# Late Time Tail of Waves on Dynamic Asymptotically Flat Spacetimes, Part I

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# Overview

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# Motivation & Introduction

# Linear and nonlinear waves

Consider a general *nonlinear wave* equation on curved background:

$$(\mathbf{g}^{-1})^{\mu\nu}\partial_\mu\partial_\nu\phi + B^\mu\partial_\mu\phi + V\phi = \mathcal{N}(\phi, \partial\phi, \partial^2\phi).$$

Examples: general relativity, gauge theory, compressible fluids, etc.

We will be interested in the *global asymptotics* of $\phi$.

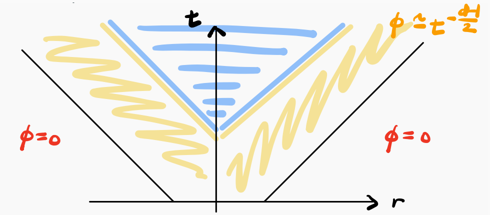
Let us begin with the simplest case, the linear wave equation on $\mathbb{R}^{1+d}$:

$$\begin{cases} (-\partial_t^2 + \Delta)\phi = 0, \\ (\phi, \partial_t\phi) \restriction_{t=0} = (g, h) \in C_c^\infty(\mathbb{R}^d). \end{cases}$$

Linear wave equation on $\mathbb{R}^{1+d}$:

$$\begin{cases} (-\partial_t^2 + \Delta)\phi = 0, \\ (\phi, \partial_t \phi) \restriction_{t=0} = (g, h) \in C_c^\infty(\mathbb{R}^d). \end{cases}$$

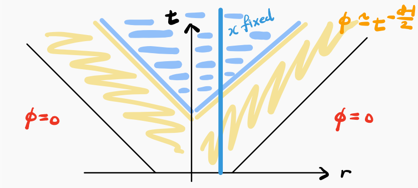
This can be explicitly solved via fundamental solution (d'Alembert, Kirchhoff, Poisson) and $\phi$ exhibits the following behavior:




- *Finite speed of propagation:* $\phi = 0$ outside of forward light (or characteristic) cone.
- *Dispersive decay:* $\sup_x |\phi(t,x)|$ is achieved in the Yellow region, where $|\phi(t,x)| \simeq t^{-\frac{d-1}{2}}$.
- *Late time tail:* $|\phi(t,x)|$ is smaller inside the Blue region.

General nonlinear wave equation on asymptotically flat background:

$$(\mathbf{g}^{-1})^{\mu\nu}\partial_\mu\partial_\nu\phi + B^\mu\partial_\mu\phi + V\phi = \mathcal{N}(\phi, \partial\phi, \partial^2\phi).$$




*Finite speed of propagation* remains true in general.

*Dispersive decay* is the main mechanism for stability of nonlinear waves.

*Late time tails* are for understanding the interaction of waves (radiation) with spatially localized objects, e.g., solitons or black holes.

**Question (late time tails/asymptotics)**

For a "generic" $\phi$, what is its asymptotics as $x$ is fixed and $t \to \infty$?

More generally, what is the asymptotics of $\phi$ along timelike curves?

# Motivation: Black holes, singularities and late time tails

Our main motivation for studying late time asymptotics comes from understanding *singularity inside black holes*.

Image credits: Event Horizon Telescope

# Penrose's incompleteness theorem & weak cosmic censorship

A celebrated theorem of Penrose – of the 2020 Nobel prize fame – says that, according to general relativity, singularities are abound:

But no singularity has ever been observed. The discrepancy explained by:

**Conjecture (Penrose, weak cosmic censorship)**

Generically, all singularities are hidden from (idealized, infinitely) far-away observers by "black holes."

# Schwarzschild and Kerr black holes

According to general relativity, gravity is described by a Lorentzian metric on $\mathcal{M}^{1+3}$ solving the *Einstein equation*:

$$Ric_{\mu\nu}[\mathbf{g}] - \frac{1}{2}\mathbf{g}_{\mu\nu} \operatorname{tr}_{\mathbf{g}} Ric = T_{\mu\nu} \quad \text{in } \mathcal{M}.$$

In suitable gauge, it is a nonlinear wave equation for $\mathbf{g}$.

