Lecture III Dynamics of smooth sol'n & forward-in-time construction of singular dynamics

$$(CSS_m) \qquad \boxed{\partial_t u + i L_u^* D_u u = 0}$$

$$\boxed{\begin{aligned} &D_u = \partial_r - \frac{(m + A_\theta[u])}{r}, \qquad L_u = D_u - \frac{2A_\theta[u(\cdot)]}{r} u \\ &A_\theta[u] = A_\theta[u,u], \qquad A_\theta[u,v] = -\frac{1}{2}\int_0^r \mathrm{Re}(\bar{u} v)\, r' dr' \end{aligned}}$$

Solitons :

$$\boxed{Q_{\lambda,\gamma} := e^{i\gamma} \frac{1}{\lambda} Q\left(\frac{r}{\lambda}\right)}$$

where

$$\boxed{Q := \sqrt{8}(m+1) \frac{r^m}{1 + r^{2(m+1)}} \quad \text{if} \quad m \geq 0}$$

Linearized Operator : $\mathcal{L}_Q$

$$\boxed{N(i\mathcal{L}_Q) = \langle \Lambda Q, iQ \rangle}, \qquad \boxed{N_g(i\mathcal{L}_Q) = \langle -i\frac{y^2}{4} Q, \rho \rangle}$$

Recall :

Exact Pseudo conformal Blow up

$$\boxed{S = [Q e^{-ib\frac{y^2}{4}}]_{\lambda,\gamma} \quad \begin{cases} \lambda = |t|, & \gamma = 0 \\ b = |t| \end{cases}}$$

Exact Rotational Instability

$$\boxed{u = [P(y; b, \eta)]_{\lambda,\gamma} \quad \begin{cases} \lambda = \sqrt{|t|^2 + \eta_0^2}, & \gamma = (m+1)\arctan\left(\frac{-t}{\eta_0}\right) \\ b = -t, & \eta = \eta_0 \end{cases}}$$

We want to study the dynamics of smth sol'ns.
How much of these exact dynamics would we see?

Consider set of id:

$$\mathcal{O}_{init} = \{ u_0 = [P(\cdot; b_0, \eta_0) + \epsilon_0]_{\lambda_0, \gamma_0} : \|\epsilon_0\|_{\dot{\mathcal{H}}^3_m} < b_0^3 \}$$

where $P(\cdot; b, \eta) = Q - ib\frac{y^2}{4}Q + (m+1)\eta\rho + O_{H^1_m}(|(b,\eta)|^{1+})$

((modified) profile, to be determined) ← needs to be cut off when $m=0$.

When $m \geq 1$:

Thm $(m \geq 1)$ [Kim-Kwon 2] $\exists b^*$ s.t. the following holds.

Let $u_0 \in \mathcal{O}_{init} \cap \{0 < b_0 < b^*\}$. Then $\exists \tilde{\eta}_0 \in (-b_0^{1+}, b_0^{1+})$

s.t. $\tilde{u}_0 = [P(\cdot; b_0, \tilde{\eta}_0) + \epsilon_0]_{\lambda_0, \gamma_0}$ leads to $\tilde{u}$: H^3_m-sol'n with $T_+ < +\infty$

s.t. $$\begin{cases} \tilde{u} - Q_{\lambda,\gamma} - z^* \xrightarrow{t \nearrow T_+} 0 \quad \text{in } L^2 \\ \lambda(t) \overset{t \nearrow T_+}{\sim} \ell (T_+ - t) \ (\ell > 0) \\ \gamma(t) \overset{t \nearrow T_+}{\sim} \gamma^* \end{cases}$$

← "codim 1 inv. set" leading to pseudo conf. blow up

When $m = 0$, $S \in H^1_m$. But we have:

Thm $(m=0)$ [Kim-Kwon-O.] $\exists b^*$ s.t. the following holds.

