Late Time Tail of Waves on Dynamic Asymptotically Flat Spacetimes, Part II

Sung-Jin Oh (UC Berkeley),
based on joint work with Jonathan Luk (Stanford)

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Seoul National University

Overview

Back to the main questions

Price's law: Prediction for the generic late-time tail for $\Box_{\mathbf{g}_{Sch}}\phi = 0$.

Question 1: Going beyond Price's law

What happens to Price's law if we change the setting?

To summarize what we have seen:

- Price's law generalizes to linear stationary equations, but there might be (possibly many orders of) cancellations, leading to anomalously fast decay.
- For *nonlinear* and/or *dynamical* (i.e., nonstationary) perturbations of the equation, slower decay rates may emerge.

We will now state the Main Theorem, which rigorously answers:

Question 2: Determination of late-time asymptotics

Given a general (nonlinear) wave equation in odd space dimension, how are the late-time asymptotics determined?

Main theorem

Main Theorem, executive summary

Main Theorem (Luk–O., 2024) in one sentence

There is an algorithm to determine the late time asymptotics of generic solutions to (possibly nonlinear) wave equations.

- Main Theorem reduces the PDE problem to (recurrence relations consisting of) ODEs.
- Main Theorem always gives an late-time tail upper bound $t^{-(d-1)}$ (cf. $d \geq 2$ even)
- Linearity + stationary always give cancellations in the ODE $\rightsquigarrow$ Price's law!
  Moreover, *anomalous cancellations* in ODEs could give faster decay.
- However, these cancellations in ODEs are highly unstable with respect to *nonlinear* and/or *dynamical* (i.e., *nonstationary*) perturbations of the equation.

Main Theorem

Consider $\mathcal{M} = \mathbb{R}^d \setminus \mathcal{K}$ (possibly $\mathcal{K} = \emptyset$) equipped with (τ, r, θ), where $\tau = t$ in $\{r < 2R\}$ and $\tau = t - r + C$ in $\{r > R\}$.

Main Theorem (Luk–O. (2024), simplified & summarized)

Consider the wave equation $\mathcal{P}\phi = \mathcal{N}(\phi, \partial\phi, \partial^2\phi)$ in $(d+1)$ dimensions, $d \geq 3$ odd. Assume:

1. $\mathcal{P} = (\mathbf{g}^{-1})^{\mu\nu}\partial^2_{\mu\nu} + B^\mu\partial_\mu + V$ is asymptotically flat (towards $r \to \infty$), ultimately stationary (towards $\tau \to \infty$);
2. "null condition" holds for $\mathcal{N}$ when $d = 3$,
3. ${}^{(\infty)}\mathcal{P}_0$ (ultimate stationary operator ${}^{(\infty)}\mathcal{P}$ with ∂_t dropped) is invertible with suitable estimates,
4. ϕ has C^∞_c data and the solution obeys vector field bounds $\phi = O^M_{\boldsymbol{\Gamma}}(\tau^{-\alpha_0 + \frac{d-1}{2}}(r+\tau)^{-\frac{d-1}{2}})$ for $\alpha_0 \in \mathbb{R}$; if $\mathcal{N} \neq 0$, $\alpha_0 > C(\mathcal{N})$.

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Main Theorem (Luk–O. (2024), simplified & summarized)

Then in $\{r \lesssim \tau^{1-}\}$,

$$\phi(t,r,\theta) = c_{d,J}(\mathbb{S}_{(0)}\mathfrak{L})\eta(r,\theta)\tau^{-J-\frac{d-1}{2}} + O(t^{-J-\frac{d-1}{2}-\delta}),$$

where $\mathbb{S}_{(\ell)}$ is projection to the ℓ-th spherical harmonics; $\mathbb{S}_{(0)}$ is the spherical average. For the key objects J, $\mathfrak{L}$ and η:

- Start with expansion $r^{\frac{d-1}{2}}\phi = \mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 \dots$, which obey recurrence ODE relations. $\rightsquigarrow \mathring{\Phi}_j = \mathring{\Phi}_j[\mathring{\Phi}_0]$.
- **Correction rule:** $\lim_{u\to\infty} \mathbb{S}_{(\ell)}\mathring{\Phi}_j(u,\theta) = 0$ if $j \leq \frac{d-3}{2} + \ell$.
- Let J be the smallest integer such that

$$\mathfrak{L} := \lim_{u\to\infty} \mathring{\Phi}_J(u,\theta) \neq 0.$$

- The spatial profile η solves ${}^{(\infty)}\mathcal{P}_0\eta = 0$ with suitable boundary condition at spatial infinity (i.e., $\eta \to 1$ as $|x| \to +\infty$).