Explicit "black hole" solutions (for the vacuum case $T_{\mu\nu} = 0$):

- Schwarzschild $\mathbf{g}_M$, $M > 0$: represents static black hole with mass $M > 0$;
- Kerr, $\mathbf{g}_{M,a}$, $0 < |a| < M$: represents stationary but rotating black hole with mass $M > 0$ and rotation parameter $a$

*Black hole* refers to a subregion of such a spacetime that cannot reach infinitely faraway observers (e.g., us!) even by null curves.

# Schwarzchild and Kerr black holes

In the *black hole exterior*, $\mathbf{g}_{M,a}$ and $\mathbf{g}_M$ are very close. There has been amazing developments concerning their nonlinear stability (Giorgi–Klainerman–Szeftel–Shen, Dafermos–Holzegel–Rodnianski–Taylor, etc.)

But the *black hole interiors* are completely different!

All observers that fell into the Schwarzschild black hole interior reaches a curvature singularity $\mathcal{S}$, past which $\mathbf{g}$ is not even $C^0$ extendible (Sbierski (2018)).

Meanwhile, all observers that fell into the Kerr black hole interior reaches a smooth Cauchy horizon $\mathcal{CH}^+$ (boundary of maximal globally hyperbolic development of the initial data), through which $\mathbf{g}$ is $C^\infty$ extendible(!), and the Einstein (vacuum) equation does *not* uniquely determine $\mathbf{g}$.

**Question**

How do singularities in *generic* (i.e., physical) black holes behave?

# Strong cosmic censorship

**Conjecture (Penrose, strong cosmic censorship)**

For **generic** asymptotically flat initial data, the maximal Cauchy development solving the Einstein vacuum equations is **inextendible as a suitably regular Lorentzian manifold**.

In particular, regarding Kerr, strong cosmic censorship says

- the smooth Cauchy horizon in Kerr black hole interior is nongeneric (and therefore nonphysical); and thus, a small perturbation of Kerr data leads to singularities inside the black hole.

Regarding "suitably regular":

- By Dafermos (2003; for spherical symmetric model problem) and Dafermos–Luk (2019), "suitably regular" $\neq$ "with $C^0$ metric," as opposed to Schwarzschild singularity.
- Christodoulou proposed: "suitably regular" $=$ "with $H^1_{loc}$ metric"; $\mathbf{g} \in H^1_{loc}$ is the lowest regularity needed for weak formulation of Einstein.

# Late time tails and SCC

**Conjecture (Strong cosmic censorship near Kerr)**

For *generic* small perturbations of the Kerr data, $g$ is not extendible as a “suitably regular” metric past the Cauchy horizon.

In Luk–O. (2019), we resolved the strong cosmic censorship conjecture in spherically symmetric model (Einstein–Maxwell–uncharged scalar field, two-ended asymptotically flat data).

In our approach, the first step was to analyze the black hole exterior and prove that *generic data leads to nontrivial late time tail* on the boundary of the black hole region.

The nontriviality of the tail was then used as the input for the instability of the Cauchy horizon in the black hole interior; see also Dafermos (2005).

As in Luk–O. (2019), we expect the generic late time tail in the black hole exterior to play an important role in resolving this conjecture.

# Discussion & main questions

# Late time tails for nonlinear and/or dynamical problems

Model: general nonlinear wave equation on a.f. background:

$$(\mathbf{g}^{-1})^{\mu\nu}\partial_\mu\partial_\nu\phi + B^\mu\partial_\mu\phi + V\phi = \mathcal{N}(\phi, \partial\phi, \partial^2\phi).$$

*Late time tails* are important for understanding the interaction of waves (radiation) with spatially localized objects, e.g., black holes or solitons. For this, it is important to allow for *dynamical* spacetimes and *nonlinear* equation. Also, $d = 3$ is of interest.

