0^1 \Big(\frac{1}{i}\partial^j \widehat{U}^l + \frac{1}{i}\partial^l \widehat{U}^j\Big)(\sigma\xi)\, d\sigma \Big)\Big] \tag{209}
$$

$$
r_1^{jl}[U] = \mathcal{F}^{-1}\Big[\frac{1}{2}\Big(\int_0^1 (1-\sigma)(i\xi_k)(-\partial^j \partial^k \widehat{U}^l - \partial^l \partial^k \widehat{U}^j)(\sigma\xi)\, d\sigma \Big)\Big], \tag{210}
$$

$$
r_2^{jl}[U] = \mathcal{F}^{-1}\Big[-\Big(\int_0^1 (1-\sigma)(i\xi_k)(-\partial^j \partial^l \widehat{U}^k)(\sigma\xi)\, d\sigma \Big)\Big]. \tag{211}
$$

Computing the inverse Fourier transform, we arrive at the formal formulae

$$r_0^{jl}[U] = -\frac{1}{2}\int_0^1 \left(\frac{x^j}{\sigma}U^l(\frac{x}{\sigma}) + \frac{x^l}{\sigma}U^j(\frac{x}{\sigma})\right)\frac{d\sigma}{\sigma^d}, \tag{212}$$

$$r_1^{jl}[U] = \frac{1}{2}\frac{\partial}{\partial x^k}\int_0^1 (1-\sigma)\left(\frac{x^j x^k}{\sigma^2}U^l(\frac{x}{\sigma}) + \frac{x^l x^k}{\sigma^2}U^j(\frac{x}{\sigma})\right)\frac{d\sigma}{\sigma^d}, \tag{213}$$

$$r_2^{jl}[U] = -\frac{\partial}{\partial x^k}\int_0^1 (1-\sigma)\left(\frac{x^l x^j}{\sigma^2}U^k(\frac{x}{\sigma})\right)\frac{d\sigma}{\sigma^d}. \tag{214}$$

Thus, for $a = 0, 1, 2$, the values of $r_a^{jl}[U]$ at a point $x \in \mathbb{R}^3$ are given formally as weighted integrals of U and ∇U along the ray emanating from x away from the origin.

In fact, when interpreted correctly, these expressions already give us a *distributional* solution to (200) with the desired support property

$$\operatorname{supp} r^{jl} \subseteq B(\bar\rho; 0), \tag{215}$$

but with a singularity at $x = 0$. Indeed, given a test function $\varphi \in C_c^\infty$, we will define

$$\langle r_0^{jl}[U], \varphi\rangle := -\lim_{\delta\to 0+}\frac{1}{2}\int_\delta^1\int\left(\frac{x^j}{\sigma}U^l(\frac{x}{\sigma}) + \frac{x^l}{\sigma}U^j(\frac{x}{\sigma})\right)\varphi(x)\,dx\frac{d\sigma}{\sigma^d}, \tag{216}$$

$$\langle r_1^{jl}[U], \varphi\rangle := -\frac{1}{2}\lim_{\delta\to 0+}\int_\delta^1 (1-\sigma)\int\left(\frac{x^j x^k}{\sigma^2}U^l(\frac{x}{\sigma}) + \frac{x^l x^k}{\sigma^2}U^j(\frac{x}{\sigma})\right)\partial_k\varphi(x)\,dx\frac{d\sigma}{\sigma^d}, \tag{217}$$

$$\langle r_2^{jl}[U], \varphi\rangle := \lim_{\delta\to 0+}\int_\delta^1 (1-\sigma)\int\left(\frac{x^l x^j}{\sigma^2}U^k(\frac{x}{\sigma})\right)\partial_k\varphi(x)\,dx\frac{d\sigma}{\sigma^d}. \tag{218}$$

These are well-defined (tempered) distributions on $\mathbb{R}^d$. Indeed, by a simple change of variables, we see that

$$|\langle r_0^{jl}[U], \varphi\rangle| \le C_{U,\bar\rho}\|\varphi\|_{C_x^0}, \quad |\langle r_1^{jl}[U], \varphi\rangle| + |\langle r_2^{jl}[U], \varphi\rangle| \le C_{U,\bar\rho}\|\nabla\varphi\|_{C_x^0}.$$

The support property (215) and smoothness outside $\{x = 0\}$ follow immediately from the definition. Moreover, a straightforward computation with distributions shows that

$$\partial_j r^{jl}[U] = U^l - \left(\int U^l(x)\,dx\right)\delta_0 - \frac{1}{2}\left(\int (x^l U^j - x^j U^l)(x)\,dx\right)\partial_j\delta_0. \tag{219}$$

Thus, under the assumption (208), we see that $r^{jl}[U]$ is a distributional solution to (200).

Unfortunately, $r^{jl}[U]$ as defined above apparently has a singularity at $x = 0$. We will overcome this difficulty by exploiting translation invariance of (200); more precisely, we will conjugate $r^{jl}[U]$ by translations and take a smooth average of the resulting formulae, ultimately 'smearing out' the singularity.

Given $y \in \mathbb{R}^d$, let us conjugate the operators r_0^{jl}, r_1^{jl} and r_2^{jl} by translation by y. Then we are led to the conjugated operator $^{(y)}r^{jl} = {}^{(y)}r_0^{jl} + {}^{(y)}r_1^{jl} + {}^{(y)}r_2^{jl}$, which is formally defined by

$$^{(y)}r_0^{jl}[U] = -\frac{1}{2}\int_0^1 \frac{(x-y)^j}{\sigma}U^l(\frac{x-y}{\sigma}+y) + \frac{(x-y)^l}{\sigma}U^j(\frac{x-y}{\sigma}+y)\frac{d\sigma}{\sigma^d}, \tag{220}$$

$$^{(y)}r_1^{jl}[U] = \frac{1}{2}\frac{\partial}{\partial x^k}\int_0^1 (1-\sigma)\frac{(x-y)^j(x-y)^k}{\sigma^2}U^l(\frac{x-y}{\sigma}+y)\frac{d\sigma}{\sigma^d} \tag{221}$$
$$+ \frac{1}{2}\frac{\partial}{\partial x^k}\int_0^1 (1-\sigma)\frac{(x-y)^l(x-y)^k}{\sigma^2}U^j(\frac{x-y}{\sigma}+y)\frac{d\sigma}{\sigma^d},$$

$$^{(y)}r_2^{jl}[U] = -\frac{\partial}{\partial x^k}\int_0^1 (1-\sigma)\frac{(x-y)^l(x-y)^j}{\sigma^2}U^k(\frac{x-y}{\sigma}+y)\frac{d\sigma}{\sigma^d}. \tag{222}$$

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These are to be interpreted as in (216)–(218) as distributions. Note that as long as $y \in B(\bar{\rho}; 0)$, the distribution $^{(y)}r^{jl}[U]$ satisfies the desired support property (215). Motivated by this consideration, let us take a smooth function $\zeta(y)$ which is supported in $B(\bar{\rho}; 0)$ and satisfies

$$\int \zeta(y)dy = 1 \tag{223}$$

We now define the solution operator $^{(\zeta)}\widetilde{R}^{jl}[U]$ by averaging $^{(y)}r^{jl}[U]$ against ζ, i.e.,

$$^{(\zeta)}\widetilde{R}^{jl}[U](x) = \int {}^{(y)}r^{jl}(x)\zeta(y)\, dy. \tag{224}$$

We will finally obtain the solution operator $R^{jl}[U]$ of Theorem 10.1 by making an appropriate choice of ζ depending on time.

From the discussion above, we see that $^{(\zeta)}\widetilde{R}^{jl}[U]$ inherits the desirable properties of $r^{jl}[U]$. Indeed, assuming (208), $^{(\zeta)}\widetilde{R}^{jl}[U]$ is a (distributional) solution to (200) satisfying the support property

$$\operatorname{supp}{}^{(\zeta)}\widetilde{R}^{jl}[U] \subseteq B(\bar{\rho}; 0). \tag{225}$$

As we shall see below, thanks to averaging with respect to ζ, $^{(\zeta)}\widetilde{R}^{jl}[U]$ will moreover turn out to be smooth in the spatial variables provided U is smooth as well (see in particular the calculations (240) and (241) below).

10.3 Formula for $R^{jl}[U]$ and basic properties

Let U^l be a vector field satisfying the hypotheses (202) and (203) of Theorem 10.1. Denote by $\bar{v}_\epsilon(t)$ the value of the coarse scale velocity v_ϵ at $\Phi_{t-t(I)}(t(I), x(I))$. For $t \in [t(I) - \bar{\tau}, t(I) + \bar{\tau}]$, this point is exactly the center of the cross-section $\hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t(I), x(I)) \cap \{t\} \times \mathbb{R}^d$.

Recall from the previous subsection that we need to choose a (spatially) smooth function ζ with integral 1 in order to determine our solution operator for (200). We shall define a function $\zeta = \zeta(t, x)$ adapted to $\hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t(I), x(I))$ according to the following procedure: given a smooth function $\widetilde{\zeta} = \widetilde{\zeta}(x)$ with $\operatorname{supp}\widetilde{\zeta} \subseteq B(\bar{\rho}; x(I))$ and $\int \widetilde{\zeta}(x)dx = 1$, let ζ be the solution to the transport equation

$$\begin{cases} (\partial_t + \bar{v}_\epsilon^j(t)\partial_j)\zeta(t, x) = 0 & \text{for } t \in [t(I) - \bar{\tau}, t(I) + \bar{\tau}], \\ \qquad\qquad \zeta(t(I), x) = \widetilde{\zeta}(x). \end{cases} \tag{226}$$

Note that ζ satisfies the support property

$$\operatorname{supp}\zeta \subseteq \hat{C}_{v_\epsilon}(\bar{\tau}, \bar{\rho}; t(I), x(I)). \tag{227}$$

and also satisfies $\int_{\mathbb{R}^d} \zeta(t, y)dy = 1$ at all times t.

Moreover, choosing $\widetilde{\zeta}$ to be a bump function adapted to $B(\bar{\rho}; x(I))$, the following estimates hold for ζ:

$$\|\nabla^\beta \zeta\|_{C^0_{t,x}} \le C_\beta \bar{\rho}^{-d-|\beta|} \qquad \text{for all } |\beta| \ge 0. \tag{228}$$

We are now ready to define the solution operator $R^{jl}[U]$. Let $R^{jl}[U] := R_0^{jl}[U] + R_1^{jl}[U] + R_2^{jl}[U]$,

where

$$R_0^{jl}[U] = -\frac{d}{2}\int_0^1 \int \zeta(t,y)\frac{(x-y)^j}{\sigma}U^l(t,\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma \qquad (229)$$

$$-\frac{d}{2}\int_0^1 \int \zeta(t,y)\frac{(x-y)^l}{\sigma}U^j(t,\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma,$$

$$R_1^{jl}[U] = \frac{1}{2}\int_0^1 \int (\partial_k\zeta)(t,y)\frac{(x-y)^j(x-y)^k}{\sigma^2}U^l(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma \qquad (230)$$

$$+\frac{1}{2}\int_0^1 \int (\partial_k\zeta)(t,y)\frac{(x-y)^l(x-y)^k}{\sigma^2}U^j(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma,$$

$$R_2^{jl}[U] = -\int_0^1 \int (\partial_k\zeta)(t,y)\frac{(x-y)^j(x-y)^l}{\sigma^2}U^k(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma. \qquad (231)$$

The following Proposition summarizes the basic properties of the operator $R^{jl}[U]$.