Remark 1: On assumptions

Assumptions for Main Theorem

3. ${}^{(\infty)}\mathcal{P}_0$ (ultimate stationary operator ${}^{(\infty)}\mathcal{P}$ with ∂_t dropped) is invertible with suitable estimates,
4. ϕ has C_c^∞ data and the solution obeys some (weak) decay estimates.

- We allow for quasilinear nonlinearity, but Condition 2 leaves out weak null condition (e.g., $\Box\phi + \phi\Delta\phi = 0$). In this case, a different picture arises [Luk–O.–Yu, forthcoming]!
- Condition 3 $\Leftrightarrow$ "no eigenvalue or resonance at zero energy."
- Well-known obstructions to local decay, such as trapping, superradiance, discrete spectrum etc., would enter in the proof of the weak (vector field) decay estimate – more precisely, in the proof of ILED. However, they need *not* be considered in our result, which is conditional on the weak decay estimate.

Remark 2: Analysis of higher radiation fields

Recurrence ODE relations for $\mathring{\Phi}_j$: If $\Box\phi = f = r^{-1-\frac{d-1}{2}}(r^{-1}\mathring{F}_1 + \cdots)$,

$$\partial_u \mathring{\Phi}_j = -\frac{1}{2j}\left((j-1)j - \frac{(d-1)(d-3)}{4} + \mathring{\Delta}\right)\mathring{\Phi}_{j-1} + \frac{1}{2j}\mathring{F}_j.$$

Projecting to ℓ-th spherical harmonics,

$$\boxed{\partial_u \mathbb{S}_{(\ell)}\mathring{\Phi}_j = -\frac{1}{2j}\left(j - \frac{d-1}{2} - \ell\right)\left(j + \frac{d-1}{2} + \ell - 1\right)\mathbb{S}_{(\ell)}\mathring{\Phi}_{j-1} + \frac{1}{2j}\mathbb{S}_{(\ell)}\mathring{F}_j.}$$

The recurrence equations are problem specific (through $\mathring{F}_j$).

An important observation: *some part can be prescribed!* For instance, in $d = 3$, given any $\chi(u) \in C_c^\infty$, one can perturb $\mathbb{S}_{(0)}\mathring{\Phi}_0(u)$ approximately by $\epsilon\chi(u)$. This is why we can ensure $\mathfrak{L} := \lim_{u\to\infty} \mathring{\Phi}_J(u,\theta) \neq 0$ *generically* without knowing the Friedlander radiation field $\mathring{\Phi}_0$ (cf. Luk–O. (2019)).

Remark 3: Strong Huygens principle & the Correction Rule

If $\Box\phi = f = r^{-1-\frac{d-1}{2}}(r^{-1}\mathring{F}_1 + \cdots)$,

$$\partial_u \mathbb{S}_{(\ell)}\mathring{\Phi}_j = -\tfrac{1}{2j}\left(j - \tfrac{d-1}{2} - \ell\right)\left(j + \tfrac{d-1}{2} + \ell - 1\right)\mathbb{S}_{(\ell)}\mathring{\Phi}_{j-1} + \tfrac{1}{2j}\mathbb{S}_{(\ell)}\mathring{F}_j.$$

Correction rule: $\lim_{u\to\infty}\mathbb{S}_{(\ell)}\mathring{\Phi}_j(u,\theta) = 0$ if $j \le \frac{d-3}{2} + \ell$.

In particular, $\lim_{u\to\infty}\mathring{\Phi}_j(u,\theta) = 0$ for $j = 0, \ldots, \frac{d-3}{2}$

$\rightsquigarrow$ $J \ge \frac{d-1}{2}$, or $J + \frac{d-1}{2} \ge d-1$ (at least as fast as $d \ge 2$ even).

Since d is odd, there is cancellation $j - \frac{d-1}{2} - \ell = 0$ if $j = \frac{d-1}{2} + \ell$.

In particular, if $f = 0$ (i.e., $\Box\phi = 0$), then $\mathbb{S}_{(\ell)}\mathring{\Phi}_j \equiv 0$ for $j \ge \frac{d-1}{2} + \ell$ from NPC (and since ϕ has C_c^∞ data). With Correction Rule, $J = +\infty$

$\rightsquigarrow$ Strong Huygens Principle!

Clearly, the argument leading to $J = +\infty$ is very unstable, and is generically broken if $\mathbb{S}_{(\ell)}\mathring{F}_j$ is nontrivial.