Interestingly, it turns out that

- the fact that $d = 3$ is odd makes the late time tail behavior nontrivial to determine, even for the *linear* wave equation on *stationary* black hole spacetimes (see **Price's law**, see below);
- for higher spin fields (i.e., tensor-valued waves such as *electromagnetic* or *gravitational* fields), late time tails behave *very differently* in the presence of *nonlinear* and/or *dynamic* (i.e., nonstationary) background (as our work shows).

# Even vs. odd space dimensions

To see why the parity of $d$ matters, consider again

$$(-\partial_t^2 + \Delta)\phi = 0 \quad \text{in } \mathbb{R}^{1+d}$$

with $(\phi, \partial_t \phi) \restriction_{t=0} = (g, h) \in C_c^\infty(\mathbb{R}^d)$, $\mathrm{supp}(g, h) \subseteq B_R(0)$.

**When $d \geq 2$ is even**

$$\boxed{\phi(t, x) \sim C t^{-(d-1)} \text{ as } t \to \infty}$$

for $x$ fixed as $t \to \infty$; $C \neq 0$ generically.

This is the expected rate from scaling: $d - 1 = (d + 1) - 2$.

This rate is expected to be *stable* under perturbations (Gajic (2023), Hintz (2023), ...).

**When $d \geq 3$ is odd**

$$\boxed{\phi(t, x) = 0}$$

for $x$ fixed as $t$ large enough, due to the *Strong Huygens Principle* (SHP).

In fact, $\phi \equiv 0$ in the entire region where $|x| < t - R$.

SHP is expected to be *unstable*, but no easy expectation for the rate.

# Instability of SHP and generation of late time tail

Let $d \geq 3$ be odd. While SHP holds for $\Box\phi = 0$, it is highly unstable!

In $(u = t - r, r, \theta \in \mathbb{S}^{d-1})$ coordinates, consider

$$\Box\phi + a(r)\partial_r^2\phi = 0 \quad \text{ in } \mathbb{R}^{d+1}$$

where $a = \epsilon r^{-1}\chi_{>1}(r)$ and $\epsilon$ is constant. Then as long as $\epsilon \neq 0$,

$$\boxed{\phi(t,x) \sim Ct^{-d} \text{ as } t \to \infty}$$

for $x$ fixed as $t \to \infty$; $C \neq 0$ generically.

Note that a tail is generated, but it is faster than the even dimensional case!

For data supported on the $\ell$-th spherical harmonics, (if $\epsilon \neq 0$)

$$\boxed{\phi(t,x) \sim Ct^{-d-2\ell} \text{ as } t \to \infty}$$

for $x$ fixed as $t \to \infty$; $C \neq 0$ generically.

# Price's law

PHYSICAL REVIEW D VOLUME 5, NUMBER 10 15 MAY 1972

**Nonspherical Perturbations of Relativistic Gravitational Collapse.**
**I. Scalar and Gravitational Perturbations***

Richard H. Price†
*California Institute of Technology, Pasadena, California 91109*
(Received 12 April 1971; revised manuscript received 27 December 1971)

**Schwarzschild black hole exterior**: $\mathcal{M} = (0,\infty)_t \times (2M,\infty)_r \times \mathbb{S}^2$,

$$\mathbf{g}_M = -(1-\tfrac{2M}{r})\mathrm{d}t^2 + (1-\tfrac{2M}{r})^{-1}\mathrm{d}r^2 + r^2\mathbf{g}_{\mathbb{S}^2}.$$

**"Price's law" on $(3+1)$-dimensional Schwarzschild black holes**

Assume:

- $\Box_{\mathbf{g}_M}\phi = 0$,
- $\phi$ initially compactly supported,
- $\phi$ supported on the spherical harmonics of degree $\ell$.

Then Price's prediction was: $\boxed{|\phi(t,r,\theta)| \simeq t^{-2\ell-3}.}$

Through the work of many contributors in the past few decades, Price's prediction had been made into a theorem.

Consider $\Box_{\mathbf{g}_M}\phi = 0$ on Schwarzschild spacetime, $\phi$ initially smooth and compactly supported.

