THE NASH C^1 ISOMETRIC EMBEDDING THEOREM

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Abstract. The goal of this expository talk is to present a proof of the remarkable Nash(–Kuiper) C^1 embedding theorem, which states that the unit sphere $\mathbb{S}^2$ can be 'crumpled' in a C^1 fashion into an arbitrarily small ball in $\mathbb{R}^3$. Note that such a statement is obviously false if one replaces C^1 by 'smooth', by consideration of curvature! This theorem turned out to be more than a mere curiosity; its proof foreshadowed an important technique called 'convex integration', which found remarkable applications in a wide array of fields, such as symplectic topology, calculus of variations and fluid dynamics.

1. Introduction

This talk will concern the following startling result:

Theorem 1.1 (Nash [8], Kuiper [6]). *Let* (M, g) *be any* 2*-surface,* $N \geq \dim M + 1$ *and* $u : M \to \mathbb{R}^N$ *an embedding which is strictly* short*, i.e., the length of every vector in* M *(strictly) shrinks under* ∇u*. Then* u *can be* uniformly *approximated by* C^1 isometric embeddings*.*

An example of a short map is the homothety $\mathbb{S}^2 \to \epsilon\mathbb{S}^2$ for any $0 < \epsilon < 1$. In particular, this theorem tells us that there exist C^1 isometric embedding of the standard sphere into a ball of any radius B_ϵ.

Remark 1.2. In fact, more strikingly, it is a classical result that any C^2 isometric embedding $u : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ must agree with the standard embedding $\mathbb{S}^2 \hookrightarrow \{x \in \mathbb{R}^3 : |X| = 1\}$ up to a translation and a rotation.

This theorem is proved using an iteration scheme introduced by Nash, now often called *convex integration*, which has found surprising applications in different contexts. Examples include symplectic topology [4], calculus of variations [7], and incompressible Euler equations [2, 3, 5].

The foremost goal of today's talk is to explain in detail the proof of a simpler version of Theorem 1.1 (namely Theorem 2.1); see Section 2. The outline of the proof follows the excellent lecture notes [9] of Székelyhidi, except we try to motivate the steps of the argument differently. Various extensions of this proof, including the proof of Theorem 1.1, will be sketched in Section 3.

2. A model result

Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit 2-disk, and let $g = g_{ij}(x)$ be a metric on D (i.e., for every $x \in D$, $g_{ij}(x)$ is a positive definite matrix). Recall that a map $u : D \to \mathbb{R}^n$ for is

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an *immersion* if the linear map $\nabla u(x)$ (viewed as an $n \times 2$ matrix) is injective for every x. The metric on D induced by u takes the form

$$\nabla u^{\top}(x)\nabla u(x) = \begin{pmatrix} \nabla_1 u \cdot \nabla_1 u & \nabla_1 u \cdot \nabla_2 u \\ \nabla_2 u \cdot \nabla_1 u & \nabla_2 u \cdot \nabla_2 u \end{pmatrix}.$$

When $\nabla u^{\top}\nabla u = g$ at every point $x \in D$, then the map u is *isometric*. The map u is *short* [resp. strictly short] if

$$\nabla u^{\top}\nabla u - g \leq 0 \quad \text{[resp. } < 0]$$

at every point $x \in D$.

Our goal will be to give a (more-or-less) complete proof of the following result:

Theorem 2.1 (Baby Nash). *Let $n \geq 4$ $(= 2+2)$ and $u : D \to \mathbb{R}^n$ be a strictly short immersion. The for any $\epsilon > 0$, there exists a C^1 isometric immersion $\tilde{u} : D \to \mathbb{R}^n$ such that $\|u - \tilde{u}\|_{C^0(D)} < \epsilon$.*

Remark 2.2. Note that Theorem 2.1 requires the ambient Euclidean space to have codimension at least 2; it is essentially the version proved by Nash [8]. Refinement to codimension 1, as stated in Theorem 1.1, is the contribution of Kuiper [6] (see Section 3.2).

We will achieve this by an iteration procedure, where each step consists of adding a highly oscillating(!) correction to u, designed to make the deviation from being an isometry smaller. To motivate this procedure, we start with some basic computation.