Generalization of Price's law, linear stationary case

Stationary tails such as Price's law can be interpreted as an improved decay statement originating from *further cancellations in the ODE*. In general:

Theorem (Luk-O. (2024)).

For a solution ϕ to a *linear stationary* problem $\mathcal{P}\phi = 0$ satisfying the assumptions of the Main Theorem,

$$|\phi|(t,x) \leq Ct^{-d}.$$

This result is deeply related to some conserved quantities known as *Newman–Penrose constants* (cf. Angelopoulos–Aretakis–Gajic).

We have already seen an example of linear stationary problem where the rate is even faster (higher dimensional Schwarzschild).

However, these cancellations are very unstable; they are easily destroyed by nonlinear and/or nonstationary perturbations of the equation.

Generalization of Price's law, nonlinear case

The rate $t^{-(d-1)}$ is, however, very stable – this observation forms the basis of *our generalization of Price's law*.

Conjecture

Generic solutions to the Einstein vacuum equations which converge to an asymptotically flat stationary solution (e.g., Kerr) decay with an exact rate of t^{-6}.

Price's law predicts t^{-7} for linearized gravity around Schwarzschild; see Ma–Zhang (2021) and Millet (2023) for proofs. Both predictions are based on the idea that dynamic gravitational perturbations are supported in spherical modes $\ell \geq 2$.

We make an analogous conjecture for Maxwell with t^{-4} decay (Price's law predicts t^{-5}; see Ma–Zhang (2021), Millet (2023)). This is consistent with the general upper bound proved by Metcalfe–Tataru–Tohaneanu (2017).

Remark 4: Conditions on $\mathring{\Phi}_0$ from rapidly decaying data

The **correction rules**

$$(C_j) \qquad \lim_{u\to\infty} \mathbb{S}_{(\geq j-\frac{d-3}{2})} \mathring{\Phi}_j(u,\theta) = 0$$

for $j = 0, 1, \ldots$ gives necessary conditions for $\mathring{\Phi}_0$ (via recurrence ODE relations) arising from $\phi \in C_c^\infty$. More generally, data with decay $(\langle r\rangle\partial)^I \phi = o(r^{-J_0})$ gives (C_j) with $j = 0, \ldots, J_0$.

An interesting question is the converse, i.e., **scattering theory**:
if $\mathring{\Phi}_0$ satisfies the correction rule (and $\mathring{\Phi}_0$ vanishes rapidly as $u \to -\infty$), does there exist a solution ϕ with *(spatially) rapidly decaying* data?

- For $\Box\phi = r^{-3}\mathring{F}(u,\theta) + \cdots$ with $d = 3$, the necessary condition $\mathbb{S}_{(\geq 1)}\mathring{\Phi}_1 \to 0$ becomes

  $$\int (\mathring{\slashed{\Delta}}\mathring{\Phi}_0 - \mathbb{S}_{(\geq 1)}\mathring{F})(u,\theta)\,\mathrm{d}u = 0,$$

  which was derived and imposed by Lindblad–Schlue (2023) in the study of scattering theory.

Analysis of a model case

Example: $(\Box - V)\phi = 0$, $V = \frac{\epsilon}{r^3}$ for $r \gg 1$

Consider the problem $\mathcal{P}\phi := (\Box_{\mathbb{R}^{1+d}} + V)\phi = 0$.

For $\mathcal{P}$, we need two facts:

- W.r.t. $(u := t - r, r, \theta)$ in $\{r > R_0\} = \mathcal{M}_{med} \cup \mathcal{M}_{wave}$,

$$\boxed{\mathcal{P}\phi = \Box_{\mathbb{R}^{1+d}}\phi + \mathcal{P}^{rem}\phi, \quad \mathcal{P}^{rem}\phi := -\epsilon r^{-3}\phi}$$

- Define τ to agree with u in $\{r > R_0\}$, and $\tau = t$ near $\{x = 0\}$. On $\Sigma_\tau = \{\tau = const\}$,

$$\boxed{{}^{(\infty)}\mathcal{P}_0 \text{ is invertible (between some weighted Sobolev spaces on } \Sigma_\tau)}$$

Recall: In this example, ${}^{(\infty)}\mathcal{P}_0 = -\Delta + V$.
Remark: On the Schwarzschild black hole, ${}^{(\infty)}\mathcal{P}_0$ is only *degenerate elliptic* ($\because \mathcal{H}^+$), but such a statement still follows, for instance, from ILED for "$\omega = 0$".