Proposition 10.1. *Let U^l be a vector field on $\mathbb{R}\times\mathbb{R}^d$ satisfying the hypotheses* (202) *and* (203) *of Theorem 10.1. Define $R^{jl}[U] := R_0^{jl}[U] + R_1^{jl}[U] + R_2^{jl}[U]$ by* (229), (230) *and* (231). *Then $R^{jl}[U]$ possesses the following properties:*

1. *$R^{jl}[U]$ is symmetric in j,l and depends linearly on U.*

2. *$R^{jl}[U]$ solves the symmetric divergence equation, i.e.,*

$$\partial_j R^{jl}[U] = U^l.$$

3. *$R^{jl}[U]$ has the support property*

$$\operatorname{supp} R^{jl}[U] \subseteq \hat{C}_{v_\epsilon}(\bar{\tau},\bar{\rho};t(I),x(I)). \qquad (232)$$

4. *The following differentiation formulae hold for $R_a^{jl}[U]$ ($a = 0,1,2$):*

$$\nabla^\beta R_0^{jl}[U] = -\frac{d}{2}\sum_{\beta_1+\beta_2=\beta}\int_0^1 \int (\nabla^{\beta_1}\zeta)(t,y)\frac{(x-y)^j}{\sigma}(\nabla^{\beta_2}U^l)(t,\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma \qquad (233)$$

$$-\frac{d}{2}\sum_{\beta_1+\beta_2=\beta}\int_0^1 \int (\nabla^{\beta_1}\zeta)(t,y)\frac{(x-y)^l}{\sigma}(\nabla^{\beta_2}U^j)(t,\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma,$$

$$\nabla^\beta R_1^{jl}[U] = \frac{1}{2}\sum_{\beta_1+\beta_2=\beta}\int_0^1 \int (\nabla^{\beta_1}\partial_k\zeta)(t,y)\frac{(x-y)^j(x-y)^k}{\sigma^2}(\nabla^{\beta_2}U^l)(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma \qquad (234)$$

$$+\frac{1}{2}\sum_{\beta_1+\beta_2=\beta}\int_0^1 \int (\nabla^{\beta_1}\partial_k\zeta)(t,y)\frac{(x-y)^l(x-y)^k}{\sigma^2}(\nabla^{\beta_2}U^j)(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma,$$

$$\nabla^\beta R_2^{jl}[U] = -\sum_{\beta_1+\beta_2=\beta}\int_0^1 \int (\nabla^{\beta_1}\partial_k\zeta)(t,y)\frac{(x-y)^j(x-y)^l}{\sigma^2}(\nabla^{\beta_2}U^k)(\frac{x-y}{\sigma}+y)\frac{dy}{\sigma^d}d\sigma. \qquad (235)$$

where the summations are over all pairs of multi-indices (β_1,β_2) such that $\beta_1 + \beta_2 = \beta$.

5. *Define the* approximate advective derivative *$\frac{\overline{\overline{D}}}{\partial t}$ to be $\frac{\overline{\overline{D}}}{\partial t} := \partial_t + \bar{v}_\epsilon(t)\cdot\nabla$. Then R^{jl} commutes with $\frac{\overline{\overline{D}}}{\partial t}$, i.e.,*

$$\frac{\overline{\overline{D}}}{\partial t}R_a^{jl}[U] = R_a^{jl}[\frac{\overline{\overline{D}}}{\partial t}U] \qquad a = 0,1,2. \qquad (236)$$

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Proof. Symmetry in j, l, linear dependence on U and the support property (232) may be easily read off from the definition (229)–(231). Next, we prove the differentiation formulae (233)–(235) and (236).

To justify the various calculations to follow (such as differentiating under the integral sign), the following lemma, whose proof will be given in the next subsection, will be useful:

Lemma 10.1. *Let $\widetilde{\zeta}$ be a non-negative smooth function with $\operatorname{supp}\widetilde{\zeta} \subseteq B(\bar{\rho}; x_0)$ such that*

$$\|\widetilde{\zeta}\|_{C_x^0} \le C_\beta A \bar{\rho}^{-d} \tag{237}$$

for some $A > 0$. Then for any $k \ge 0$ and $f \in L_x^\infty$ supported in $B(\bar{\rho}; x_0)$, we have

$$\sup_{x \in \mathbb{R}^d, \sigma \in [0,1]} \Big| \int \widetilde{\zeta}(y) \Big(\frac{|x-y|}{\sigma} \Big)^k f\Big(\frac{x-y}{\sigma} + y \Big) \frac{dy}{\sigma^d} \Big| \le C_k A \bar{\rho}^k \|f\|_{L_x^\infty}. \tag{238}$$

In order to establish (233), it suffices to prove the case $|\beta| = 1$, i.e.,

$$\begin{aligned}
\partial_m R_0^{jl}[U](t,x) = &-\frac{d}{2} \int_0^1 \int (\partial_m \zeta)(t,y) \frac{(x-y)^j}{\sigma} U^l\Big(t, \frac{x-y}{\sigma} + y\Big) \frac{dy}{\sigma} d\sigma \\
&-\frac{d}{2} \int_0^1 \int \zeta(t,y) \frac{(x-y)^j}{\sigma} (\partial_m U^l)\Big(t, \frac{x-y}{\sigma} + y\Big) \frac{dy}{\sigma} d\sigma \\
&+ \text{(Symmetric terms in } j, l).
\end{aligned} \tag{239}$$

The case of $|\beta| > 1$ will follow from an induction argument, using similar ideas. To prove (239), we first proceed as follows:

$$\begin{aligned}
\frac{\partial}{\partial x^m} R_0^{jl}[U](t,x) = &\frac{\partial}{\partial z^m} R_0^{jl}[U](t, x+z)\Big|_{z=0} \\
= &-\frac{d}{2} \frac{\partial}{\partial z^m}\Big|_{z=0} \int_0^1 \int \zeta(t,y) \frac{(x+z-y)^j}{\sigma} U^l\Big(t, \frac{x+z-y}{\sigma} + y\Big) \frac{dy}{\sigma^d} d\sigma \\
&-\frac{d}{2} \frac{\partial}{\partial z^m}\Big|_{z=0} \int_0^1 \int \zeta(t,y) \frac{(x+z-y)^l}{\sigma} U^j\Big(t, \frac{x+z-y}{\sigma} + y\Big) \frac{dy}{\sigma^d} d\sigma,
\end{aligned} \tag{240}$$

Let us concentrate on the first term on the right-hand side; the other term is symmetric to the first one in j, l. Making a change of variable $\widetilde{y} = y - z$, we get

$$\begin{aligned}
&-\frac{d}{2} \frac{\partial}{\partial z^m}\Big|_{z=0} \int_0^1 \int \zeta(t,y) \frac{(x+z-y)^j}{\sigma} U^l\Big(t, \frac{x+z-y}{\sigma} + y\Big) \frac{dy}{\sigma^d} d\sigma \\
&= -\frac{d}{2} \frac{\partial}{\partial z^m}\Big|_{z=0} \int_0^1 \int \zeta(t, \widetilde{y}+z) \frac{(x-\widetilde{y})^j}{\sigma} U^l\Big(t, \frac{x-\widetilde{y}}{\sigma} + \widetilde{y} + z\Big) \frac{d\widetilde{y}}{\sigma^d} d\sigma
\end{aligned}$$

Now differentiating under the integral sign, which is justified by (227), (228), Lemma 10.1 and the smoothness of U, we get the desired formula (239).

The proofs of (234) and (235) are similar and thus omitted. The formula (236) is also proved in a similar manner, starting from

$$\frac{\overline{\overline{D}}}{\partial t} R_a^{jl}[U](t,x) = \frac{d}{ds} R_a^{jl}[U]\Big(t+s, x + \int_t^s \bar{v}_\epsilon(s')\, ds'\Big)\Big|_{s=0}$$

for $a = 0, 1, 2$. We also use the fact that $\frac{\overline{\overline{D}}}{\partial t} \nabla^\beta \zeta = 0$ for any $|\beta| \ge 0$ by construction. We omit the details.

Now, it only remains to prove that $R^{jl}[U]$ is a (distributional) solution to (200) under the assumption (202). For this purpose, it suffices to show that

$$R^{jl}[U](t,x) = {}^{(\zeta(t,\cdot))}\widetilde{R}^{jl}[U(t,\cdot)](x).$$

where ${}^{(\zeta(t,\cdot))}\widetilde{R}^{jl}[U(t,\cdot)]$ has been defined in the previous subsection.

To arrive at the formulae (229)–(231), we need to integrate by parts the derivative ∂_k on the outside of (221) and (222) after averaging against $\zeta(y)$. More precisely, consider the expression

$$\widetilde{R}_2^{jl}[U](t,x) := \int \zeta(t,y)^{(y)} r_2^{jl}[U(t,\cdot)](x)\,dy.$$

Using (227), (228), Lemma 10.1 and the differentiation formulae that we established, it is not difficult to justify the following chain of identities:

$$\widetilde{R}_2^{jl}[U](t,x) = -\frac{\partial}{\partial x^k}\int_0^1\int(1-\sigma)\zeta(t,y)\frac{(x-y)^l(x-y)^j}{\sigma^2}U^k\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$= -\int_0^1\int(1-\sigma)(\partial_k\zeta)(t,y)\frac{(x-y)^l(x-y)^j}{\sigma^2}U^k\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma \qquad (241)$$

$$-\int_0^1\int(1-\sigma)\zeta(t,y)\frac{(x-y)^l(x-y)^j}{\sigma^2}(\partial_kU^k)\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma.$$

Note that

$$(\partial_kU^k)\left(t,\frac{x-y}{\sigma}+y\right) = -\frac{\sigma}{1-\sigma}\frac{\partial}{\partial y^k}\left[U^k\left(t,\frac{x-y}{\sigma}+y\right)\right]$$

which may be integrated by parts in y. As a result, we arrive at the formula

$$\widetilde{R}_2^{jl}[U](t,x) = -\int_0^1\int(\partial_k\zeta)(t,y)\frac{(x-y)^l(x-y)^j}{\sigma^2}U^k\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$+\int_0^1\int\zeta(t,y)\frac{(x-y)^l}{\sigma}U^j\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$+\int_0^1\int\zeta(t,y)\frac{(x-y)^j}{\sigma}U^l\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma.$$

Similarly, we compute

$$\widetilde{R}_1^{jl}[U](t,x) := \int\zeta(t,y)^{(y)}r_1^{jl}[U(t,\cdot)](x)\,dy$$

$$= \int_0^1\int(\partial_k\zeta)(t,y)\frac{(x-y)^j(x-y)^k}{\sigma^2}U^l\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$-\frac{d+1}{2}\int_0^1\int\zeta(t,y)\frac{(x-y)^l}{\sigma}U^j\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$+ (\text{Symmetric terms in } j,l),$$

and

$$\widetilde{R}_0^{jl}[U](t,x) := \int\zeta(t,y)^{(y)}r_0^{jl}[U(t,\cdot)](x)\,dy$$

$$= -\frac{1}{2}\int_0^1\int\zeta(t,y)\frac{(x-y)^l}{\sigma}U^j\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma$$

$$-\frac{1}{2}\int_0^1\int\zeta(t,y)\frac{(x-y)^j}{\sigma}U^l\left(t,\frac{x-y}{\sigma}+y\right)\frac{dy}{\sigma^d}d\sigma.$$

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It therefore follows that

$$^{(\zeta(t,\cdot))}\widetilde{R}^{jl}[U(t,\cdot)](x) = \widetilde{R}_0^{jl}[U](t,x) + \widetilde{R}_1^{jl}[U](t,x) + \widetilde{R}_2^{jl}[U](t,x) = R^{jl}[U](t,x),$$

as desired. $\qquad\square$

10.4 Estimates for the solution operator and proof of Theorem 10.1

In this subsection, we begin by deriving a key technical lemma (Lemma 10.2) which allows us to derive L^p estimates for the operator $R^{jl}[U]$ (Lemma 10.3). Next, we use Proposition 10.1 and Lemma 10.2 to establish various commutator estimates. Using the results developed so far, a proof of Theorem 10.1 is given at the end.

Lemma 10.2. *Given $\bar{\rho} > 0$, let $\widetilde{\zeta}$ be a non-negative smooth function with $\operatorname{supp}\widetilde{\zeta} \subseteq B(\bar{\rho}; x_0)$ such that*

$$\|\widetilde{\zeta}\|_{C_x^0} \leq A\bar{\rho}^{-d} \tag{242}$$

for some $A > 0$. Then the following statements hold:

1. *For any $k \geq 0$ and $f \in L_x^\infty$ supported in $B(\bar{\rho}; x_0)$, we have*

$$\sup_{x\in\mathbb{R}^d, \sigma\in[0,1]} |\int \widetilde{\zeta}(y)\Big(\frac{|x-y|}{\sigma}\Big)^k f(\frac{x-y}{\sigma} + y)\frac{dy}{\sigma^d}| \leq C_k A\bar{\rho}^k \|f\|_{L_x^\infty}. \tag{243}$$

2. *Moreover for any $k \geq 0$ and $f \in L_x^\infty$ supported in $B(\bar{\rho}; x_0)$, we have*

$$\|\int_0^1 \int \widetilde{\zeta}(y)\Big(\frac{|x-y|}{\sigma}\Big)^k f(\frac{x-y}{\sigma} + y)\frac{dy}{\sigma^d}d\sigma\|_{L_x^\infty} \leq C_k A\bar{\rho}^k \|f\|_{L_x^\infty}. \tag{244}$$

Proof. First, observe that (244) immediately follows from (243). Furthermore, we claim that it suffices to prove the latter inequality in the case $k = 0$. Indeed, by the triangle inequality, we have

$$\frac{|x-y|}{\sigma} \leq |\frac{x-y}{\sigma} + y - x_0| + |y - x_0|.$$

Note that, within the integral, the first and second terms on the right-hand side are $\leq \bar{\rho}$ by the support properties of f and $\widetilde{\zeta}$, respectively. This implies

$$\Big(\frac{|x-y|}{\sigma}\Big)^k \leq 2^k \bar{\rho}^k,$$

which implies that the $k > 0$ case of (243) follows from the $k = 0$ case.