Let $u_0 \in \mathcal{O}_{init} \cap \{0 < b_0 < b^*\}$. Then $\exists \tilde{\eta}_0 \in \left(-\frac{b_0}{|\log b_0|}, \frac{b_0}{|\log b_0|}\right)$

s.t. $\tilde{u}_0 = [P(\cdot; b_0, \tilde{\eta}_0) + \epsilon_0]_{\lambda_0, \gamma_0}$ leads to $\tilde{u}$: H^3_m-sol'n with $T_+ < +\infty$

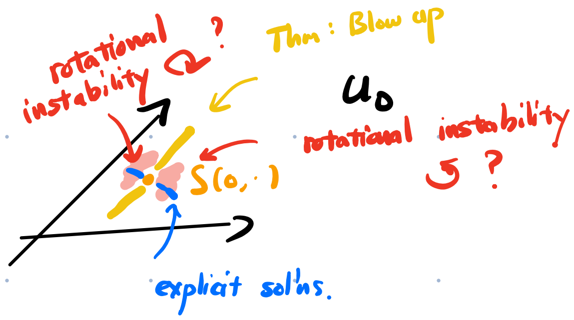
s.t. $$\begin{cases} \tilde{u} - Q_{\lambda,\gamma} - z^* \xrightarrow{t \nearrow T_+} 0 \quad \text{in } L^2 \\ \lambda(t) \overset{t \nearrow T_+}{\sim} \ell \frac{(T_+ - t)}{|\log(T_+ - t)|^2} \quad (\ell > 0) \\ \gamma(t) \overset{t \nearrow T_+}{\sim} \gamma^* \end{cases}$$

← "codim 1 inv. set" to $\log^2$-corrected pseudo conformal blow up

The exact sol'ns from Lecture I suggests

Conj. (Nonlinear rotational instability)

The blow up sol'ns constructed in the above thms can be avoided via nonlinear rotational instability.



Rmk • [Kim] : For $m \geq 1$, if u: H^3_m-sol'n, $T_+ < +\infty$, then $\forall$ finite blow up is as in Thm ($m \geq 1$). In particular,

$$\boxed{\lambda(t) \sim \ell (T_+ - t)^{\pm\lambda T_+} \text{ for some } \ell > 0.}$$

So Conj $\Longrightarrow$ "singularity formation is non-generic for $m \geq 1$."

- For $m = 0$, $\exists$ continuum of blow-up rates from $z^* = q r^\nu \chi_{\varepsilon_1}(r)$ [Kim-Kwon-O. 3]. Such sol'ns will be H^1_m, but not more regular. For smth sol'ns, $\exists$ discrete blow-up rates is expected. (see [Raphaël-Schweyer]) Maybe these are all...?

Caricature of forward-in-time construction $(m \geq 1)$ [Kim–Kwon 2]

1. Main decomposition

(based on
[Merle–Raphaël],
[Rodnianski–Sterbenz],
[Raphaël–Rodnianski]
[Merle–Raphaël–Rodnianski])

$$u = [\underbrace{P(\cdot\,; b(t), \eta(t))}_{\text{(modified) profile}} + \underbrace{\epsilon}_{\text{error}}]_{\lambda(t), \gamma(t)}$$

$$(\Leftrightarrow) \quad U = P(y; b, \eta) + \epsilon$$

where $P(y; b, \eta) \text{ "=" } Q - bi\frac{y^2}{4}Q + \eta (m+1) \rho + \mathcal{O}(b, \eta)^2$

2. Adapted coordinates ("fix Q") & Equation

$$\boxed{dt = \lambda^2 ds, \quad r = \lambda y, \quad u = \frac{e^{i\gamma}}{\lambda} U}$$

(i.e., $u = [U(s(t), \cdot)]_{\lambda, \gamma}$)

(CSS_m) $\partial_t u + i L_u^* D_u u = 0$

$$\Longleftrightarrow \quad \boxed{\underbrace{\partial_s U - \frac{\lambda_s}{\lambda} \Lambda U + \gamma_s i U}_{\text{from } \partial_t} + i L_U^* D_U U = 0.}$$