Analysis of higher radiation fields, for $(\Box_{\mathbb{R}^{1+d}} - V)\phi = 0$, $V = \frac{\epsilon}{r^3}$ for $r \gg 1$

Example 1: r^{-3}-potential in $\mathbb{R}^{1+3}$

Consider

$$(\Box - V)\phi \quad \text{in } \mathbb{R}^{1+3}, \quad V = \frac{\epsilon}{r^3} \quad \text{for } r \gg 1,$$

as above.

Assume, for simplicity, $\phi = \mathbb{S}_{(0)}\phi$ (radial).

- When $d = 3$, $\mathring{\Phi}_0 \to 0$; $\mathring{\Phi}_0$ can be prescribed.
- Recurrence ODE relations: From $Q\Phi = \epsilon r^{-3}\Phi$,

$$-2\partial_u\partial_r \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 + \cdots\right) + \partial_r^2 \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + \cdots\right)$$
$$-\epsilon r^{-3}\left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + \cdots\right) = 0,$$

we obtain

$$2\partial_u\mathring{\Phi}_1 = 0, \quad 4\partial_u\mathring{\Phi}_2 + 2\mathring{\Phi}_1 - \epsilon\mathring{\Phi}_0 = 0,$$

or equivalently,

$$\partial_u\mathring{\Phi}_1 = 0, \quad \partial_u\mathring{\Phi}_2 = -\frac{1}{2}\mathring{\Phi}_1 + \epsilon\mathring{\Phi}_0.$$

$$(\Box - V)\phi \quad \text{in } \mathbb{R}^{1+3}, \quad V = \frac{\epsilon}{r^3} \quad \text{for } r \gg 1,$$

Assume: ϵ is constant.

- When $d = 3$, $\mathring{\Phi}_0 \to 0$; $\mathbb{S}_{(0)}\mathring{\Phi}_0$ can be prescribed.
- Recurrence ODE relations:

$$\partial_u \mathring{\Phi}_1 = 0, \quad \partial_u \mathring{\Phi}_2 = -\frac{1}{2}\mathring{\Phi}_1 + \frac{\epsilon}{4}\mathring{\Phi}_0.$$

- $\partial_u \mathring{\Phi}_1 = 0 \implies \mathring{\Phi}_1 \equiv 0$. ($\mathring{\Phi}_1$ is the Newman–Penrose constant; we call $\partial_u \mathring{\Phi}_1 = 0$ Newman–Penrose cancellation)
- $\partial_u \mathring{\Phi}_2 = -\frac{1}{2}\mathring{\Phi}_1 + \epsilon\mathring{\Phi}_0 = \epsilon\mathring{\Phi}_0$. In general $\mathring{\Phi}_2 \not\to 0$ (since $\mathring{\Phi}_0$ can be prescribed).
- Hence, the main contribution far-away comes from $\mathring{\Phi}_2$, which gives a late time tail $\phi \sim \tau^{-3}$ on compact r region.

Example 2: stationary r^{-3}-potential in $\mathbb{R}^{1+5}$

(Red: deviation from $d = 3$) Consider

$$(\Box - V)\phi \quad \text{in } \mathbb{R}^{1+5}, \quad V = \frac{\epsilon}{r^3} \quad \text{for } r \gg 1,$$

as above. **Assume:** ϵ is constant, $\phi = \mathbb{S}_{(0)}\phi$.

- When $d = 5$, $\mathring{\Phi}_0, \mathring{\Phi}_1 \to 0$; $\mathring{\Phi}_1$ can be prescribed .
- Recurrence ODE relations: From $Q\Phi = \epsilon r^{-3}\Phi$,

$$\begin{aligned}
&- 2\partial_u\partial_r \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 + r^{-3}\mathring{\Phi}_3 + r^{-4}\mathring{\Phi}_4 + \cdots\right) \\
&+ \partial_r^2 \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 + r^{-3}\mathring{\Phi}_3 + \cdots\right) \\
&- 2r^{-2} \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 + r^{-3}\mathring{\Phi}_3 + \cdots\right) \\
&- \epsilon r^{-3} \left(\mathring{\Phi}_0 + r^{-1}\mathring{\Phi}_1 + r^{-2}\mathring{\Phi}_2 + \cdots\right) = 0,
\end{aligned}$$

implies

$$2\partial_u\mathring{\Phi}_1 - 2\mathring{\Phi}_0 = 0, \quad 4\partial_u\mathring{\Phi}_2 + 2\mathring{\Phi}_1 - 2\mathring{\Phi}_1 - \epsilon\mathring{\Phi}_0 = 0,$$

$$6\partial_u\mathring{\Phi}_3 + 6\mathring{\Phi}_2 - 2\mathring{\Phi}_2 - \epsilon\mathring{\Phi}_1 = 0, \quad 8\partial_u\mathring{\Phi}_4 + 12\mathring{\Phi}_3 - 2\mathring{\Phi}_3 - \epsilon\mathring{\Phi}_2 = 0.$$

$$(\Box - V)\phi \quad \text{in } \mathbb{R}^{1+5}, \quad V = \frac{\epsilon}{r^3} \quad \text{for } r \gg 1,$$

as above. **Assume:** ϵ is constant, $\phi = \mathbb{S}_{(0)}\phi$.