Therefore, it only remains to prove (243) in the case $k = 0$. We start with the bound

$$\sup_x |\int \widetilde{\zeta}(y)f(\frac{x-y}{\sigma} + y)\frac{dy}{\sigma^d}| \leq C\sigma^{-d}\bar{\rho}^d\|\widetilde{\zeta}\|_{L_x^\infty}\|f\|_{L_x^\infty} = CA\sigma^{-d}\|f\|_{L_x^\infty}. \tag{245}$$

This estimate degenerates as $\sigma \to 0$. On the other hand, making the change of variables

$$z = \frac{x-y}{\sigma} + y \tag{246}$$

we have

$$\sup_x |\int \widetilde{\zeta}(y)f(\frac{x-y}{\sigma} + y)\frac{dy}{\sigma^d}| = \sup_x |\int \widetilde{\zeta}\Big(\frac{1}{1-\sigma}x - \frac{\sigma}{1-\sigma}z\Big)f(z)\frac{dz}{(1-\sigma)^d}|$$

$$\leq C(1-\sigma)^{-d}\bar{\rho}^d\|\widetilde{\zeta}\|_{L_x^\infty}\|f\|_{L_x^\infty} = CA(1-\sigma)^{-d}\|f\|_{L_x^\infty}. \tag{247}$$

Combining (245) and (247), we obtain (243).

$\qquad\square$

As a consequence of the previous lemma and the differentiation formulae (233)–(235) and (236), we obtain the following $C^0_{t,x}$ estimates for R^{jl} and the commutator between ∇^β and R^{jl}.

Lemma 10.3 (Bounds for R^{jl}). *Let U^l be a smooth vector field on $\mathbb{R} \times \mathbb{R}^d$ satisfying the hypotheses (202) and (203) of Theorem 10.1. Define $R^{jl}[U] := R^{jl}_0[U] + R^{jl}_1[U] + R^{jl}_2[U]$ by (229), (230) and (231). Then we have*

$$\|R^{jl}[U]\|_{C^0_{t,x}} \leq C\bar{\rho}\|U\|_{C^0_{t,x}}. \tag{248}$$

Lemma 10.4 (Commutator between ∇^β and R^{jl}). *Let U^l and $R^{jl}[U]$ be as in the hypotheses of Lemma 10.3. Then for every multi-index β, we have*

$$\|[\nabla^\beta, R^{jl}][U]\|_{C^0_{t,x}} \leq C_\beta \bar{\rho} \sum_{\beta_1 + \beta_2 = \beta : \beta_2 \neq \beta} (\bar{\rho})^{-|\beta_1|} \|\nabla^{\beta_2} U\|_{C^0_{t,x}}. \tag{249}$$

These lemmas follow immediately by applying Lemma 10.2 to the differentiation formulae (233)–(235) on each time slice, keeping in mind the properties (227) and (228) of ζ. We omit the details.

In preparation for estimating the advective derivative of $R^{jl}[U]$, we prove the following general commutator estimate:

Lemma 10.5 (Commuting with vector fields). *Let $R^{jl}[U]$ be as in Lemma 10.3, and let Z and $\widetilde{U}^l$ be smooth vector fields on $\mathbb{R}^d$. Then*

$$\|[Z \cdot \nabla, R^{jl}][\widetilde{U}]\|_{C^0} \leq C\bar{\rho}^{1 + \frac{d}{q} - \frac{d}{p}} (\bar{\rho}^{-1}\|Z\|_{C^0} + \|\nabla Z\|_{C^0})\|\widetilde{U}\|_{C^0}. \tag{250}$$

Proof. We claim that, for R^{jl}_0 and R^{jl}_a, $a = 1, 2$, the following pointwise estimates hold

$$
\begin{aligned}
|[Z \cdot \nabla, R^{jl}_0][\widetilde{U}](x)| \leq{}& C\|Z\|_{C^0} \int_0^1 \int |\nabla\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big) |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma \\
&+ C\|\nabla Z\|_{C^0} \int_0^1 \int |\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big) |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma \\
&+ C\|\nabla Z\|_{C^0} \int_0^1 \int |\nabla\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big)^2 |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma
\end{aligned} \tag{251}
$$

$$
\begin{aligned}
|[Z \cdot \nabla, R^{jl}_a][\widetilde{U}](x)| \leq{}& C\|Z\|_{C^0} \int_0^1 \int |\nabla^{(2)}\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big)^2 |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma \\
&+ C\|\nabla Z\|_{C^0} \int_0^1 \int |\nabla\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big)^2 |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma \\
&+ C\|\nabla Z\|_{C^0} \int_0^1 \int |\nabla^{(2)}\zeta(y)| \Big(\frac{|x-y|}{\sigma}\Big)^3 |\widetilde{U}(\frac{x-y}{\sigma} + y)| \frac{dy}{\sigma^d} d\sigma
\end{aligned} \tag{252}
$$

From these claims, the desired estimate (250) follows by Lemma 10.2. We remark that the variable t plays no role in the proof.

The estimates (251) and (252) are all proved similarly; we give a detailed proof of (251), and omit

the details for the latter. We begin by applying the differentiation formula (233) to compute

$$[Z \cdot \nabla, R_0^{jl}][\widetilde{U}] = -\frac{d}{2} \int_0^1 \int Z^k(x) \partial_k \zeta(y) \frac{(x-y)^j}{\sigma} \widetilde{U}^l(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{253}$$

$$-\frac{d}{2} \int_0^1 \int Z^k(x) \partial_k \zeta(y) \frac{(x-y)^l}{\sigma} \widetilde{U}^j(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{254}$$

$$-\frac{d}{2} \int_0^1 \int \zeta(y) \frac{(x-y)^j}{\sigma} (Z^k(x) - Z^k(z)) \partial_k \widetilde{U}^l(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{255}$$

$$-\frac{d}{2} \int_0^1 \int \zeta(y) \frac{(x-y)^l}{\sigma} (Z^k(x) - Z^k(z)) \partial_k \widetilde{U}^j(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{256}$$

Here we write $z = \frac{x-y}{\sigma} + y$ for the argument of U.

The terms (253), (254) are immediately seen to verify (251), so it only remains to estimate the latter terms. We will focus on the term (255) since the last term is treated identically.

Starting with the identity

$$\partial_k U^l(z) = \partial_k U\left(\frac{x-y}{\sigma} + y\right) = -\frac{\sigma}{(1-\sigma)} \frac{\partial}{\partial y^k} \left[U\left(\frac{x-y}{\sigma} + y\right)\right],$$

we integrate by parts in y to obtain

$$(255) = -\frac{d}{2} \int_0^1 \int \partial_k \zeta(y) \frac{(x-y)^j}{\sigma} \frac{\sigma(Z^k(x) - Z^k(z))}{(1-\sigma)} \widetilde{U}^l(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{257}$$

$$-\frac{d}{2} \int_0^1 \int \zeta(y) \delta_k^j \frac{(Z^k(x) - Z^k(z))}{(1-\sigma)} \widetilde{U}^l(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{258}$$

$$-\frac{d}{2} \int_0^1 \int \zeta(y) \frac{(x-y)^j}{\sigma} \partial_k Z^k(z) \widetilde{U}^l(\frac{x-y}{\sigma} + y) \frac{dy}{\sigma^d} d\sigma \tag{259}$$

The estimates (251) now follow from the identity $\frac{z-x}{1-\sigma} = \frac{x-y}{\sigma}$ and the pointwise bound

$$\left(\frac{|Z^k(x) - Z^k(z)|}{(1-\sigma)}\right) \le \|\nabla Z\|_{C^0} \left(\frac{|z-x|}{(1-\sigma)}\right). \qquad \square$$

The key tool in estimating the advective derivative of $R^{jl}[U]$ will be the following estimate for the commutator $[(v_\epsilon - \overline{v}_\epsilon) \cdot \nabla, R^{jl}]$, which we derive from the commutator estimates of Lemmas 10.4 and 10.5.

Lemma 10.6 (Commutator between $(v_\epsilon - \overline{v}_\epsilon) \cdot \nabla$ and R^{jl}). *Let U^l and $R^{jl}[U]$ be as in the hypotheses of Lemma 10.3. Then for every multi-index β with $|\beta| \le L - 1$, we have*

$$\|\nabla^\beta[(v_\epsilon - \overline{v}_\epsilon) \cdot \nabla, R^{jl}][U]\|_{C^0_{t,x}} \le C_\beta \Xi e_v^{1/2} \overline{\rho} \sum_{J_0+J_1+J_2=|\beta|} (\overline{\rho})^{-J_0} \Xi^{J_1} \|\nabla^{(J_2)} U\|_{C^0_{t,x}}. \tag{260}$$

The summation is over all triplets of non-negative integers (J_0, J_1, J_2) such that $J_0 + J_1 + J_2 = |\beta|$.

Proof. In $\nabla^\beta[(v_\epsilon - \overline{v}_\epsilon) \cdot \nabla, R^{jl}][U]$, we will find that the worst case occurs when all derivatives fall on U, or when all the derivatives fall on the vector field $Y := (v_\epsilon - \overline{v}_\epsilon)$.

Observe that, for $(t,x) \in \hat{C}_{v_\epsilon}(\overline{\tau}, \overline{\rho}; t(I), x(I))$ we have the estimates

$$\|\nabla^\gamma Y\|_{C^0} \le C \Xi^{|\gamma|} e_v^{1/2} \qquad 1 \le |\gamma| \le L \tag{261}$$

$$|Y(t,x)| \le C \overline{\rho} \Xi e_v^{1/2} \tag{262}$$

Let us now decompose $\nabla^\beta[(v_\epsilon - \bar{v}_\epsilon) \cdot \nabla, R^{jl}][U] = \nabla^\beta[Y \cdot \nabla, R^{jl}][U]$ as

$$\nabla^\beta[Y \cdot \nabla, R^{jl}][U] = \nabla^\beta \left(Y^k \partial_k R^{jl}[U] \right) - \nabla^\beta R^{jl}[Y^k \partial_k U] \tag{263}$$

$$= \nabla^\beta \left(Y^k \partial_k R^{jl}[U] \right) - R^{jl}[\nabla^\beta(Y^k \partial_k U)] \tag{264}$$

$$- [\nabla^\beta, R^{jl}][Y^k \partial_k U] \tag{265}$$

The term (265) can be estimated using Lemma 10.4 by

$$\|(265)\|_{C^0_{t,x}} \leq C\bar{\rho} \sum_{\substack{\beta_1 + \beta_2 + \beta_3 = \beta \\ \beta_1 \neq 0}} (\bar{\rho})^{-\beta_1} \|\nabla^{\beta_2} Y\|_{C^0} \|\nabla^{\beta_3+1} U\|_{C^0_{t,x}} \tag{266}$$

We separate out the cases $\beta_2 = 0$ and $1 \leq |\beta_2| \leq L - 1$ according to estimates (261)-(262). In every case, we obtain

$$\|(265)\|_{C^0_{t,x}} \leq C\bar{\rho} \Xi e_v^{1/2} \sum_{J_0 + J_1 + J_2 = \beta} (\bar{\rho})^{-J_0} \Xi^{J_1} \|\nabla^{(J_2)} U\|_{C^0_{t,x}} \tag{267}$$

We estimate the term (264) by first expanding into terms of the form

$$(264) = \sum_{\beta_1 + \beta_2 = \beta} \nabla^{\beta_1} Y^k \partial_k \nabla^{\beta_2} R^{jl}[U] - R^{jl}[\nabla^{\beta_1} Y^k \partial_k \nabla^{\beta_2} U] = \sum_{\beta_1 + \beta_2 = \beta} E_{\beta_1, \beta_2} \tag{268}$$

Each term on the right hand side of (268) can be expanded as follows

$$E_{\beta_1, \beta_2} = [\nabla^{\beta_1} Y \cdot \nabla, R^{jl}][\nabla^{\beta_2} U] \tag{269}$$

$$+ \nabla^{\beta_1} Y^k \partial_k [\nabla^{\beta_2}, R^{jl}][U] \tag{270}$$

We now express (270) as a sum of commutators

$$(270) = \nabla^{\beta_1} Y^k [\partial_k \nabla^{\beta_2}, R^{jl}][U] - \nabla^{\beta_1} Y^k [\partial_k, R^{jl}][\nabla^{\beta_2} U] \tag{271}$$

Each term of the form (270) can now be bounded using Lemma 10.4 by

$$\|(270)\|_{C^0_{t,x}} \leq C\bar{\rho} \sum_{\substack{J_0 + J_1 + J_2 = |\beta| + 1 \\ J_1 + J_2 \neq |\beta| + 1}} (\bar{\rho})^{-J_0} \|\nabla^{(J_1)} Y\|_{C^0} \|\nabla^{(J_2)} U\|_{C^0_{t,x}}$$

The bound (260) for this term now follows from (261)-(262).