Let $u_1 = u + U$, where

$$U = \sum_{I \in \mathcal{I}} U_I$$

where I runs over an index set $\mathcal{I}$. We allow each component U_I^j to be complex-valued, as we would like them to oscillate like $e^{ix\cdot\xi}$; in order to ensure that U is real, we require that for every $I \in \mathcal{I}$, there exists $\bar{I} \in \mathcal{I}$ and

$$U_{\bar{I}} = \overline{U}_I, \quad \bar{\bar{I}} = I.$$

The new metric error $h_1 = g - \nabla u_1^{\top}\nabla u_1$ takes the form

$$h_1 = \underbrace{(h - \sum_I \nabla\overline{U}_I^{\top}\nabla U_I)}_{q_{met}} - \underbrace{\sum_I (\nabla u^{\top}\nabla U_I + \nabla U_I^{\top}\nabla u)}_{q_{lin}} - \underbrace{\sum_{I,J:J\neq\bar{I}} \nabla U_I^{\top}\nabla U_J}_{q_{high}}.$$

- **General form of a correction.** We will attempt to find a correction which oscillates in a fixed, single direction $\xi \in \mathbb{R}^2$, $|\xi| = 1$. Let us take

  $$U_I = W = \frac{1}{\lambda}a(x)\mathbf{n}(x)e^{\lambda ix\cdot\xi},$$

  where $a : D \to \mathbb{R}$ and $\mathbf{n} : D \to \mathbb{C}^n$ such that $\bar{\mathbf{n}} \cdot \mathbf{n} = 1$. To ensure reality, we also need to add $\bar{I} \in \mathcal{I}$ and define

  $$U_{\bar{I}} = \overline{W} = \frac{1}{\lambda}a(x)\bar{\mathbf{n}}(x)e^{-\lambda ix\cdot\xi}.$$

- **Eliminating the metric error** q_{met}**.** Note the basic computation:

$$\begin{aligned}\nabla_j W =& i\xi_j a(x)\mathbf{n}(x)e^{\lambda ix\cdot\xi} + \frac{1}{\lambda}\nabla_j(a(x)\mathbf{n}(x))e^{\lambda ix\cdot\xi}\\ =& i\xi_j a(x)\mathbf{n}(x)e^{\lambda ix\cdot\xi} + O(\frac{1}{\lambda}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\nabla_i W^*(x)\nabla_j W(x) =& (-i\xi_i a(x)e^{-\lambda ix\cdot\xi})(i\xi_i a(x)e^{\lambda ix\cdot\xi})\bar{\mathbf{n}}(x)\cdot\mathbf{n}(x) + O(\frac{1}{\lambda})\\ =& a^2(x)\xi_i\xi_j + O(\frac{1}{\lambda})\end{aligned}$$

where $(\cdot)^* = \overline{(\cdot)}^\top$. Observe that in this interaction, high oscillation cancelled to give a slowly-varying main term $a^2(x)\xi_i\xi_j$!

For simplicity, fix $x \in D$ for a moment, and imagine that the metric error h were of the form

$$h(x) = a^2(x)\xi\otimes\xi + b^2(x)\xi'\otimes\xi' + c^2(x)\xi''\otimes\xi''.$$

Then the above will eliminate the $\xi\otimes\xi$ component of h! Repeating this construction for ξ' and ξ'', one can imagine reducing $h(x)$ to of size $O(1/\lambda)$.

Remark 2.3. Observe that the *short* property of h is crucial, as $\nabla_i W^*(x)\nabla_j W(x)$ gives rise to a non-negative main term. As we will see soon below, in order to ensure that the new metric error h_1 is short, it is important to have some room and assume *strict shortness* of h.

However, a problem with this argument is that the eigenvectors ξ would in general depend on x. This problem can be fixed by the following lemma:

Lemma 2.4 (Decomposing the metric error)**.** *Let $\mathcal{P}$ be the space of all positive-definite matrices. There exists a sequence $\xi^{(k)}$ of unit vectors in $\mathbb{R}^n$ and a sequence $\Gamma_{(k)} \in C_c^\infty(\mathcal{P};[0,\infty))$ such that*

$$A_{ij} = \sum_k \Gamma^2_{(k)}(A)\,\xi_i^{(k)}\xi_j^{(k)},$$

and the sum is locally finite*, i.e., there exists $N\in\mathbb{N}$ such that for each $A\in\mathcal{P}$, at most N of $\Gamma_{(k)}(A)$ are non-zero.*

The idea of the proof is to:

(1) Construct a locally finite covering of neighborhoods $\mathcal{O}\subseteq\mathcal{P}$, in each of which matrices $A\in\mathcal{O}$ is the linear combination of $\xi\otimes\xi$'s with positive coefficients;
(2) Construct global functions $\Gamma^2_{(k)}$ on $\mathcal{P}$ by a partition of unity argument.