Plug in $U = P(y; b(s), \eta(s)) + \epsilon$

$$\partial_s \epsilon + i \mathcal{L}_P \epsilon - \frac{\lambda_s}{\lambda} \Lambda \epsilon + \gamma_s i \epsilon \quad \leftarrow \text{linear evolution for } \epsilon$$

$$= + \frac{\lambda_s}{\lambda} \Lambda P - \gamma_s i P - b_s \partial_b P - \eta_s \partial_\eta P$$

$$- i L_P^* D_P P$$

modulation & modified profile construction

$$+ i R_P(\epsilon) \quad \leftarrow \text{nonlinearity, will ignore}$$

We will impose 4 orthogonality cond's on ϵ, resembling $\epsilon \perp N_g((i\mathcal{L}_Q)^*)$ at the linear order.

Since $\underset{\parallel}{N_g((i\mathcal{L}_Q)^*)} \oplus \underbrace{N_g(i\mathcal{L}_Q)}_{= (\partial_\lambda P_{\lambda,\gamma},\ \partial_\gamma P_{\lambda,\gamma},\ \partial_b P_{\lambda,\gamma},\ \partial_\eta P_{\lambda,\gamma}) \text{ at } (\lambda,\gamma,b,\eta)=(1,0,0,0)} = L^2$, this can be

achieved by selecting $(\lambda, \gamma, b, \eta)$ via implicit ftn thm.

Goal: $\underline{\underline{A}}$ Construct P & ODE system for b, η $(+\lambda, \gamma)$

so that $\lambda(s) \to 0$, $b(s) \to 0$, as $s \to +\infty$.

& $\underline{\underline{B}}$ also so that $\epsilon(s) \to 0$

via $\underline{\underline{C}}$ bootstrap & shooting (choosing η to avoid instability)

Rmk: Compared to the global (asymptotic) stability pf, note that using linear ODE's for λ, γ, b, η

$$\lambda_s + b = 0, \quad b_s = 0, \quad \gamma_s - (m+1)\eta = 0, \quad \eta_s = 0$$

should NOT be possible, because it predicts self-similar blow up $\left(\begin{array}{c} b > 0 \text{ const } \& \ \lambda_t + \dfrac{b}{\lambda} = 0 \\ \Leftarrow \text{ self-similar} \end{array} \right)$

We need to incorporate the nonlinearity.

(Recall also: $S \leftrightarrow \lambda_s + b = 0,\ b_s = b^2,\ \gamma_s = 0 \quad (\eta = 0)$)

Reminiscent of center mfd theory near equilibrium.

$\leftrightarrow$ modified profile construction

A (maybe too) simple ODE model, $\mathbb{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3$

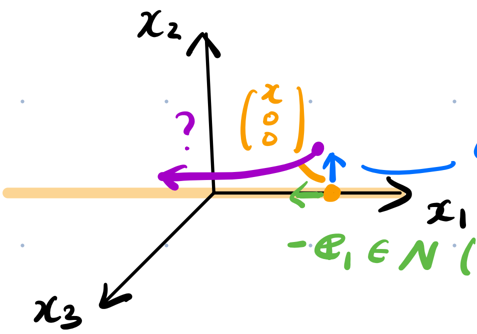
$$\boxed{\dot{\mathbb{x}} = f(\mathbb{x}), \quad f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 + x_3^2 \\ -x_3 - x_2^2 + x_3^2 \end{pmatrix}}$$

Equilibria : $\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \right\}$ $\longrightarrow$ think : $\begin{cases} x_1 \sim \log \lambda \\ x_2 \sim b \\ x_3 \sim \epsilon \end{cases}$

Linearization around $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$: $Df\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

$$N\left(Df\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}\right) = (\mathbb{e}_1),$$

$$N_g\left(Df\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}\right) = (\mathbb{e}_1, \mathbb{e}_2)$$



$\mathbb{e}_2$, $Df\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \mathbb{e}_2 = \mathbb{e}_1$