The remaining terms from (269) all have a commutator form $[Z \cdot \nabla, R^{jl}][\widetilde{U}]$ where $Z = \nabla^{\beta_1} Y$ and $\widetilde{U} = \nabla^{\beta_2} U$. Applying Lemma 10.5, we have

$$\|(270)\|_{C^0_{t,x}} \leq C\bar{\rho} \sum_{J_1 + J_2 = |\beta|} ((\bar{\rho})^{-1} \|\nabla^{(J_1)} Y\|_{C^0} + \|\nabla^{(J_1+1)} Y\|_{C^0}) \|\nabla^{(J_2)} U\|_{C^0_{t,x}} \tag{272}$$

Note that at most $|\beta| + 1 \leq L$ derivatives fall on Y. Recalling once more the estimates (261)-(262), we obtain Lemma 10.6. $\qquad \square$

We are now ready to give a proof of Theorem 10.1.

Proof of Theorem 10.1. In view of Proposition 10.1 and Lemmas 10.3 and 10.4, we are only left to establish the estimate (207). The idea is to write the advective derivative as

$$\partial_t + v_\epsilon \cdot \nabla = \frac{\overline{\overline{D}}}{\partial t} + (v_\epsilon - \overline{v}_\epsilon)^k \partial_k \,.$$

Then using the fact that $[\frac{\overline{\overline{D}}}{\partial t}, R^{jl}] = 0$, for any multi-index β with $0 \le |\beta| \le L - 1$, we have

$$\nabla^\beta (\partial_t + v_\epsilon \cdot \nabla) R^{jl}[U] = \nabla^\beta (R^{jl}[(\partial_t + v_\epsilon \cdot \nabla)U]) + \nabla^\beta [(v_\epsilon - \overline{v}_\epsilon)^k \partial_k, R^{jl}][U].$$

Applying Lemmas 10.3–10.6 and using (204), the desired estimate (207) follows. $\square$

11 Perturbations of Smooth Euler Flows

In this section, we illustrate how the Main Lemma (Lemma 3.1) can be used to establish Theorem 1.1 on the perturbation of smooth Euler flows. The basic strategy is the same as in Section 11 of [Ise12] and the construction in [DLS12a]; namely, we iterate the Main Lemma to produce a sequence of solutions $(v_{(k)}, p_{(k)}, R_{(k)})$ to the Euler-Reynolds equations, which converges to a solution (v, p) to the Euler equations as $k \to \infty$ with the desired properties. However, there are a few notable differences compared to [Ise12].

First, as discussed in Section 3, the condition $N \ge \Xi^\eta$ in the Main Lemma of [Ise12], which forced the frequency $\Xi_{(k)}$ to grow double-exponentially in k, is absent from our Main Lemma. We are therefore able to choose frequencies which grow only exponentially in k; see (289). Having this property makes our solutions closer to the physical picture of turbulence, as discussed in §1.1.4. We also remark that the exponential growth of frequency makes our proof of Theorem 1.1 simpler compared to that in [Ise12], as the evolution laws for the parameters (289), (290) and (291) are more straightforward.

Second, in the present case we need to construct an appropriate energy density function $e_{(k)}(t, x)$ at each step in order to apply the Main Lemma. In contrast, in [Ise12] only an energy function $e_{(k)}(t)$, which is the integral in x of the energy density) had to be constructed. In order to achieve the required point-wise bound (18) for $e_{(k)}(t, x)$, we employ the machinery of mollification along the flow of $v_{(k)}$. Note, however, that the only apriori information on $v_{(k)}$ we have is that $\nabla^m v_{(k)} \in C^0_{t,x}$ for $m = 1, \ldots, L$ (from its frequency and energy levels), which is far weaker than those on v_ϵ in the previous applications of mollification along the flow. This information turns out to be just sufficient for our construction; see Subsections 11.1 and 11.3.

In Subsection 11.1, we discuss the procedure of mollification along the flow of a vector field with limited regularity, which is used to construct the energy density function $e_{(k)}^{1/2}$. In Subsection 11.2, we reduce the proof of Theorem 1.1 to constructing a sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ of solutions to the Euler-Reynolds system that satisfies certain claims, i.e., Claims 1–5. In Subsection 11.3, we present the construction of the sequence $(v_{(k)}, p_{(k)}, R_{(k)})$, and in Subsection 11.4, we verify the claims made in Subsection 11.2 with such sequence, thereby concluding the proof of Theorem 1.1.

11.1 Mollification along the flow of a vector field with limited regularity

Let $L \ge 1$, and $v = (v^1, v^2, v^3)$ be a vector field on $\mathbb{R} \times \mathbb{R}^3$ whose frequency and energy levels are below (Ξ, e_v) to order L in C^0 ($L \ge 1$), in the sense that the following estimate holds.

$$\|\nabla^m v\|_{C^0_{t,x}} \le \Xi^k e_v^{1/2} \qquad\qquad m = 1, \ldots, L. \tag{273}$$

Recall that *the flow of v* is the map $^{(v)}\Phi_s(t, x) = (t + s, {}^{(v)}\Phi'_s(t, x))$, where $^{(v)}\Phi'_s$ is the unique solution to the ODE

$$\partial_s {}^{(v)}\Phi'_s(t, x) = v(t + s, {}^{(v)}\Phi'_s(t, x)), \quad {}^{(v)}\Phi'_0(t, x) = x. \tag{274}$$

As ∇v is uniformly bounded on $\mathbb{R} \times \mathbb{R}^3$ by (273), $^{(v)}\Phi'_s(t,x)$ extends indefinitely in s. By continuous dependence on parameters for ODEs, it follows that $^{(v)}\Phi'_s(t,x), \partial_s{}^{(v)}\Phi'_s(t,x)$ are continuous in $(t,x,s) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. Moreover, by differentiating the ODE (274) in x, we have that $\nabla^m(^{(v)}\Phi'_s)$ is continuous in (t,x,s) for $m = 1, \dots L$. In fact, the following Lemma can be read off from [Ise12, Proof of Proposition 18.1].

Lemma 11.1. *Let v be a vector field on $\mathbb{R} \times \mathbb{R}^3$ whose frequency and energy levels are below (Ξ, e_v) to order L in C^0 ($L \geq 1$), in the sense that (273) holds. Then for every $1 \leq m \leq L$, there exist constants $C_{a,1}, C_{a,2} > 0$ such that $\nabla^m(^{(v)}\Phi'_s)$ obeys the estimate*

$$|\nabla^m(^{(v)}\Phi'_s)(t,x)| \leq C_{m,1} e^{C_{m,2}\Xi e_v^{1/2}s}\Xi^{m-1}. \tag{275}$$

It is also true that $\partial_t(^{(v)}\Phi'_s)$ is continuous. However, this property does not follow directly by differentiating (274), as we have not assumed anything about $\partial_t v$. Rather, it is a consequence of the following Lemma.

Lemma 11.2. *Let v be a vector field on $\mathbb{R} \times \mathbb{R}^3$ whose frequency and energy levels are below (Ξ, e_v) to order L in C^0 ($L \geq 1$), in the sense that (273) holds. Then for every $(t,x,s) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ and $\sigma \in \mathbb{R}$, we have*

$$^{(v)}\Phi_s(^{(v)}\Phi_\sigma(t,x)) = {}^{(v)}\Phi_{s+\sigma}(t,x). \tag{276}$$

Moreover, $\partial_t(^{(v)}\Phi'_s(t,x))$ is continuous in $(t,x,s) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, and

$$(\partial_t + v(t,x) \cdot \nabla)^{(v)}\Phi'_s(t,x) = \partial_s{}^{(v)}\Phi'_s(t,x) = v(t, {}^{(v)}\Phi'_s(t,x)). \tag{277}$$

Proof. Equation (276) can be proved by differentiating both sides by s, and observing that both sides solve the same ODE with the same data at $s = 0$. Then the continuity of $\partial_t(^{(v)}\Phi'_s)$ and (277) follow by differentiating at $\sigma = 0$ and using the ODE (274). $\square$

Given a smooth function F on $\mathbb{R} \times \mathbb{R}^3$ with compact support, we define its *mollification* $^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}$ in space and along the flow of v by the formula

$$^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}(t,x) := \iint F(^{(v)}\Phi_s(t,x) + (0,h))\, \eta_{\bar{\rho}}(h)\eta_{\bar{\tau}}(s)\, dh ds. \tag{278}$$

where $\bar{\tau}, \bar{\rho}$ are mollification parameters, $\eta_{\bar{\tau}}(s) = \frac{1}{\bar{\tau}}\eta_0(s/\bar{\tau})$ and $\eta_{\bar{\rho}}(h) = \frac{1}{\bar{\rho}^3}\eta_1(h/\bar{\rho})$. Here, η_0, η_1 are smooth, compactly supported functions on $\mathbb{R}$ and $\mathbb{R}^3$, respectively, such that $\int_{\mathbb{R}} \eta_0\, dt = \int_{\mathbb{R}^3} \eta_1\, dx = 1$, supp $\eta_0 \subseteq \{t : |t| \leq 1\}$ and supp $\eta_1 \subseteq \{x : |x| \leq 1\}$.

The main result of this subsection is Proposition 11.1 below regarding the regularity of $^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}$ when v merely satisfies $\nabla^m v \in C^0_{t,x}$ for $m = 1, \dots L$. In this Proposition, we consider not only smooth functions F, but also locally integrable functions, as our construction involves applying formula (278) to a function which belongs to $L^\infty_{t,x}$ (the characteristic function of a measurable subset of $\mathbb{R} \times \mathbb{R}^3$). Within the proof of Proposition 11.1 below, we show that the formula (278) gives a well defined, continuous function of (t,x) whenever F is locally integrable, and we establish bounds on the regularity of $^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}$ under the assumption that F belongs to $L^\infty_{t,x}$. In particular, the value of (278) is well-defined at every point (t,x) and is independent of the almost-everywhere equivalence class of F.

Proposition 11.1. *Let v be a vector field on $\mathbb{R} \times \mathbb{R}^3$ whose frequency and energy levels are below (Ξ, e_v) to order L in C^0 ($L \geq 1$), in the sense that (273) holds. Then for every locally integrable F on $\mathbb{R} \times \mathbb{R}^3$, the following statements hold.*

1. *For $0 \leq k \leq 1$, $0 \leq m + k \leq L$, $\nabla^m \partial_t^k(^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}})$ is continuous in $(t,x) \in \mathbb{R} \times \mathbb{R}^3$.*

2. *Suppose furthermore that $F \in L^\infty(\mathbb{R} \times \mathbb{R}^3)$. Then there exist constants $C_1, C_2 > 0$, which depends only on L, such that for every $1 \le m \le L$, the following quantitative estimates hold.*

$$\|\nabla^m({}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho})\|_{C^0_{t,x}} \le C_1 e^{C_2 \Xi e_v^{1/2} \bar\tau}[(\bar\rho)^{-m} + \Xi^m]\|F\|_{L^\infty_{t,x}} \tag{279}$$

$$\|\nabla^{m-1}(\partial_t + v \cdot \nabla)({}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho})\|_{C^0_{t,x}} \le C_1 e^{C_2 \Xi e_v^{1/2} \bar\tau}[(\bar\rho)^{-(m-1)} + \Xi^{(m-1)}]\bar\tau^{-1}\|F\|_{L^\infty_{t,x}} \tag{280}$$

Proof of Proposition 11.1. For convenience, we shall omit (v) in ${}^{(v)}\Phi_s$. We omit the proof of Statement 1, which is very similar to the standard convolution case. Hence it only remains to establish Statement 2. We note that this statement does not follow from those established in [Ise12], since F is only assumed to be in $L^\infty_{t,x}$.