We postpone the detailed proof until the end of this section, as it is not too much related to the remainder of the argument

- **Interlude.** So far, observe that we have not specified our choice of the complex vector $\mathbf{n}$. Our goal is to show that by choosing $\mathbf{n}$ wisely, the error terms q_{lin} and q_{high} vanish up to terms of order $O(1/\lambda)$.
- **Linearization error.** We compute

$$\nabla_i u^\top\nabla_j W = i\xi_j a(x)e^{ix\cdot\xi}\nabla_i u\cdot\mathbf{n} + O(\frac{1}{\lambda})$$

To cancel the first term, we are motivated to choose $\mathbf{n}(x) \perp \nabla_j u(x) = 0$, or equivalently,

$$\mathbf{n}(x) \perp T_{u(x)}u(D).$$

This property is not difficult to satisfy, since the immersion has co-dimension 1. A similar computation applies to $\nabla_i W^\top \nabla_j u$, and we obtain

$$\nabla_i u^\top \nabla_j W + \nabla_i W^\top \nabla_j u = O(\frac{1}{\lambda}). \tag{2.1}$$

- **High-high interference.** In the present case, these are $\nabla W^\top \nabla W$ and $\nabla \overline{W}^\top \nabla \overline{W}$.

$$\nabla_i W^\top \nabla_j W = (-a^2(x)\xi_i\xi_j e^{2ix\cdot\xi})\mathbf{n}\cdot\mathbf{n} + O(\frac{1}{\lambda})$$

Here the idea is to use the fact that (1) $\mathbf{n}$ is a complex-valued vector and (2) the immersion has co-dimension ≥ 2, to cook up $\mathbf{n}$ such that

$$\mathbf{n}(x)\cdot\mathbf{n}(x) = 0.$$

The following choice would work:

$$\mathbf{n}(x) = \frac{1}{i\sqrt{2}}\zeta(x) + \frac{1}{\sqrt{2}}\eta(x),$$

where $\eta(x), \zeta(x)$ are unit (real-)vectors which are normal to $T_{u(x)}u(D)$.

- **Final form of a correction.** At the end, we arrive at the vector

$$W(x) = \frac{a(x)}{\lambda}\Big(\sin(\lambda x\cdot\xi)\zeta(x) + \cos(\lambda x\cdot\xi)\eta(x)\Big). \tag{2.2}$$

obeying the following properties:

(1) *Small C^0 norm*

$$\|W\|_{C^0} \leq C\frac{\|a\|_{C^0}}{\lambda}, \tag{2.3}$$

(2) *Main term in ∇W.*

$$\nabla W = a(x)(\cos(\lambda x\cdot\xi)\zeta(x) + \sin(\lambda x\cdot\xi)\eta(x)) + O_{\|a\|_{C^0},\|\nabla a\|_{C^0},\|\nabla\zeta\|_{C^0},\|\nabla\eta\|_{C^0}}(\frac{1}{\lambda}) \tag{2.4}$$

(3) *Small metric error.*

$$\nabla_i W^\top \nabla_j W(x) - a^2(x)\xi_i\xi_j = O(\frac{1}{\lambda}). \tag{2.5}$$

(4) *Small linearization error.*

$$\nabla_i u^\top \nabla_j W + \nabla_i W^\top \nabla_j u = O(\frac{1}{\lambda}). \tag{2.6}$$

(5) *Small high-high interference.*

$$\nabla_i W^\top \nabla_j W = O(\frac{1}{\lambda}). \tag{2.7}$$

Remark 2.5. Here is another way of arriving at (2.2). For $\gamma = (\gamma_1, \gamma_2) : D \times \mathbb{T} \to \mathbb{R}^2$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, define

$$W(x) = \frac{1}{\lambda}\Big(\gamma_1(x, \lambda x \cdot \xi)\zeta(x) + \gamma_2(x, \lambda x \cdot \xi)\eta(x)\Big).$$