$-\mathbb{e}_1 \in N\left(Df\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}\right)$

$$\boxed{\text{Goal : } \exists\ \mathbb{x}(s) \longrightarrow (-\infty, 0, 0) \text{ as } s \longrightarrow +\infty}$$

A. Modified profile construction

$$\mathbb{P}(b)_x = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + b\,\mathbb{e}_2 + b^2 \mathbb{T}, \quad \mathbb{T} = \begin{pmatrix} 0 \\ T_2 \\ T_3 \end{pmatrix}$$

$\longleftarrow$ ok, because x_1 does not appear in eqns for x_2, x_3.

$$= \begin{pmatrix} x \\ b \\ 0 \end{pmatrix} + b^2 \mathbb{T}$$

Here, $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_x := \begin{pmatrix} u_1 + x \\ u_2 \\ u_3 \end{pmatrix}$ $\longleftarrow$ (imitating modulation by λ)

Let x & b depend on s. Compute :

$$\dot{\mathbb{P}}_x - f(\mathbb{P}_x) = \begin{pmatrix} \dot{x} \\ \dot{b} \\ 0 \end{pmatrix} + 2b\dot{b}\begin{pmatrix} 0 \\ T_2 \\ T_3 \end{pmatrix} - \begin{pmatrix} b + b^2 T_2 \\ b^2 T_3 + b^4 T_3^2 \\ -b^2 T_3 - (b + b^2 T_3)^2 + (b^2 T_3)^2 \end{pmatrix}$$

$O(b^2)$ terms in $f(\mathbb{P})$: $\begin{pmatrix} T_2 \\ T_3 \\ -T_3 - 1 \end{pmatrix}$ $Df\begin{pmatrix} x \\ \vdots \\ \end{pmatrix}T$

nonlinearity

→ Can we choose T_2, T_3 to cancel this ?

Issue : $\begin{pmatrix} T_2 \\ T_3 \\ -T_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ is not solvable.

But we can also use $\dot b$! If $\dot b = c_2 b^2 + \cdots$,
$O(b^2)$ terms cancel if

$$\begin{pmatrix} T_2 \\ T_3 \\ -T_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix}$$

$\rightarrow$ $T_2 = 0$, $c_2 = -1$, $T_3 = -1$

& $\dot{\mathbb{P}}_b - f(\mathbb{P}_b) = \begin{pmatrix} 0 \\ O(b^4) \\ O(b^3) \end{pmatrix}$

Note : $\begin{cases} \dot x = -b \\ \dot b = -b^2 \end{cases}$ leads to $x \xrightarrow{s\to\infty} -\infty$, $b \xrightarrow{s\to\infty} 0$ (pseudoconformal blow up)

Now consider :

$$\mathbf{x}(s) = (\mathbb{P}(b) + \epsilon)_x = \begin{pmatrix} x \\ b \\ -b^2 + \epsilon \end{pmatrix}$$

$$\Rightarrow \begin{cases} \dot x = -b \\ \dot b = -b^2 + O(\epsilon) + O(b^4) \\ \dot\epsilon = -\epsilon + O(b^3) \end{cases}$$

B. Decay

↑ linear decay !

Exercise

→ can show $\begin{cases} x \to -\infty \\ b \to 0 \\ \epsilon \to 0 \end{cases}$ as $s \to \infty$ (by ⊆ Bootstrap for this ODE)

if $\epsilon_0 = O(b^3)$ & $b > 0$ small

A. Modified Profile Construction (See [Kim, Kim-Kwon-O.1])

Construct :

$$\begin{cases} \hat{P} = Q - ib\frac{y^2}{4}Q + (m+1)\eta\rho + \sum_{j,k} b^j \eta^k T_{j,k}(y) \\ b_s = \sum_{j\geq 2} c_j b^j \\ \eta_s = \sum_{k\geq 2} d_k \eta^k \end{cases}$$