Let β be a multi-index with $|\beta| = m$. Then differentiating under the integral sign (which is justified for F smooth) and applying the chain rule, we see that

$$\nabla^\beta[{}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho}](t, x) = \int \nabla^\beta[(\eta_{\bar\rho} * F)(\Phi_s(t, x))]\eta_{\bar\tau}(s)\, ds$$

is a linear combination of terms of the form

$$\int [\partial_{j_1} \cdots \partial_{j_K}(\eta_{\bar\rho} * F)](\Phi_s(t, x)) \prod_{i=1}^K \nabla^{\beta_i}\Phi_s^{j_i}(t, x)\, \eta_{\bar\tau}(s)\, ds \tag{281}$$

where $0 \le K \le m$ and $\beta_1, \ldots, \beta_K$ are multi-indices such that $\beta_1 + \cdots + \beta_K = \beta$. Using Lemma 11.1, the standard convolution estimate

$$\partial_{j_1} \cdots \partial_{j_K}(\eta_{\bar\rho} * F) \le C(\bar\rho)^{-K}\|F\|_{L^\infty_{t,x}},$$

and the fact that $\int |\eta_{\bar\tau}|\, ds \le C$ (independent of $\bar\tau$), we see that the C^0 norm of (281) is bounded from above by

$$\le C e^{C\Xi e_v^{1/2} \bar\tau}(\bar\rho)^{-K}\|F\|_{L^\infty_{t,x}} \prod_{i=1}^K \Xi^{|\beta_i|-1} \le C e^{C\Xi e_v^{1/2} \bar\tau}(\Xi^m + (\bar\rho)^{-m})\|F\|_{L^\infty_{t,x}}$$

where the last inequality follows from Young's inequality, using the fact that $0 \le K \le m$ and $|\beta_1| + \cdots + |\beta_K| = m$. This estimate proves (279). To prove (280), note that

$$(\partial_t + v(t, x) \cdot \nabla)^{(v)}\widetilde{F}_{\bar\tau, \bar\rho}(t, x) = -\iint F(\Phi_s(t, x) + (0, h))\eta_{\bar\rho}(h)\frac{d}{ds}\eta_{\bar\tau}(s)\, dhds. \tag{282}$$

Indeed, for every $\sigma \in \mathbb{R}$, we have

$${}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho}(\Phi_\sigma(t, x)) = \iint F(\Phi_{s+\sigma}(t, x) + (0, h))\eta_{\bar\rho}(h)\eta_{\bar\tau}(s)\, dhds,$$

by (276). Making a change of variable $s' = s + \sigma$ and differentiating at $\sigma = 0$, we obtain (282). Then (280) can be proved in a similar manner as before, using the fact that $\int |\frac{d}{ds}\eta_{\bar\tau}|\, ds \le C(\bar\tau)^{-1}$. The estimates of Proposition 11.1 for $F \in L^\infty$ now follow by from the case where F is smooth by a straightforward approximation argument, as in the proof of continuity of ${}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho}$. $\qquad\square$

Proposition 11.1 will be used later to obtain the desired *upper* bounds for the energy density. In order to obtain the desired lower bound, we need to know about the locality of the mollification ${}^{(v)}\widetilde{F}_{\bar\tau, \bar\rho}$. This property can be described succinctly by using Eulerian cylinders adapted to v (Definition 3.2), as the following lemma shows.

Lemma 11.3 (Locality of the mollification). *Let v be a vector field on $\mathbb{R} \times \mathbb{R}^3$ whose frequency and energy levels are below (Ξ, e_v) to order L in C^0 ($L \geq 1$), in the sense that (273) holds. Also, let F be a locally integrable function on $\mathbb{R} \times \mathbb{R}^3$ and $\bar{\tau}, \bar{\rho} > 0$. Then for every $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, the mollification $^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}(t, x)$ depends only on the values of v and F on $\hat{C}_v(\bar{\tau}, \bar{\rho}; t, x)$. Furthermore, the advective derivative $(\partial_t + v \cdot \nabla)[^{(v)}\widetilde{F}_{\bar{\tau},\bar{\rho}}](t, x)$ also depends only on the values of v and F on $\hat{C}_v(\bar{\tau}, \bar{\rho}; t, x)$ as well.*

Proof. This follows from the definition (278), the identity (282) and our choice of $\eta_{\bar{\tau}}, \eta_{\bar{\rho}}$. □

11.2 Reduction of Theorem 1.1

We are now ready to begin the proof of Theorem 1.1. In this Subsection, we reduce the proof of Theorem 1.1 to constructing a sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ of solutions to the Euler-Reynolds system that satisfies certain claims (Claims 1–5).

From the hypotheses of Theorem 1.1, recall that we are given positive numbers $\epsilon, \delta > 0$, a smooth solution $(v_{(0)}, p_{(0)})$ to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$ and pre-compact open sets $\Omega_{(0)}, \mathcal{U}$ such that $\Omega_{(0)} \neq \emptyset$ and

$$\overline{\Omega_{(0)}} \subseteq \mathcal{U}. \tag{283}$$

From these inputs, we shall produce in the following Subsections (Subsections 11.3 and 11.4) a sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ of solutions to the Euler-Reynolds system which satisfies the following Claims:

Claim 1 (Vanishing of the Euler-Reynolds stress). *The Euler-Reynolds stress $R_{(k)}$ converges uniformly to zero, i.e., $\|R_{(k)}\|_{C^0} \to 0$ as $k \to \infty$.*

Claim 2 (Compact support in space-time). *There exists a pre-compact set $\Omega_{(\infty)} \subseteq \mathbb{R} \times \mathbb{R}^3$ such that $\overline{\Omega_{(\infty)}} \subseteq \mathcal{U}$ and for every $k \geq 0$,*

$$\operatorname{supp}\left(v_{(k)} - v_{(0)},\, p_{(k)} - p_{(0)}\right) \subseteq \Omega_{(\infty)}.$$

Claim 3 (Hölder regularity of the solution). *For $\alpha = \frac{1}{5} - \epsilon$, the sequence $(v_{(k)}, p_{(k)})$ is Cauchy in $C_{t,x}^\alpha \times C_{t,x}^{2\alpha}$ as $k \to \infty$. Moreover, for every $k \geq 0$, we have*

$$\|v_{(k)} - v_{(0)}\|_{C_{t,x}^\alpha} + \|p_{(k)} - p_{(0)}\|_{C_{t,x}^{2\alpha}} \leq \frac{\delta}{2}. \tag{284}$$

We state Claims 4-5 using the notation

$$I[\Omega_{(0)}] := \{t \in \mathbb{R} \ : \ \Omega_{(0)} \cap \{t\} \times \mathbb{R}^3 \neq \emptyset\}$$
$$S_{t_\star}[\Omega_{(0)}] := \{x \in \mathbb{R}^3 \ : \ (t_\star, x) \in \Omega_{(0)}\}.$$

Claim 4 (Increase of local energy). *For every $t_\star \in I[\Omega_{(0)}]$ and smooth, compactly supported function ψ such that $\psi \equiv 1$ on $S_{t_\star}[\mathcal{U}]$, we have*

$$\int \psi(x) \frac{|v_{(k+1)}(t_\star, x)|^2}{2}\, dx > \int \psi(x) \frac{|v_{(k)}(t_\star, x)|^2}{2}\, dx \tag{285}$$

for every $k \geq 0$.

Claim 5 (Irregularity of the solution). *For any $t_\star \in I[\Omega_0]$ and $B(\rho_\star; x_\star) \subseteq S_{t_\star}[\Omega_{(0)}]$, let $\psi = \psi(x)$ be a smooth function on $\mathbb{R}^3$ such that $\operatorname{supp} \psi \subseteq B(\rho_\star; x_\star)$, $\psi \geq 0$ and $\int \psi(x)\, dx = 1$. Then for every $u \in W_x^{1/5,1}(B(\rho_\star; x_\star)) \cup C_x^{1/5}(B(\rho_\star; x_\star))$, there exists $k_\star = k_\star(\rho_\star, t_\star, x_\star, v_{(0)}, \psi, u) \geq 0$ such that*

$$\int \psi(x) \frac{|(v_{(k+1)} - u)(t_\star, x)|^2}{2}\, dx > \int \psi(x) \frac{|(v_{(k)} - u)(t_\star, x)|^2}{2}\, dx \tag{286}$$

holds for all $k \geq k_\star$.

Assuming these Claims, Theorem 1.1 follows rather immediately.

Proof of Theorem 1.1 assuming Claims 1–5. By Claims 1 and 3, it follows that $(v, p) := \lim_{k \to \infty} (v_{(k)}, p_{(k)})$ exists in $C_{t,x}^{1/5-\epsilon} \times C_{t,x}^{2(1/5-\epsilon)}$, and is a solution to the incompressible Euler equations. Moreover, by (284), it follows that

$$\|v - v_{(0)}\|_{C_{t,x}^{1/5-\epsilon}} + \|p - p_{(0)}\|_{C_{t,x}^{2(1/5-\epsilon)}} \leq \frac{\delta}{2} < \delta,$$

which proves Statement 2. Statements 1 and 4 of Theorem 1.1 then follow from Claims 2 and 4, respectively. Finally, for every $t_\star \in I[\Omega_0]$, $B(\rho_\star; x_\star) \subseteq S_{t_\star}[\Omega_{(0)}]$ and $u \in W_x^{1/5,1}(B(\rho_\star; x_\star)) \cup C_x^{1/5}(B(\rho_\star; x_\star))$, Claim 5 shows that there exists a non-negative function ψ supported in $B(\rho_\star; x_\star)$ and $k_\star \geq 0$ such that the quantity

$$\int \psi(x) \frac{|(v_{(k+1)} - u)(t_\star, x)|^2}{2}\, dx$$

is *strictly increasing* for $k \geq k_\star$. Since this integral is non-negative for every k, it follows that

$$\int \psi(x) \frac{|(v - u)(t_\star, x)|^2}{2}\, dx > 0.$$

Thus, $v \neq u$ on $B(\rho_\star; x_\star)$. Since u can be an arbitrary function in $W_x^{1/5,1}(B(\rho_\star; x_\star))$ or in $C_x^{1/5}(B(\rho_\star; x_\star))$, it follows that v belongs to neither $W_x^{1/5,1}(B(\rho_\star; x_\star))$ nor $C_x^{1/5}(B(\rho_\star; x_\star))$. As $t_\star \in I[\Omega_{(0)}]$ and $B(\rho_\star; x_\star) \subseteq S_{t_\star}[\Omega_{(0)}]$ can be arbitrary, Statement 3 follows. $\square$

The following Subsections will be devoted to the construction of a sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ which satisfies the above claims. More precisely, the construction process itself will be described in Subsection 11.3, and the Claims 1–5 will be verified for the constructed sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ in Subsection 11.4.

11.3 Construction of $(v_{(k)}, p_{(k)}, R_{(k)})$

In this Subsection, we describe the construction of the sequence $(v_{(k)}, p_{(k)}, R_{(k)})$, which will be shown to satisfy Claims 1–5 in Subsection 11.4. The basic scheme is as follows: Given $(v_{(k)}, p_{(k)}, R_{(k)})$ with frequency and energy levels below $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$, along with sets $\Omega_{(k)}, \widetilde{\Omega}_{(k)}$ such that

$$\operatorname{supp}\,(v_{(k)} - v_{(0)}, p_{(k)} - p_{(0)}, R_{(k)}) \subseteq \Omega_{(k)}, \tag{287}$$

$$\hat{C}_{v_{(0)}}(5\theta_{(k)}, 5000\Xi_{(k)}^{-1}; \Omega_{(k)}) \subseteq \widetilde{\Omega}_{(k)} \subseteq \overline{\widetilde{\Omega}_{(k)}} \subseteq \mathcal{U} \tag{288}$$

(where $\theta_{(k)} = \Xi_{(k)}^{-1} e_{v,(k)}^{-1/2}$) we use the Main Lemma to produce $(v_{(k+1)}, p_{(k+1)}, R_{(k+1)})$ with frequency and energy levels below $(\Xi_{(k+1)}, e_{v,(k+1)}, e_{R,(k+1)})$ satisfying the ansatz

$$\Xi_{(k+1)} = C_0 Z^{5/2} \Xi_{(k)} \tag{289}$$

$$e_{v,(k+1)} = e_{R,(k)} \tag{290}$$

$$e_{R,(k+1)} = \frac{e_{R,(k)}}{Z} \tag{291}$$

where C_0 is the constant in the Main Lemma and Z is a parameter to be specified. Note that $\Xi_{(k)}$ grows exponentially, and $e_{v,(k)}$ and $e_{R,(k)}$ decay exponentially. We also construct $\Omega_{(k+1)}, \widetilde{\Omega}_{(k+1)}$ satisfying (287), (288) and furthermore

$$\widetilde{\Omega}_{(k+1)} \subseteq \widetilde{\Omega}_{(k)}. \tag{292}$$

11.3.1   The base case

Here, we choose the parameters $\Xi_{(0)}, e_{v,(0)}, e_{R,(0)}$. We will also choose $\widetilde{\Omega}_{(0)}$ so that (288) holds. These choices will serve as the base step for the construction sketched above.

Remark. In general, one can construct solutions by taking the initial frequency and energy levels $\Xi_{(0)}$ and $e_{v,(0)}$ to be any values for which the bounds (11)-(12) hold for the initial velocity and pressure $(v_{(0)}, p_{(0)})$. With such a choice of parameters, it is natural to regard the pair $(\Xi_{(0)}^{-1}, e_{v,(0)}^{1/2})$ as a characteristic length scale and velocity for the solutions constructed by our procedure. In our proof below, we will take a more specific choice of $(\Xi_{(0)}, e_{v,(0)})$ that is convenient for proving Claims 1–5.