**Theorem (Price's law)**

1. (Dafermos–Rodnianski (2005), Tataru (2013), Donninger–Schlag–Soffer (2012), Metcalfe–Tataru–Tohaneanu (2012))

$$\boxed{|\phi(t, r, \theta)| \leq C(r)t^{-3}.}$$

2. (Angelopoulos–Aretakis–Gajic (2018, 2021), Hintz (2020)) Generic $\phi$ supported on spherical harmonics $\geq \ell$ obeys, for some $c \neq 0$,

$$\boxed{\phi(t, r, \theta) = ct^{-3-2\ell}Y_\ell(\theta) + O_r(t^{-3-2\ell-\delta}),}$$

where $Y_\ell$ is a spherical harmonic.

**Theorem (Price's law)**

For $\Box_{\mathbf{g}_M}\phi = 0$, $\phi$ initially smooth and compactly supported:

1. $\boxed{|\phi|(t,r,\theta) \le C(r)t^{-3}.}$
2. Generic $\phi$ supported on spherical harmonics $\ge \ell$ obeys
   $\boxed{\phi(t,r,\theta) = ct^{-3-2\ell}Y_\ell(\theta) + O_r(t^{-3-2\ell-\delta}).}$

Remarks:

1. More generally in $(3+1)$ dimensions:
   - Theorem holds on a large class of asymptotically flat, stationary (and spherically symmetric in case $\ell \ge 1$) spacetimes, including Kerr.
   - Upper bound also holds on dynamical spacetimes.
2. Restriction to spherical harmonics $\ge \ell$ is a proxy for considering spin-$\ell$ fields (e.g., electromagnetic and gravitational fields, which are spin 1 and 2 respectively).
3. Restriction to spherical harmonics $\ge 1$ is similar to considering $d > 3$ odd. Indeed, if $\phi$ supported on spherical harmonics $\ell$, then $\Box_{\mathbb{R}^{1+3}}\phi = 0 \Leftrightarrow \Box_{\mathbb{R}^{1+3+2\ell}}(r^{-\ell}\phi) = 0$.

See also:

Barack–Ori, Bičák, Bizoń–Chmaj–Rostworowski, Blaksley–Burko, Burko–Khanna, Casals–Ottewill, Ching–Leung–Suen–Young, Gómez–Winicour–Schmidt, Gundlach–Price–Pullin, Hod, Krivan–Laguna–Papadopoulos–Andersson, Leaver, Lucietti–Murata–Reall–Tanahashi, Marsa–Choptuik, Poisson, Szpak, Zenginoǧlu–Khanna–Burko, ..., Aretakis, Baskin–Vasy–Wunsch, Dafermos–Rodnianski, Donninger–Schlag–Tataru, Gajic, Gajic–Kehrberger, Guillarmou–Hassell–Sikora, Kehrberger, Lindblad, Looi, Looi–Xiong, Luk–O., Ma, Ma–Zhang, Millet, Morgan, Morgan–Wunsch, Moschidis, Oliver–Sterbenz, Schlue, Van de Moortel, ...

# Main questions

**Question 1: Going beyond Price's law**

What happens to Price's law if we change the setting?

In particular, what happens if we go beyond the $(1+3)$-dimensional, linear, and stationary setting? More generally,

**Question 2: Determination of late-time asymptotics**

Given a general (nonlinear) wave equation in odd space dimension, how are the late-time asymptotics determined?

Later, I will present our *Main Theorem*, which answers Question 2 for a large class of nonlinear wave equations on dynamical background.

But first, I will present several applications of the Main Theorem to answer Question 1, which turns out to be quite subtle.

# Examples & some conjectures

# Example 0: Anomalous decay in the linear stationary case

Consider

$$\Box\phi = c(r)\phi \quad \text{in } \mathbb{R}^{d+1},$$

$c$ small, smooth, $c(r) = \epsilon r^{-3}$ for large $r$, $\epsilon$ constant.

- When $d = 3$, $\boxed{\phi \sim \epsilon C(r) t^{-3}}$.

So $\phi \sim \epsilon C(r) t^{-5}$ when $d = 5$?

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- When $d = 3$, $\boxed{\phi \sim \epsilon C(r) t^{-3}}$.