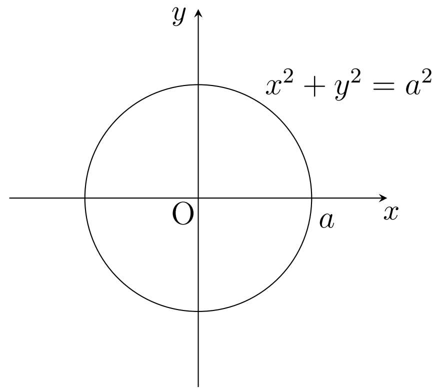
Denote by $\dot\gamma$ the derivative with respect to t. Then

$$\nabla W^\top(x)\nabla W(x) = \Big(\dot\gamma_1^2(x, \lambda x \cdot \xi) + \dot\gamma_2^2(x, \lambda x \cdot \xi)\Big)\xi \otimes \xi + O(\frac{1}{\lambda})$$

For each fixed x, we need to find $\gamma(x, \cdot)$ such that

(1) $\dot\gamma_1^2 + \dot\gamma_2^2 = a^2$,
(2) $t \mapsto \dot\gamma(x,t)$ is 2π-periodic and $\int \dot\gamma \, \mathrm{d}t = 0$.

Note that the latter condition is equivalent to $t \mapsto \gamma(x,t)$ is 2π-periodic. Note that $\dot\gamma$ is required to solve an inclusion with average zero. In particular, the origin *must* lie on the *convex hull* of the circle of radius a; this is where the term convex integration originated.



Following the language of Nash '54, adding a correction is called a *step*; as we have seen, each step eliminates one component of h up to an error of size $O(1/\lambda)$. Invoking Lemma 2.4 and iterating steps, we can reduce the $C^0(D)$ norm of the metric error; this is called a *stage*, and forms a main 'atom' for iteration. The precise statement we need is as follows:

Lemma 2.6 (Stage: Main iteration lemma)**.** *Let $u : D \to \mathbb{R}^n$ be a smooth strictly short immersion, such that $h := g - \nabla u^\top \nabla u$ obeys*

$$\|h\|_{C^0(D)} \le e_h$$

for some $e_h > 0$. Then for any $\varepsilon > 0$, there exists a smooth strictly short immersion $u_{[1]}$ of the form $u_{[1]} = u + U$, where

$$\|\nabla U\|_{C^0(D)} \le C e_h^{1/2}, \tag{2.8}$$

$$\|U\|_{C^0(D)} \le \varepsilon, \tag{2.9}$$

and $h_{[1]} := g - \nabla u_{[1]}^\top \nabla u_{[1]}$ obeys

$$\|h_{[1]}\|_{C^0(D)} \le \varepsilon. \tag{2.10}$$

Proof. By Lemma 2.4, we have

$$h(x) = \sum_k \Gamma_{(k)}^2(h(x))\xi^{(k)} \otimes \xi^{(k)}.$$

where for each $h(x)$, there are at most K many summands which are nonzero.

By compactness of $h(D) \subseteq \mathcal{P}$, there exist only finitely many summands which are nonvanishing functions. We re-enumerate these summands as

$$\Gamma^2_{(1)}(h(x))\xi^{(1)} \otimes \xi^{(1)}, \dots, \Gamma^2_{(N)}(h(x))\xi^{(N)} \otimes \xi^{(N)}.$$

By taking the trace, we see that

$$\|\Gamma_{(j)}(h)\|_{C^0} \leq \|h\|^{1/2}_{C^0} \leq e_h^{1/2}.$$

We now add N corrections (i.e., N steps) of the form (2.2) to u, oscillating in the directions $\xi^{(1)}, \dots, \xi^{(N)}$, to cancel these errors. More precisely, given a small parameter $\delta > 0$ (to be fixed soon), we recursively define $u_j = u_{j-1} + (1-\delta)^{1/2} U_j$, where $u_0 = u$ and

$$U_j = \frac{\Gamma_{(j)}(h(x))}{\lambda_j}\Big(\sin(\lambda_j x \cdot \xi^{(j)})\zeta_j(x) + \cos(\lambda_j x \cdot \xi^{(j)})\eta_j(x)\Big).$$

Here, ζ_j, η_j are defined with respect to u_j, and λ_j will also be chosen depending on u_j.

The purpose of $\delta > 0$ is to preserve strict shortness, so that we may iterate stages. We fix $0 < \delta < \frac{\varepsilon}{2(1+e_h)}$ so that $h \geq \delta I$, which is possible since u is assumed to be strictly short. Later, we will verify that $h_{[1]} \geq \frac{1}{2}\delta^2 I$.