- When $d = 5$, $\mathring{\Phi}_0, \mathring{\Phi}_1 \to 0$; $\mathring{\Phi}_1$ can be prescribed.
- Recurrence ODE relations:

$$\partial_u \mathring{\Phi}_1 = \mathring{\Phi}_0, \quad \partial_u \mathring{\Phi}_2 = 0 \cdot \mathring{\Phi}_1 + \frac{\epsilon}{4}\mathring{\Phi}_0,$$

$$\partial_u \mathring{\Phi}_3 = -\frac{2}{3}\mathring{\Phi}_2 + \frac{\epsilon}{6}\mathring{\Phi}_1, \quad \partial_u \mathring{\Phi}_4 = -\frac{5}{4}\mathring{\Phi}_3 + \frac{\epsilon}{8}\mathring{\Phi}_2.$$

- $\partial_u \left(\mathring{\Phi}_2 - \frac{\epsilon}{4}\mathring{\Phi}_1\right) = 0 \implies \mathring{\Phi}_2 = \frac{\epsilon}{4}\mathring{\Phi}_1 \to 0$! ($\mathring{\Phi}_2 - \frac{\epsilon}{4}\mathring{\Phi}_1$ is the Newman–Penrose constant; a similar conservation law holds for any linear stationary problem (generalized Price's law))
- $\partial_u \mathring{\Phi}_3 = -\frac{2}{3}\mathring{\Phi}_2 + \frac{\epsilon}{6}\mathring{\Phi}_1 = -\frac{\epsilon}{6}\mathring{\Phi}_1 + \frac{\epsilon}{6}\mathring{\Phi}_1 = 0!! \implies \mathring{\Phi}_3 = 0$ (anomalous cancellation).
- $\partial_u \mathring{\Phi}_4 = -\frac{5}{4}\mathring{\Phi}_3 + \frac{\epsilon}{8}\mathring{\Phi}_2 = \frac{\epsilon^2}{32}\mathring{\Phi}_1$. In general $\mathring{\Phi}_4 \not\to 0$ (since $\mathring{\Phi}_1$ can be prescribed).
- Hence, the main contribution far-away comes from $\mathring{\Phi}_4$, which gives a late time tail $\phi \sim \tau^{-6}$ on compact r region.

Example 3: dynamic r^{-3}-potential in $\mathbb{R}^{1+5}$

Consider

$$(\Box - V)\phi \quad \text{in } \mathbb{R}^{1+5}, \quad V = \frac{\epsilon(u)}{r^3} \quad \text{for } r \gg 1,$$

as above. **Assume:** $\epsilon = \epsilon(u)$, $\phi = \mathbb{S}_{(0)}\phi$.

- When $d = 5$, $\mathring{\Phi}_0, \mathring{\Phi}_1 \to 0$; $\mathring{\Phi}_1$ can be prescribed.
- Recurrence ODE relations:

$$\partial_u \mathring{\Phi}_1 = \mathring{\Phi}_0, \quad \partial_u \mathring{\Phi}_2 = \frac{\epsilon}{4}\mathring{\Phi}_0,$$
$$\partial_u \mathring{\Phi}_3 = -\frac{2}{3}\mathring{\Phi}_2 + \frac{\epsilon}{6}\mathring{\Phi}_1, \quad \partial_u \mathring{\Phi}_4 = -\frac{5}{4}\mathring{\Phi}_3 + \frac{\epsilon}{8}\mathring{\Phi}_2.$$

- $\partial_u \left(\mathring{\Phi}_2 - \frac{\epsilon}{4}\mathring{\Phi}_1\right) = -\frac{\partial_u \epsilon}{4}\mathring{\Phi}_1$. In general $\mathring{\Phi}_2 \not\to 0$ (since $\mathring{\Phi}_1$ can be prescribed).
- Hence, the main contribution far-away comes from $\mathring{\Phi}_2$, which gives a slower late time tail $\phi \sim \tau^{-4}$ on compact r region.

Thank you for your attention!