Choice of $e_{v,(0)}$ and $e_{R,(0)}$.   We choose

$$e_{v,(0)} = 1, \quad e_{R,(0)} = Z^{-1} \tag{293}$$

where $Z > 1$ is a large parameter to be chosen later; in fact, it will be finally fixed in the Proof of Claim 4 in Subsection 11.4. We take $\Xi_{(0)} > 1$ sufficiently large so that

$$(v_{(0)}, p_{(0)}, R_{(0)}) \text{ has frequency and energy levels below } (\Xi_{(0)}, 1, \frac{1}{Z}). \tag{294}$$

This choice of $\Xi_{(0)}$ can be made independently of the choice of Z, since $R_{(0)} = 0$.

Choice of $\widetilde{\Omega}_{(0)}$ and $\Xi_{(0)}$.   We choose

$$\widetilde{\Omega}_{(0)} := \hat{C}_{v_{(0)}}(5\theta_{(0)}, 5000\Xi_{(0)}^{-1}; \Omega_{(0)}), \tag{295}$$

which makes the first inclusion in (288) hold automatically. Since $\Omega_{(0)}$ is pre-compact, we may ensure that the last inclusion in (288) holds as well by choosing $\Xi_{(0)} > 1$ larger if necessary. We remark that (287) also holds, since the left-hand side is empty for $k = 0$.

11.3.2   Choosing the parameters for $k \geq 1$

Here, we describe the choice of parameters needed to apply the Main Lemma in order to construct $(v_{(k+1)}, p_{(k+1)}, R_{(k+1)})$, except for the choice of the energy density $e_{(k)}(t, x)$.

From (21) of the Main Lemma, the Ansatz (291) and base case (293), we are led to the choices

$$\frac{e_{v,(k)}}{e_{R,(k)}} = Z, \qquad N_{(k)} = Z^2 \Big(\frac{e_{v,(k)}}{e_{R,(k)}}\Big)^{1/2} = Z^{5/2}. \tag{296}$$

for $k \geq 0$. Note that $Z > 1$ is enough to ensure (19). Accordingly, we choose $\Xi_{(k+1)}$ to be

$$\Xi_{(k+1)} = C_0 N_{(k)} \Xi_{(k)} = C_0 Z^{5/2} \Xi_{(k)},$$

where $C_0 > 1$ is the constant given by the Main Lemma, which depends only on $M > 0$. The latter constant will be chosen to be $M = C_1 e^{C_2}$, where C_1, C_2 are constants in Proposition 11.1; see (300).

Remark. The size of the constant C_0 in the Main Lemma determines whether the constructed solution (v, p) belongs to $C_{t,x}^{1/5} \times C_{t,x}^{2/5}$ or not. In our proof of the Main Lemma, recall that C_0 was chosen to be sufficiently large in order to absorb many implicit constants that arose in the proof. In particular, $C_0 > 1$, and as we shall see below, this inequality forces the constructed solution (v, p) to fail to belong to $C_{t,x}^{1/5} \times C_{t,x}^{2/5}$ locally, as stated in Theorem 1.1 (see also Claim 5). On the other hand, if we had $C_0 \leq 1$, then it would follow that (v, p) belongs to $C_{t,x}^{1/5} \times C_{t,x}^{2/5}$, by a slight variant of our proof of Claim 3 below.

At this point, we take $Z > 1$ to be sufficiently large to make sure that the space- and time-scales $\Xi_{(k)}^{-1}$, $\theta_{(k)}$ decrease sufficiently fast to be used in the construction of $e_{(k)}(t, x)$ below. In particular, our Z will satisfy the hypothesis of the following lemma.

Lemma 11.4. *Let* $\Xi_{(k)}$, $e_{v,(k)}$, $e_{R,(k)}$, $N_{(k)}$ *and* $\theta_{(k)}$ *be chosen inductively according to* (289), (290), (291) *and* (296) *from the case* $k = 0$ *given above. Then there exists* $Z_0 > 0$ *such that if* $Z \geq Z_0$, *then we have*

$$\Xi_{(k+1)}^{-1} \leq \frac{1}{5000} \Xi_{(k)}^{-1}, \quad \theta_{(k+1)} \leq \frac{1}{500} \theta_{(k)}. \tag{297}$$

Proof. The first inequality follows from (289), by taking $C_0 Z_0^{5/2} \geq 5000$. To prove the second inequality, note that

$$\theta_{(k+1)} = \Xi_{(k+1)}^{-1} e_{v,(k+1)}^{-1/2} = C_0^{-1} Z^{-2} \Xi_{(k)}^{-1} e_{v,(k)}^{-1/2} = C_0^{-1} Z^{-2} \theta_{(k)}. \tag{298}$$

Thus, taking $C_0 Z_0^2 \geq 500$, the second inequality follows. $\qquad\square$

11.3.3 Choosing the energy density

We now describe how to choose the energy density $e_{(k)}(t, x)$, which satisfies the hypotheses (17) and (18) of the Main Lemma. This choice allows us to invoke the Main Lemma to produce $(v_{(k+1)}, p_{(k+1)}, R_{(k+1)})$ with frequency and energy levels below $(\Xi_{(k+1)}, e_{v,(k+1)}, e_{R,(k+1)})$ satisfying (290) and (291).

Recall that we are given $\Omega_{(k)}$, $\widetilde{\Omega}_{(k)}$ satisfying (287), (288). Let $\chi_{(k)}$ be the characteristic function of $\hat{C}_{v_{(k)}}(2\theta_{(k)}, 2\Xi_{(k)}^{-1}; \Omega_{(k)})$. Note that $\chi_{(k)}$ is a locally integrable function[8] Define $e_{(k)}^{1/2}$ to be $(K e_{R,(k)})^{1/2}$ times the mollification of $\chi_{(k)}$ in space and along the flow of $v_{(k)}$, with parameters $\bar{\tau} = \frac{1}{100} \theta_{(k)}$ and $\bar{\rho} = \frac{1}{100} \Xi_{(k)}^{-1}$. More precisely,

$$e_{(k)}^{1/2}(t, x) := (K e_{R,(k)})^{1/2} \left({}^{(v_{(k)})} \widetilde{[\chi_{(k)}]}_{\frac{1}{100}\theta_{(k)}, \frac{1}{100}\Xi_{(k)}^{-1}}\right)(t, x). \tag{299}$$

The desired upper bound (18) follows from

$$\|\nabla^m (\partial_t + v_{(k)} \cdot \nabla)^r e_{(k)}^{1/2}\|_{C^0} \leq M \Xi_{(k)}^m (\Xi_{(k)} e_{v,(k)}^{1/2})^r e_{R,(k)}^{1/2} \qquad 0 \leq r \leq 1, 0 \leq m + r \leq L \tag{300}$$

which in turn is an immediate consequence of Proposition 11.1 (with $M = C_1 e^{C_2}$).

Next, we verify that the desired lower bound holds, i.e.,

$$e_{(k)}(t, x) \geq K e_{R,(k)} \quad \text{for} \quad (t, x) \in \hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \Omega_{(k)}). \tag{301}$$

By (39) in Lemma 4.3, we see that

$$\hat{C}_{v_{(k)}}\left(\frac{1}{100}\theta_{(k)}, \frac{1}{100}\Xi_{(k)}^{-1}; \hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \Omega_{(k)})\right) \subseteq \hat{C}_{v_{(k)}}(2\theta_{(k)}, 2\Xi_{(k)}^{-1}; \Omega_{(k)}).$$

---

[8] Strictly speaking, one must check at this point that the set $\hat{C}_{v_{(k)}}(2\theta_{(k)}, 2\Xi_{(k)}^{-1}; \Omega_{(k)})$ and the function $\chi_{(k)}$ are measurable. This point can be proven by noting that $\hat{C}_{v_{(k)}}(2\theta_{(k)}, 2\Xi_{(k)}^{-1}; \Omega_{(k)})$ is a countable union of compact subsets of $\mathbb{R} \times \mathbb{R}^3$. of (t, x).

Thus, for every $(t, x) \in \hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \Omega_{(k)})$, we have $\chi_{(k)} \equiv 1$ on $\hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; t, x)$. Using Lemma 11.3 (Locality of the mollification) to replace $\chi_{(k)}$ by 1, and noting that the mollification of the latter is trivially $\equiv 1$, we conclude that (301) holds, in fact, with equality.

11.3.4 Controlling the enlargement of support

To continue the construction, we need to choose $\Omega_{(k+1)}$ and $\widetilde{\Omega}_{(k+1)}$ so that (287), (288), (292) hold. We define $\Omega_{(k+1)}$, $\widetilde{\Omega}_{(k+1)}$ to be appropriate $v_{(0)}$-adapted cylindrical neighborhoods of $\Omega_{(k)}$, i.e.,

$$\Omega_{(k+1)} := \hat{\Gamma}_{v_{(0)}}(4\theta_{(k)}, 2000\Xi_{(k)}^{-1}; \Omega_{(k)}), \quad \widetilde{\Omega}_{(k+1)} := \hat{C}_{v_{(0)}}(5\theta_{(k)}, 5000\Xi_{(k)}^{-1}; \Omega_{(k)}). \tag{302}$$

We first establish (287) for $k + 1$. By construction, note that

$$\operatorname{supp} e_{(k)} \subseteq \hat{\Gamma}_{v_{(k)}}\left(\frac{1}{100}\theta_{(k)}, \frac{1}{100}\Xi_{(k)}^{-1}; \hat{C}_{v_{(k)}}(2\theta_{(k)}, 2\Xi_{(k)}^{-1}; \Omega_{(k)})\right)$$

By (41), (39) of Lemma 4.3 and Lemma 4.2 (Equivalence of Eulerian and Lagrangian Cylinders), we have

$$\hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \operatorname{supp} e_{(k)}) \subseteq \hat{\Gamma}_{v_{(k)}}(4\theta_{(k)}, 2000\Xi_{(k)}^{-1}; \Omega_{(k)}).$$

Since $\operatorname{supp}(v_{(k)} - v_{(0)}) \subseteq \Omega_{(k)}$, Lemma 4.6 applies and it follows that

$$\hat{\Gamma}_{v_{(k)}}(4\theta_{(k)}, 2000\Xi_{(k)}^{-1}; \Omega_{(k)}) = \hat{\Gamma}_{v_{(0)}}(4\theta_{(k)}, 2000\Xi_{(k)}^{-1}; \Omega_{(k)}) = \Omega_{(k+1)}.$$

As $(V_{(k)}, P_{(k)}, R_{(k)})) = (v_{(k+1)} - v_{(k)}, p_{(k+1)} - p_{(k)}, R_{(k)})$ produced by the Main Lemma is supported in $\hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \operatorname{supp} e_{(k)})$, we see that (287) holds for $k + 1$.

Next, by (288) for k, we see that (292) holds, i.e., $\widetilde{\Omega}_{(k+1)} \subseteq \widetilde{\Omega}_{(k)}$. In particular, note that the last inclusion in (288) holds for $\widetilde{\Omega}_{(k+1)}$.

Finally, we need to verify that the first inclusion in (288) holds for $k + 1$. By (297), it suffices to show that

$$\hat{C}_{v_{(0)}}(\theta_{(k)}, \Xi_{(k)}; \Omega_{(k+1)}) \subset \widetilde{\Omega}_{(k+1)}. \tag{303}$$

Note that we use $v_{(0)}$ instead of $v_{(k)}$ on the left-hand side. Applying (40) in Lemma 4.3, (288) for k, and using the fact that $e^{5\theta_k \|\nabla v_{(0)}\|_{C^0}} \leq e^{\frac{1}{100}} \leq 2$ by (297), the desired inclusion (303) follows.

11.4 Verification of Claims $1 - 5$

Here, we complete the proof of Theorem 1.1 by establishing the Claims 1–5, which were made in Subsection 11.2.