Recall the estimates (2.3)–(2.7). For any fixed $\varepsilon > 0$, by choosing λ_j sufficiently large depending[1] on u_{j-1}, e_h and $\varepsilon > 0$, we may ensure that

$$\|U_j\|_{C^0} \ll \varepsilon, \tag{2.11}$$

$$\nabla U_j = \Gamma_{(j)}(h)(\cos(\lambda_j x \cdot \xi^{(j)})\zeta - \sin(\lambda_j x \cdot \xi^{(j)})\eta) + \mathrm{err}_j, \quad \|\mathrm{err}_j\|_{C^0} \ll \varepsilon, \tag{2.12}$$

$$h_j = h_{j-1} - (1-\delta)\Gamma^2_{(k)}(h)\xi^{(j)}\xi^{(j)} + \mathrm{err}'_j, \quad \|\mathrm{err}'_j\|_{C^0} \ll \delta^2. \tag{2.13}$$

where $h_j = g - \nabla u_j^\top \nabla u_j$. The implicit constants in $\ll$ may be chosen depending on N.

To conclude the proof, we now verify that $U = U_1 + \cdots + U_N$ and $u_{[1]} = u_N = u + U$ obey the desired conclusions. Summing up (2.11), we clearly have (2.9). To obtain (2.8) with a constant C independent of N, we recall from Lemma 2.4 that for each x, there are at most finitely many (say K) terms $\Gamma_{(j)}(h(x))$ which are nonzero. It follows that

$$|\nabla U(x)| \leq K e_h^{1/2} + \sum_{j=1}^{N} \|\mathrm{err}_j\|_{C^0} \leq 2K e_h^{1/2},$$

if the small implicit constants are chosen appropriately. Finally, summing up (2.13), we see that

$$h_N = h - (1-\delta)\sum_j \Gamma^2_{(k)}(h)\xi^{(j)}\xi^{(j)} + \sum_{j=1}^{N} \mathrm{err}'_j = \delta h + \sum_{j=1}^{N} \mathrm{err}'_j.$$

Fixing the small implicit constants appropriately, we may ensure that $u_{[1]} = u_N$ is still strictly short, i.e., $h_{[1]} = h_N \geq \frac{1}{2}\delta^2 I$ as desired, whereas (2.10) also hold. □

We are now ready to iterate stages (Lemma 2.4) to conclude the proof of Theorem 2.1.

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[1] In addition to $\|u_{j-1}\|_{C^1}$, whose size is kept in track in the iteration, we note that the choice of λ_j depends on the higher order norm $\|u_{j-1}\|_{C^2}$, in order to control $\nabla\zeta_j$ and $\nabla\eta_j$. While this is not an issue for the present construction, this 'loss of derivative' necessitates a careful smoothing procedure if one is interested in the constructing a Hölder regular isometric immersion.

Proof of Theorem 2.1 using Lemma 2.6. Let $e_{h,[k]} > 0$ be a sequence such that

$$\sum_k e_{h,[k]} \leq \epsilon, \quad \sum_k e_{h,[k]}^{1/2} < \infty.$$

By Lemma 2.6, we obtain a sequence of smooth, strictly short maps $u_{[k]}$ such that $u_{[0]} = u$ and

$$\begin{aligned} \|g - \nabla u_{[k]}^\top \nabla u_{[k]}\|_{C^0} \leq & e_{h,[k]} \\ \|\nabla u_{[k+1]} - \nabla u_{[k]}\|_{C^0} \leq & C e_{h,[k]}^{1/2} \\ \|u_{[k+1]} - u_{[k]}\|_{C^0} \leq & e_{h,[k+1]}, \end{aligned}$$

from which the theorem is obvious. □

Appendix: Proof of Lemma 2.4. We proceed in two steps, following the ideas outlined earlier.