So $\phi \sim \epsilon C(r) t^{-5}$ when $d = 5$? No!

**Theorem (Luk–O. (2024))**

When $d = 5$,

$$\boxed{|\phi| \sim \epsilon C(r) t^{-6},}$$

where $C \neq 0$ generically.

# Example 1: Anomalous decay in the linear stationary case

Consider higher dimensional Schwarzschild black holes:

$$\mathbf{g}_M = -\Big(1 - \frac{2M}{r^{d-2}}\Big)\,\mathrm{d}t^2 + \Big(1 - \frac{2M}{r^{d-2}}\Big)^{-1}\mathrm{d}r^2 + r^2\mathbf{g}_{\mathbb{S}^{d-1}}.$$

For generic stationary metrics with $r^{-d+2}$ decay, solution has $t^{-2d+3}$ tail.

- So $\phi \sim Ct^{-7}$ on $(1+5)$-dimensional Schwarzschild?

# Example 1: Anomalous decay in the linear stationary case

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For generic stationary metrics with $r^{-d+2}$ decay, solution has $t^{-2d+3}$ tail.

- So $\phi \sim Ct^{-7}$ on $(1+5)$-dimensional Schwarzschild? No!

In fact, heuristic/numerical arguments suggest *anomalous faster decay* Cardoso–Yoshida–Dias–Lemos (2003), Bizoń– Chmaj–Rostworowski (2007). Indeed, we prove

**Theorem (Luk–O. (2024))**

Solutions arising from compactly supported data to $\Box_{\mathbf{g}_M}\phi = 0$ on $(5+1)$-dimensional Schwarzschild spacetime satisfies

$$\boxed{\phi(t,r,\theta) \sim Ct^{-10},}$$

where $c \neq 0$ generically.

# Example 2: Dynamical black hole background

However, on *dynamical* backgrounds, slower tails emerge! For $\Box\phi = c\phi$, if $c = \frac{\epsilon(u,r)}{r^3}$ on $\mathbb{R}^{1+5}$, then if $\partial_u \epsilon \neq 0$, $\boxed{\phi \sim \epsilon C(r) t^{-4}}$ (Luk–O. (2024); cf. $t^{-6}$ when $\partial_u \epsilon = 0$).

For a generic dynamical spacetime tending to the $(5+1)$-dimensional Schwarzschild spacetime, $\boxed{\phi \sim \epsilon t^{-6}}$ (Luk–O. (2024); cf. $t^{-10}$ in the stationary case)

More interesting is the next example from general relativity in dimension $(1+3)$.

# Example 2: Dynamical black hole background

**Theorem (Luk–O. (2024))**

Let $(\mathcal{M}, \mathbf{g})$ be a "generic" $(1+3)$-dimensional spherically symmetric spacetime **converging** to Schwarzschild. Consider $\phi$ solving $\Box_{\mathbf{g}}\phi = 0$ with compactly supported data supported in spherical harmonics $\geq \ell$. Then

$$\boxed{\phi(t,r,\theta) = \begin{cases} ct^{-3} + O(t^{-3-\delta}) & \ell = 0, \\ ct^{-2-2\ell}Y_\ell(\theta) + O(t^{-2-2\ell-\delta}) & \ell \geq 1, \end{cases}}$$

where "generically" $c \neq 0$.

Observe that *the $\ell = 0$ tail remains the same*. This was also verified for the spherically symmetric Einstein–Maxwell–Scalar Field system by **Gautam (2024)**, leading to a stronger version of SCC (mass inflation).

However, when $\ell \neq 0$, the tail is slower than Price's law decay (which was $t^{-3-2\ell}$); *these should be relevant for higher spin fields!*

Our result rigorously verifies (and settles) the scenario suggested by Bizoń–Chmaj–Rostworowski, etc., and numerically observed by Gundlach–Price–Pullin (1994).