Proof of Claim 1: Vanishing of the Euler-Reynolds stress. This claim is obvious from construction, since

$$\|R_{(k)}\|_{C^0} \leq e_{R,(k)} \to 0 \text{ as } k \to \infty. \qquad \square$$

Proof of Claim 2: Compact support in space-time. Let $\Omega_{(\infty)} := \cup_{k=1}^{\infty} \Omega_{(k)}$. By construction, for every $k \geq 0$ we have

$$\operatorname{supp}(v_{(k)} - v_{(0)}, p_{(k)} - p_{(0)}) \subseteq \Omega_{(k)} \subseteq \Omega_{(\infty)}.$$

Note furthermore that $\overline{\Omega_{(\infty)}} \subseteq \overline{\widetilde{\Omega}_{(0)}} \subseteq \mathcal{U}$, from which the claim follows. $\qquad \square$

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Proof of Claim 3: Hölder regularity of the solution. Note that $v_{(K)} = v_{(0)} + \sum_{k=1}^{K} V_{(k)}$ and $p_{(K)} = p_{(0)} + \sum_{k=1}^{K} P_{(k)}$, where

$$\|V_{(k)}\|_{C^0} \leq C e_{R,(k)}^{1/2} = C Z^{-(k+1)/2}, \tag{304}$$

$$\|P_{(k)}\|_{C^0} \leq C e_{R,(k)}^{1/2} = C Z^{-(k+1)}, \tag{305}$$

$$\|\nabla_{t,x} V_{(k)}\|_{C^0} \leq C C_0 N_{(k)} \Xi_{(k)} e_{R,(k)}^{1/2} = C C_0^{k+1} Z^{5(k+1)/2} Z^{-(k+1)/2}, \tag{306}$$

$$\|\nabla_{t,x} P_{(k)}\|_{C^0} \leq C C_0 N_{(k)} \Xi_{(k)} e_{R,(k)} = C C_0^{k+1} Z^{5(k+1)/2} Z^{-(k+1)}, \tag{307}$$

by the Main Lemma and the base case $e_{R,(0)} = Z^{-1}$. The estimates (306), (307) for the time derivative ∂_t follow by writing

$$\partial_t = (\partial_t + v_{(k)} \cdot \nabla) - v_{(k)} \cdot \nabla = (\partial_t + v_{(k)} \cdot \nabla) - (v_{(k)} - v_{(0)}) \cdot \nabla - v_{(0)} \cdot \nabla$$

and noting that the advective derivative obeys an even more favorable estimate than needed, while the terms $(v_{(k)} - v_{(0)})$ and $v_{(0)}$ are bounded uniformly on $\Omega_{(k)}$, independent of k. The uniform boundedness of $(v_{(k)} - v_{(0)}) = \sum_{k'=0}^{k-1} V_{(k')}$ follows by summing (304) in k'. Also, the C^0 norm of $v_{(0)}$ over $\Omega_{(k)}$ is also bounded uniformly in k, as $v_{(0)}$ is smooth and the sets $\Omega_{(k)}$ are contained in a fixed compact set $\overline{\widetilde{\Omega}_{(0)}}$ by Claim 2. Therefore the constants in (306)-(307) are independent of k.

By interpolation of (304)-(307), we obtain the following upper bounds on the $C_{t,x}^\alpha$ norm of $V_{(k)}$ and $P_{(k)}$.

$$\|V_{(k)}\|_{C_{t,x}^\alpha} \leq C C_0^{\alpha(k+1)} Z^{\frac{5\alpha-1}{2}(k+1)}, \tag{308}$$

$$\|P_{(k)}\|_{C_{t,x}^{2\alpha}} \leq C C_0^{2\alpha(k+1)} Z^{(5\alpha-1)(k+1)}. \tag{309}$$

Therefore, for $\alpha = 1/5 - \epsilon$, choosing $Z > 1$ sufficiently large so that

$$C_0^{2\alpha} Z^{-5\epsilon} < 1, \tag{310}$$

we see that the bounds (308)-(309) for $V_{(k)}$ and $P_{(k)}$ can be summed in a geometric series, and therefore $(v_{(k)}, p_{(k)})$ is Cauchy in $C_{t,x}^\alpha \times C_{t,x}^{2\alpha}$. Moreover, taking Z even larger, we can ensure that the sum

$$\sum_{k \geq 0} \|V_{(k)}\|_{C_{t,x}^\alpha} + \|P_{(k)}\|_{C_{t,x}^{2\alpha}}$$

is arbitrarily small, which proves (284). $\qquad\square$

Proof of Claim 4: Increase of local energy. The proof below closely follows the argument of [Ise12, §11.2.7]. We begin by reducing our consideration to a specific ψ for each $t_\star \in I[\Omega_{(0)}]$. Indeed, by Claim 2 which has been already verified, the following statement holds: If ψ, ψ' are two smooth, compactly supported, smooth function on $\mathbb{R}^3$ such that $\psi \equiv \psi'$ on $S_{t_\star}[\mathcal{U}]$, then for every $k \geq 1$ we have

$$\int (\psi' - \psi)(x) \frac{|v_{(k)}(t_\star, x)|^2}{2} \, dx = \int (\psi' - \psi)(x) \frac{|v_{(0)}(t_\star, x)|^2}{2} \, dx.$$

Therefore, it suffices to verify (285) for a *specific* $\psi_{t_\star}$ for each $t_\star \in I[\Omega_{(0)}]$. By the pre-compactness of $\Omega_{(0)}$ and $\mathcal{U}$, there exists a smooth, compactly supported $\psi_{t_\star} = \psi_{t_\star}(x)$ for each $t_\star \in I[\Omega_{(0)}]$ so that $\psi_{t_\star} \equiv 1$ on $S_{t_\star}[\mathcal{U}]$ and

$$\sup_{t_\star \in I[\Omega_{(0)}]} \left(\|\psi_{t_\star}\|_{L_x^1} + \|\nabla \psi_{t_\star}\|_{L_x^1} \right) \leq C < \infty \tag{311}$$

for some $C = C(\Omega_{(0)}, \mathcal{U})$.

We are now ready to prove (285). Here we will often omit the x variable for functions $f(t_\star, x) = f(t_\star)$ depending on x. Recalling that $v_{(k+1)} = v_{(k)} + V_{(k)}$, we compute

$$\int \psi_{t_\star} \frac{|v_{(k+1)}(t_\star)|^2}{2}\,dx - \int \psi_{t_\star} \frac{|v_{(k)}(t_\star)|^2}{2}\,dx$$
$$= \int \psi_{t_\star} e_{(k)}(t_\star)\,dx + \int \psi_{t_\star}\Big(\frac{|V_{(k)}(t_\star)|^2}{2} - e_{(k)}(t_\star)\Big)\,dx + \int \psi_{t_\star}\, v_{(k)} \cdot V_{(k)}(t_\star)\,dx \tag{312}$$

Given $t_\star \in I[\Omega_{(0)}]$, let $x_\star$ be a point in $\mathbb{R}^3$ such that $(t_\star, x_\star) \in \Omega_{(0)}$. From the construction, note that $e_{(0)}(t,x) \geq K e_{R,(0)}$ on $\hat{C}_{v_{(0)}}(\theta_{(0)}, \Xi_{(0)}^{-1}; \Omega_{(0)})$. In particular, we have

$$e_{(k)}(t_\star, x) \geq K e_{R,(k)} \text{ on } B(\Xi_{(0)}^{-1}; x_\star), \tag{313}$$

for $k = 0$. Next, again by construction in Subsection 11.3, note that $\hat{C}_{v_{(0)}}(\theta_{(0)}, \Xi_{(0)}^{-1}; \Omega_{(0)}) \subseteq \Omega_{(1)} \subseteq \Omega_{(k)}$ for every $k \geq 1$; therefore, (313) holds for $k \geq 1$ as well. Thus, we conclude that for every $k \geq 0$, we have

$$\int \psi_{t_\star} e_{(k)}(t_\star)\,dx \geq c e_{R,(k)}. \tag{314}$$

for some constant $c > 0$ which depends on $\Xi_{(0)}^{-1}$ and K, but does not depend on k, Z or $t_\star$.

On the other hand, by the Main Lemma, we have the bound

$$\Big|\int \psi_{t_\star}\Big(\frac{|V_{(k)}(t_\star)|^2}{2} - e_{(k)}(t_\star)\Big)\,dx\Big| \leq C \frac{e_{v,(k)}^{1/2} e_{R,(k)}^{1/2}}{N_{(k)}}\Big(\|\psi_{t_\star}\|_{L_x^1} + \Xi_{(k)}^{-1}\|\nabla\psi_{t_\star}\|_{L_x^1}\Big)$$
$$\leq C Z^{-2} e_{R,(k)}, \tag{315}$$

where we used (311) and the fact that $\Xi_{(k)}^{-1} < 1$ on the last line. Next, we have

$$\Big|\int \psi_{t_\star}\, v_{(k)} \cdot V_{(k)}(t_\star)\,dx\Big| = \Big|\int \psi_{t_\star}\, v_{(k)} \cdot \nabla \times W_{(k)}(t_\star)\,dx\Big|$$
$$\leq \Big|\int \psi_{t_\star} \nabla \times v_{(k)} \cdot W_{(k)}(t_\star)\,dx\Big| + \Big|\int (\nabla\psi_{t_\star} \times v_{(k)}) \cdot W_{(k)}(t_\star)\,dx\Big|. \tag{316}$$

In this case, $\nabla\psi_{t_\star} = 0$ on supp $W_{(k)}(t_\star)$ by hypothesis, and therefore the second term on the last line vanishes. Therefore, by (311), we have

$$\Big|\int \psi_{t_\star}\, v_{(k)} \cdot V_{(k)}(t_\star)\,dx\Big| \leq C \frac{e_{v,(k)}^{1/2} e_{R,(k)}^{1/2}}{N_{(k)}}\|\psi_{t_\star}\|_{L_x^1} \leq C Z^{-2} e_{R,(k)}. \tag{317}$$

In conclusion, we have

$$\int \psi_{t_\star} \frac{|v_{(k+1)}(t_\star)|^2}{2}\,dx - \int \psi_{t_\star} \frac{|v_{(k)}(t_\star)|^2}{2}\,dx \geq c e_{R,(k)} + C Z^{-2} e_{R,(k)}. \tag{318}$$

Taking Z sufficiently large, we obtain the desired claim. $\qquad\square$

Proof of Claim 5: Irregularity of the solution. The idea of the proof below is similar to that of Claim 4. An important difference, however, is that not only do we take $Z \geq Z_\star$ for some large $Z_\star > 1$ (as in Claim 4), but we also take $k \geq k_\star$ for a sufficiently large $k_\star \geq 0$. In this proof, we shall say that a

constant is *universal* if it is independent of the given $\rho_\star, t_\star, x_\star, v_{(0)}, \psi$ and u in the hypotheses of Claim 5. A constant $C > 0$ that occurs below is always universal, unless otherwise stated. It is important to note that $Z_\star$ is also universal, whereas $k_\star$ is not.

Let $t_\star, x_\star, \rho_\star, \psi$ and u be given as in the hypotheses of Claim 5. Let us assume that $u \in W_x^{1/5,1}(B(\rho_\star; x_\star))$ since the proof in the case where $u \in C_x^{1/5}(B(\rho_\star; x_\star))$ is identical. Below, we shall use the shorthand $B := B(\rho_\star; x_\star)$.

As in the proof of Claim 4, we begin by computing

$$
\int \psi \frac{|(v_{(k+1)} - u)(t_\star)|^2}{2} \, dx - \int \psi \frac{|(v_{(k)} - u)(t_\star)|^2}{2} \, dx
$$
$$
= \int \psi e_{(k)}(t_\star) \, dx + \int \psi \Big(\frac{|V_{(k)}(t_\star)|^2}{2} - e_{(k)}(t_\star) \Big) \, dx + \int \psi(v_{(k)} - u) \cdot V_{(k)}(t_\star) \, dx. \tag{319}
$$

Since $\operatorname{supp} \psi \subseteq B \subseteq S_{t_\star}[\Omega_{(0)}] \subseteq S_{t_\star}[\Omega_{(k)}]$, we have by (301)

$$
\int \psi e_{(k)}(t_\star) \, dx \geq K e_{R,(k)}. \tag{320}
$$

For the second term on the right-hand side of (319), we have

$$
\Big| \int \psi \Big(\frac{|V_{(k)}(t_\star)|^2}{2} - e_{(k)}(t_\star) \Big) \, dx \Big| \leq C \frac{e_{v,(k)}^{1/2} e_{R,(k)}^{1/2}}{N_{(k)}} \Big(\|\psi\|_{L_x^1} + \Xi_{(k)}^{-1} \|\nabla \psi\|_{L_x^1} \Big) \tag{321}
$$
$$
\leq C Z_\star^{-2} e_{R,(k)} + C \|\nabla \psi\|_{L_x^1} Z_\star^{-2} \Xi_{(k)}^{-1} e_{R,(k)}.
$$

To estimate the third term on the right-hand side of (319), we first write

$$
\int \psi(v_{(k)} - u) \cdot V_{(k)}(t_\star) \, dx = \int \psi(v_{(k)} - u_\epsilon) \cdot V_{(k)}(t_\star) \, dx + \int \psi(u_\epsilon - u) \cdot V_{(k)}(t_\star) \, dx, \tag{322}
$$

where $u_\epsilon = \int u(x - y) \eta_\epsilon(y) \, dy$ is a mollification of u, $\eta_\epsilon(y) = \epsilon^{-3} \eta(y/\epsilon)$ and η is a smooth compactly supported function such that $\int \eta = 1$. Here we have assumed that the ϵ-neighborhood of the support of ψ is contained in B, which will be true for sufficiently small ϵ chosen in the proof below. For the last term on the right-hand side of (322), we estimate

$$
\Big| \int \psi(u_\epsilon - u) \cdot V_{(k)}(t_\star) \, dx \Big| \leq C \epsilon^{1/5} e_{R,(k)}^{-1/2} \|\psi\|_{C_x^0} \|u\|_{W_x^{1/5,1}} e_{R,(k)},
$$

where we have used the elementary convolution estimate $\|u_\epsilon - u\|_{L_x^1} \leq C \epsilon^{1/5} \|u\|_{W^{1/5,1}}$.