so that, under $\frac{\lambda_s}{\lambda} + b = 0$, $\gamma_s = -(m+1)\eta$,

after applying cutoff

$$P = Q + \left[-ib\frac{y^2}{4}Q + (m+1)\eta\rho + \sum_{j,k} b^j\eta^k T_{j,k}(y)\right]\chi(y/B)$$

the error

$$\boxed{\Psi_P := -\frac{\lambda_s}{\lambda}\Lambda P + \gamma_s iP + b_s\partial_b P + \eta_s\partial_\eta P + iL_P^* D_P P}$$

is sufficiently small in an appropriate sense

(so that we can close the ODEs) (in some Sobolev space from decay est for ϵ ?)

- P is constructed recursively, adjust to ensure solvability

$$\mathcal{L}_Q T_{j,k} = F_{j,k}\left((T_{j',k'}, c_{j'}, d_{k'})_{j'+k'<j+k}\right)$$

- When evaluating Ψ_P, spatial decay of P also matters.

  One should think "gaining b loses y^2 (inverting $\mathcal{L}_Q$)"

  $\longrightarrow$ natural cutoff is $B = b^{1/2}$, which is roughly the self-similar scale. ↖ usual choice

B. Decay for ϵ For forward analysis, we need decay.

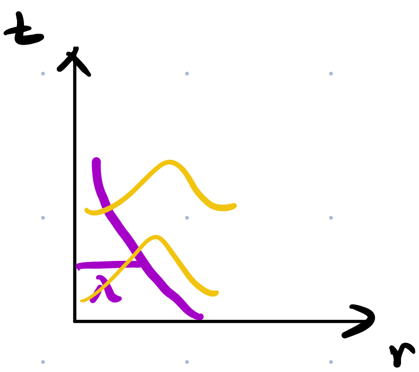
① Decay of the linear level (cf. Schörkhuber's lecture)

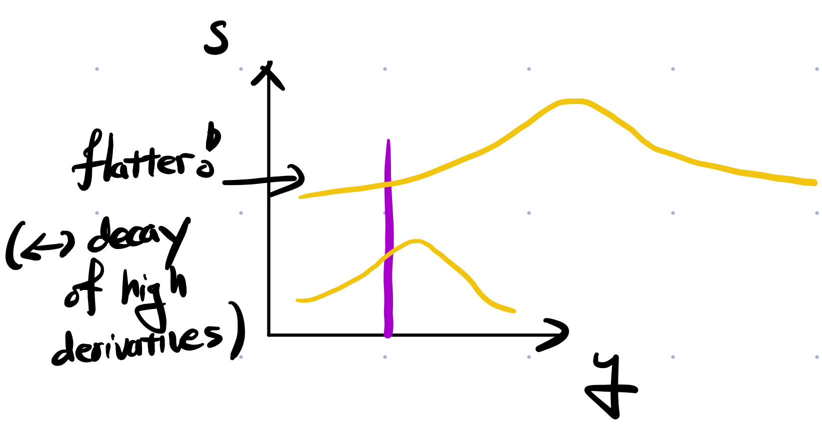
$(t,\ r,\ u)$		$(s,\ y,\ U)$
$\partial_t u - i\Delta u = 0$	$dt = \lambda^2 ds$ $r = \lambda y$ $u = \frac{1}{\lambda} U$ $(\gamma = 0)$	$\partial_s U - \frac{\lambda_s}{\lambda}\Lambda U - i\Delta U = 0$

$$\partial_s\left(\frac{1}{\lambda^k}\|U(s,y)\|_{\dot H^k_m}\right) = 0$$

$$\partial_t \|u(t,x)\|_{\dot H^k_m} = 0 \quad \underset{\#}{\overset{b}{\rightleftarrows}} \quad \Longrightarrow \|U(s,y)\|_{\dot H^k_m} \lesssim \lambda^k \|U(s_0,y)\|_{\dot H^k_m}$$

$\left(\to \text{control } \left\|\frac{1}{r^k} U(s,y)\right\|_{L^2} \text{ with orthogonality cond.}\right)$





$\longrightarrow$ "separation of scales" between radiation (~ 1) & soliton $(\sim \lambda)$

② How to understand $(\partial_s + i\mathcal{L}_Q)$?