- Note that $\mathcal{P}$ is a convex open subset of $\mathbb{R}^{n\times n}_{sym}$, which is a vector space of dimension $N = \frac{n(n+1)}{2}$. Recall Carathèodory's theorem:
  Any point x in the convex hull of $K \subseteq \mathbb{R}^N$ may be written as the convex combination of a subset $K' \subseteq K$ consisting of at most $N+1$ many points.
  Using this theorem, we may easily construct a locally finite covering of $\mathcal{P}$ by neighborhoods $\mathcal{O}_i$, each which is the convex hull of an $(N+1)$-point set $\{A_{i,1}, \dots, A_{i,N+1}\}$. Since each $A_{i,j}$ is symmetric and positive-definite, it admits a decomposition of the form $A_{i,j} = \sum_{k=1}^n c_{i,j,k}^2 \xi_{i,j,k} \otimes \xi_{i,j,k}$. It follows that every element $A \in \mathcal{O}_i$ admits a decomposition of the form
  $$A = \sum_{j,k} d_{j,k}^2 \xi_{i,j,k} \otimes \xi_{i,j,k}$$
  where $d_{j,k}$ can be chosen to depend smoothly on $A \in \mathcal{O}_i$.
- To find global functions $\Gamma_k(A)$, consider a quadratic partition of unity ψ_i subordinate to $\{\mathcal{O}_i\}$ (i.e., $\sum \psi_i^2 = 1$). Applying the above decomposition for each $\psi_i A$, we see that
  $$A = \sum_{i,j,k} (\psi_i(A) d_{j,k}(A))^2 \xi_{i,j,k} \otimes \xi_{i,j,k},$$
  which is the desired decomposition (note that the i-sum is locally finite, where as the j,k-sums are finite).

3. Extensions

Finally, we sketch some extensions of Theorem 2.1.

3.1. **Extension to embedding of manifolds.**
It is not difficult to extend Theorem 2.1 in the following ways.
To immersions from a general surface: Reduce to coordinate patches by a partition of unity.
From immersions to embeddings: Since M is compact, we can find $\varepsilon > 0$ such that

$$\inf_{x,y} \operatorname{dist}(u(x), u(y)) \geq \epsilon.$$

Now perform the construction within a $\frac{1}{100}\epsilon$ neighborhood.

3.2. **Kuiper's refinement: Codimension 1 embedding.** By modifying the form of the correction, we can achieve the same construction in the codimension 1 setting. Let $\eta : D \to \mathbb{R}^3$ be the unit normal vector field on $u(D)$, and let

$$\zeta = \nabla u(\nabla u^\top \nabla u)^{-1}\xi.$$

Take

$$U = \frac{1}{\lambda}\Big(\gamma_1(x, \lambda x \cdot \xi)\tilde{\zeta}(x) + \gamma_2(x, \lambda x \cdot \xi)\tilde{\eta}(x)\Big)$$

where

$$\tilde{\zeta} = \frac{\zeta}{|\zeta|^2}, \quad \tilde{\eta} = \frac{\eta}{|\zeta|}.$$

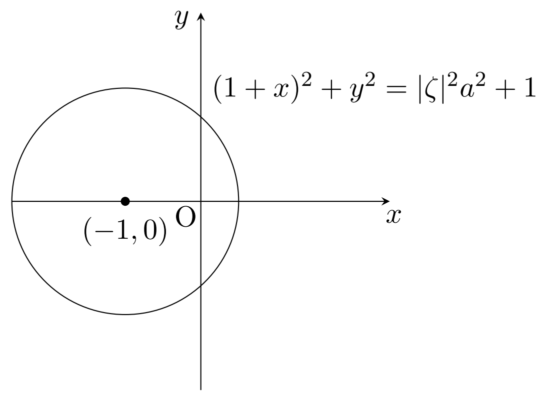
and let $u_1 = u + U$. This leads to

$$\nabla u_1^\top \nabla u_1 = \nabla u^\top \nabla u + \frac{1}{|\zeta|^2}(2\dot{\gamma}_1 + \dot{\gamma}_1^2 + \dot{\gamma}_2^2)\xi \otimes \xi + O(\frac{1}{\lambda})$$

Hence for each x and $a = a(x) \in \mathbb{R}$, we now need γ to obey

(1) $(1 + \dot{\gamma}_1)^2 + \dot{\gamma}_2^2 = |\zeta|^2 a^2 + 1$,
(2) $t \mapsto \dot{\gamma}(x, t)$ is 2π-periodic and $\int \dot{\gamma}\, \mathrm{d}t = 0$.

and such that $|\dot{\gamma}| \leq C|a|$. This is possible since the *convex* hull of $\{(x, y) : (1 + x)^2 + y^2 = |\zeta|^2 a^2 + 1\}$ contains 0 (**Exercise:** Construct such $\dot{\gamma}$!). This feature in the design of the correction is one of the reasons why Nash's technique became known as *convex integration.*



3.3. **Hölder continuous embeddings.** Finally, for refinement of the convex integration techniques to produce a Hölder $C^{1,\alpha}$-continous immersion, see [1].

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