Indeed, from **Gundlach–Price–Pullin (1994)** (revisited by Bizoń–Chmaj–Rostworowski (2008)):

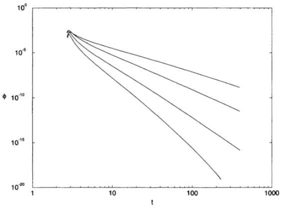


FIG. 12. Log-log plots of test fields $\varphi_l^m(r = 10, t_B)$ for different multipole indices $l$ on one background spacetime. The background is evolved from Gaussian initial data for $\phi$ with amplitude 0.02 (noncollapsing). The initial data for the test field are in each case a Gaussian (the test field amplitude 1.0 has no significance). From top to bottom the curves correspond to $l = 0, 1, 2, 3$. The best fits for the power-law exponents are $-2.77$, $-3.95$, $-5.94$, and $-8.34$, compared to predictions of $-3$, $-5$, $-7$, and $-9$.

Recall: Price's law on exact

Schwarzschild is
$\phi_{(\geq \ell)} \sim C t^{-3-2\ell} Y_\ell$.

# Generalization of Price's law

As we have seen, *nonstationarity* (i.e., dynamical spacetime) should lead to a behavior *different* from the linear stationary case. In fact, *nonlinearity* also has a similar effect (see Ex. 3). Hence, we make:

**Conjecture**

Generic solutions to the Einstein vacuum equations which converge to an asymptotically flat stationary solution (e.g., Kerr) decay with an exact rate of $t^{-6}$.

Price's law predicts $t^{-7}$ for linearized gravity around Schwarzschild; see Ma–Zhang (2021) and Millet (2023) for proofs. Both predictions are based on the idea that dynamic gravitational perturbations are supported in spherical modes $\ell \geq 2$.

We make an analogous conjecture for Maxwell with $t^{-4}$ decay (Price's law predicts $t^{-5}$; see Ma–Zhang (2021), Millet (2023)). This is consistent with the general upper bound proved by Metcalfe–Tataru–Tohaneanu (2017).

# Example 3: Effect of nonlinearity

Nonlinear equation with "null condition"' in $(3+1)$ dimensions:

$$\Box\phi^I = \Gamma^I_{JK}(\phi)(\partial_t\phi^J\partial_t\phi^K - \sum_{i=1}^{3}\partial_i\phi^J\partial_i\phi^K), \quad I, J, K = 1, \cdots, N.$$

**Theorem (Christodoulou, Klainerman (1986))**

Suppose the initial data $(\phi, \partial_t\phi)$ are $C_c^\infty$ and sufficiently small, then the solutions are global-in-time and decay.

Nonlinear decay of $O(t^{-2})$ is known by Christodoulou.

But in many interesting cases, this is *not* the generic tail. In determination of the tail, both dispersion and structure of nonlinearity are important!

**Example:** There are equations for which $\phi = O(t^{-\infty})$. (e.g., $\Box\phi = (\partial_t\phi)^2 - \sum_{i=3}^{3}(\partial_i\phi)^2$, equivalent to $\Box\psi = 0$ for $\phi = e^{\psi}$.)

Consider the wave map system:

- $(\Sigma, h)$ a 2-dimensional Riemannian manifold
- $\phi : \mathbb{R}^{3+1} \to \Sigma$ a wave map:

$$\Box\phi^I = \Gamma^I_{JK}(\phi)(\partial_t\phi^J\partial_t\phi^K - \sum_{i=1}^{3}\partial_i\phi^J\partial_i\phi^K).$$

Perturbations of the constant map $\mathbb{R}^3 \mapsto p \in \Sigma$ satisfies the global stability theorem.

For wave maps, late time tail depends on curvature of $(\Sigma, h)$ at $p$!

$\phi : \mathbb{R}^{3+1} \to \Sigma$ a wave map, perturbation of the constant map to $p \in \Sigma$.

**Theorem (Luk–O. (2024))**

1. Suppose the Gauss curvature $K(p) \neq 0$. Then, for an open and dense subclass of small initial data, the solution obeys
$$t^{-3} \lesssim \mathrm{dist}(\phi(t,x), p) \lesssim t^{-3}.$$
2. Suppose $K(p) = 0$. Then small-data solutions obey
$$\mathrm{dist}(\phi(t,x), p) \lesssim t^{-4}.$$

Thank you for your attention!