Finally, we estimate the first term on the right-hand side of (322). Integrating by parts and using the triangle inequality, we may write

$$
\Big| \int \psi(v_{(k)} - u_\epsilon) \cdot V_{(k)}(t_\star) \, dx \Big| \leq \int |\psi \nabla \times v_{(k)} \cdot W_{(k)}(t_\star)| \, dx + \int |\psi \nabla \times u_\epsilon \cdot W_{(k)}(t_\star)| \, dx
$$
$$
+ \int |(\nabla \psi \times v_{(k)}) \cdot W_{(k)}(t_\star)| \, dx + \int |(\nabla \psi \times u_\epsilon) \cdot W_{(k)}(t_\star)| \, dx \tag{323}
$$
$$
=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
$$

For I_1, we estimate

$$
\mathrm{I}_1 \leq C \frac{e_{v,(k)}^{1/2} e_{R,(k)}^{1/2}}{N_{(k)}} \|\psi\|_{L_x^1} \leq C Z_\star^{-2} e_{R,(k)}. \tag{324}
$$

We estimate I_2 by

$$I_2 \leq C\|\nabla u_\epsilon\|_{L_x^1} \frac{e_{R,(k)}^{1/2}}{\Xi_{(k)}N_{(k)}}\|\psi\|_{C_x^0} \leq C\epsilon^{-4/5}N_{(k)}^{-1}\Xi_{(k)}^{-1}e_{R,(k)}^{-1/2}\|\psi\|_{C_x^0}\|u\|_{W_x^{1/5,1}}\,e_{R,(k)},\qquad(325)$$

where we have used the convolution estimate $\|\nabla u_\epsilon\|_{L_x^1} \leq C\epsilon^{-4/5}\|u\|_{W_x^{1/5,1}}$.

To estimate I_3, we begin by noting that

$$\|v_{(k)} - v_{(0)}\|_{C^0(B)} \leq \sum_{j=0}^{k-1} e_{R,(j)}^{-1/2} \leq (Z_\star^{1/2} - 1)^{-1}.$$

Note also that $v_{(0)}$ is bounded on B, as it is smooth and B is compact. Therefore, we have

$$\begin{aligned}
I_3 &\leq C\|\nabla\psi\|_{L_x^1}(\|v_{(k)} - v_{(0)}\|_{C^0(B)} + \|v_{(0)}\|_{C^0(B)})\frac{e_{R,(k)}^{1/2}}{\Xi_{(k)}N_{(k)}} \\
&\leq C\|\nabla\psi\|_{L_x^1}\Big((Z_\star^{1/2} - 1)^{-1} + \|v_{(0)}\|_{C^0(B)}\Big)\theta_{(k+1)}e_{R,(k)}.
\end{aligned}\qquad(326)$$

Finally, for I_4, we have

$$I_4 \leq C\|\nabla\psi\|_{C_x^0}\|u_\epsilon\|_{L_x^1}\frac{e_{R,(k)}^{1/2}}{\Xi_{(k)}N_{(k)}} \leq C\|\nabla\psi\|_{C_x^0}\|u\|_{L_x^1}\theta_{(k+1)}e_{R,(k)}\qquad(327)$$

Putting everything together, we arrive at

$$\begin{aligned}
(319) >&Ke_{R,(k)} - CZ_\star^{-2}e_{R,(k)} \\
&- C(\epsilon^{1/5}e_{R,(k)}^{-1/2} + \epsilon^{-4/5}N_{(k)}^{-1}\Xi_{(k)}^{-1}e_{R,(k)}^{-1/2})\|\psi\|_{C_x^0}\|u\|_{W_x^{1/5,1}}\,e_{R,(k)} \\
&- C_\star(\Xi_{(k)}^{-1} + \theta_{(k+1)})e_{R,(k)} \\
=:&Ke_{R,(k)} - E_1 - E_2 - E_3.
\end{aligned}$$

where $C > 0$ is a universal constant and $C_\star > 0$ can depend on $\rho_\star$, $Z_\star$, $\Xi_{(0)}$, $\|v_{(0)}\|_{C^0(B)}$, $\|\nabla\psi\|_{L_x^1}$, $\|\nabla\psi\|_{C_x^0}$ and $\|u\|_{L_x^1}$. Taking $Z_\star \geq 2(C/K)^{1/2}$, we have

$$-E_1 \geq -\frac{1}{4}Ke_{R,(k)}.$$

Next, choosing $\epsilon = N_{(k)}^{-1}\Xi_{(k)}^{-1}$ and recalling the evolution laws for parameters (289)–(291) and (296), we see that

$$-E_2 \geq -CC_0^{-k/5}\Xi_{(0)}^{-1/5}\|\psi\|_{C_x^0}\|u\|_{W_x^{1/5,1}}e_{R,(k)}.$$

At this point, observe that $C_0^{-k/5} \to 0$ as $k \to \infty$ (since $C_0 > 1$), and also that $\Xi_{(k)}^{-1}$, $\theta_{(k+1)} \to 0$. Therefore, choosing $k \geq k_\star$ sufficiently large (but non-universal), we have

$$-E_2 - E_3 \geq -\frac{1}{4}Ke_{R,(k)}.$$

This bound concludes the proof. $\qquad\square$

A h-Principle for incompressible Euler on Euclidean space

In this Appendix, we observe that our construction leads to a result of "h-principle" type given in Theorem A.1 below. To motivate this theorem, recall that every finite energy weak solution to Euler with appropriate integrability conserves linear and angular momentum. Furthermore, note that if v_n is a sequence of finite energy solutions to Euler with appropriate uniform integrability (say, the family $\{(1 + |x|)v_n(t)\}_{n,t}$ is uniformly integrable in x), then the weak limit $v_n \rightharpoonup v$, provided that it exists, also conserves linear and angular momentum. Theorem A.1 essentially says that there are no other conservation laws closed under taking weak limits. More precisely, this theorem shows that every smooth, divergence free vector field on $\mathbb{R} \times \mathbb{R}^3$ which conserves both linear and angular momentum can be realized as a weak limit of a sequence of $C_{t,x}^{1/5 - \epsilon}$ Euler flows in the $L_{t,x}^\infty$ weak-* topology. We note that the space $L_{t,x}^\infty$ cannot be improved for this type of result in terms of regularity, and the result below implies weak convergence in L^p spaces for $1 < p < \infty$ as well.

Theorem A.1. *Let $\epsilon > 0$ and let $\mathcal{U}$ be a bounded, convex, open subset of $\mathbb{R} \times \mathbb{R}^3$. Let $v^l \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^3)$ be a smooth vector field with compact support in $\mathcal{U}$ such that for all $t \in \mathbb{R}$ we have*

$$\partial_l v^l(t, x) = 0 \qquad \forall \, x \in \mathbb{R}^3$$

$$\frac{d}{dt} \int_{\mathbb{R}^3} v^l(t, x) \, dx = 0 \qquad \forall \, l = 1, 2, 3$$

$$\frac{d}{dt} \int_{\mathbb{R}^3} (x^k v^l(t, x) - x^l v^k(t, x)) \, dx = 0 \qquad \forall \, 1 \le k < l \le 3$$

Then there exists a sequence of solutions to incompressible Euler in the class $(v_{(k)}, p_{(k)}) \in C_{t,x}^{1/5 - \epsilon} \times C_{t,x}^{2(1/5 - \epsilon)}(\mathbb{R} \times \mathbb{R}^3)$ such that $\operatorname{supp} v_{(k)} \cup \operatorname{supp} p_{(k)} \subseteq \mathcal{U}$ for all $k \in \mathbb{N}$ and $v_{(k)} \rightharpoonup v$ in $L_{t,x}^\infty$ weak-.*

Theorem A.1 contributes to the growing literature on h-principle type results in fluid equations, for which we refer the reader to [DLS12b, Cho13, CS14, IV14] for further discussion. The result helps to express the point that the only results that appear to be closed under weak limits for low regularity solutions to these equations can be viewed as conservation laws or as time regularity statements. Here we will outline the main ideas of the proof of Theorem A.1, and we will refer the reader to [IV14] for a detailed proof of an analogous result for active scalar equations.

A.1   Sketch of proof of Theorem A.1

Let $\epsilon > 0$ and let $\mathcal{U}$ be a bounded, convex, open subset of $\mathbb{R} \times \mathbb{R}^3$. Let $v^l \in \mathcal{C}_c^\infty(\mathcal{U})$ be an incompressible velocity field which conserves both linear and angular momentum, as in the statement of Theorem A.1. Consider the vector field $U^l = \partial_t v^l + \partial_j(v^j v^l)$. One can interpret $U^l(t, x)$ as the force per unit volume (or unit mass) acting on a particle at the point (t, x), since $U^l = \partial_t v^l + v^j \partial_j v^l$ by incompressibility.

Choose a smooth, symmetric tensor field $R^{jl} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3)$ with compact support in $\mathcal{U}$ such that

$$\partial_j R^{jl} = U^l \tag{328}$$

As we have seen, it is necessary for $U^l(t, \cdot)$ to be L^2-orthogonal to both translation and rotation vector fields at all times $t \in \mathbb{R}$ in order for such a tensor field to exist. For the vector field U^l above, the orthogonality conditions are equivalent to the conservation laws assumed in Theorem A.1, since the term $\partial_j(v^j v^l)$ is already the divergence of a symmetric tensor. With these conditions satisfied, we can construct the desired R^{jl} using the operators constructed in Proposition 10.1 (where we take the ambient velocity field to be 0 so that the operator is time-independent).

With this choice of R^{jl}, we may view v^l as part of a smooth solution $(v_{(0)}, p_{(0)}, R_{(0)})$ to the Euler-Reynolds equations with velocity field $v_{(0)}^l = v^l$, pressure $p_{(0)} = 0$ and stress tensor $R_{(0)}^{jl} = R^{jl}$ as chosen

above. The proof of Theorem A.1 now proceeds along the same lines as the proof of Theorem 1.1 given in Section 11. Namely, beginning with $(v_{(0)}, p_{(0)}, R_{(0)})$, one generates a sequence of Euler Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ by repeated application of Lemma 3.1 such that the sequence $(v_{(k)}, p_{(k)})$ converges in $C_{t,x}^{1/5-\epsilon} \times C_{t,x}^{2(1/5-\epsilon)}$ to a solution $(\hat{v}, \hat{p})$ of incompressible Euler. This sequence of Euler Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ is dictated by the choice of the sequence of frequency energy levels $(\Xi_{(k)}, e_{v,(k)} e_{R,(k)})$, which obey the iteration rules (289)-(291). The solution $(\hat{v}, \hat{p})$ is thus determined completely by the choice of initial frequency energy levels $(\Xi_{(0)}, e_{v,(0)}, e_{R,(0)})$ and the choice of the frequency $\Xi_{(1)}$ applied in the first stage of the iteration.

The key point in achieving solutions $\hat{v}$ which are close to the given $v = v_{(0)}$ in $L_{t,x}^\infty$ weak-* is that the initial frequency $\Xi_{(1)}$ (and all subsequent frequencies) may be chosen arbitrarily large in the first stage of the iteration while maintaining a uniform bound on $\|\hat{v} - v\|_{L_{t,x}^\infty}$ that is independent of the choice of $\Xi_{(1)}$. In fact, one can arrange that $\hat{v} - v = \nabla \times W$ where $\|W\|_{C^0} \leq \Xi_{(1)}^{-1} e_{R,(0)}^{1/2}$ can be made arbitrarily small, while maintaining a bound of the form $\|\hat{v} - v\|_{C^0} \leq C e_{R,(0)}^{1/2}$ and uniform control over the support of $\hat{v} - v$. To arrange that the support of the iteration remains inside a precompact subset of Ω, one may choose a larger frequency level $\Xi_{(0)}$ if necessary, since the choice of a sufficiently large frequency level at the beginning of the iteration will cause the time and spatial scales of the entire iteration to become arbitrarily small. Choosing a sequence of $\Xi_{(1)}$ tending to ∞, one obtains the desired sequence[9] of solutions $\hat{v}$. We refer to [IV14, Proof of Theorem 9.1] for a detailed implementation of this technique.

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[9]One may also construct continuous families of such $\hat{v}$ if desired using the construction employed here.

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