$i\mathcal{L}_Q$ is non-selfadjoint ... How to understand this?

Nonlinear conjugation (cf. supersymmetry transform or Darboux transform)

Self-dual form of CSS In $(z,\bar z) = (x^1+ix^2,\ x^1-ix^2)$,

& $\tilde A = (A_t - \frac{1}{2}|\phi|^2)\,dt + A_1\,dx^1 + A_2\,dx^2$

$$(CSS)\quad \begin{cases} i\tilde D_t\phi + 4\tilde D_z\tilde D_{\bar z}\phi = 0 \\ \tilde F_{t\bar z} = \bar\phi\,\partial_{\bar z}\phi \\ \tilde F_{z\bar z} = \frac{1}{4i}|\phi|^2 \end{cases}$$

Useful: $\tilde D_{\bar z} = \frac{1}{2}(D_1 + iD_2)$, $\quad [\tilde D_\mu, \tilde D_\nu]\phi = i\overset{\overset{(d\tilde A)_{\mu\nu}}{\|}}{\tilde F_{\mu\nu}}\phi$.

Take $\tilde D_{\bar z}$ of the eqn, introduce $\phi_1 := \tilde D_{\bar z}\phi$

$$i\tilde D_t\overbrace{\tilde D_{\bar z}\phi}^{\phi_1} + 4\tilde D_z\tilde D_{\bar z}\overbrace{\tilde D_{\bar z}\phi}^{\phi_1}$$

$$\downarrow\qquad + i\overbrace{[\tilde D_{\bar z},\tilde D_t]\phi}^{-i\bar\phi\,\partial_{\bar z}\phi} + 4\overbrace{[\tilde D_{\bar z},\tilde D_z]}^{i\frac{1}{4i}|\phi|^2}D_{\bar z}\phi = 0$$

$$(CSS')\quad \boxed{i\tilde D_t\phi_1 + 4\tilde D_z\tilde D_{\bar z}\phi_1 = 0}$$

$\downarrow$ $\phi = ue^{im\theta}$, Coulomb, $\quad \phi_1 = u_1e^{i(m+1)\theta}$

$$(CSS'_m)\quad \boxed{i\partial_t u_1 - \overset{2\tilde D_z}{A_u^*}\,\overset{2\tilde D_{\bar z}}{A_u}\,u_1 - \overbrace{\int_r^\infty \mathrm{Re}(\bar u u_1)\,dr'}^{\tilde A_t}\,u_1 = 0}$$

$$\boxed{A_u = \partial_r - \frac{m+1+A_\theta[u]}{r}}$$

Linearize: $u = Q + \epsilon$,

$$u_1 = D_{Q+\epsilon}(Q+\epsilon) = \overbrace{L_Q \epsilon}^{\epsilon_1^{lin}} + \cdots$$

$$\rightsquigarrow \quad i\partial_t \epsilon_1^{lin} - A_Q^* A_Q \epsilon_1^{lin} = 0.$$

Key observation: $A_Q^* A_Q$ is self-adjoint & $\mathbb{C}$-linear!

We have proved: $\boxed{L_Q \, i L_Q^* = i A_Q^* A_Q.}$

$H = A_Q^* A_Q$ has kernel, but

$\downarrow$ (super symm transform / Darboux, due to [Raphaël-Rodnianski] for harmonic maps)

$\tilde{H} = A_Q A_Q^*$ for $\epsilon_2 := A_Q L_Q \epsilon^{lin}$ is "repulsive".

$\longrightarrow$ we use energy + virial for ϵ_2^{lin}

$$\longleftrightarrow \quad \|\epsilon\|_{\dot{H}^3_m}^2$$

Rmk. Amazingly,

$$L_Q^* i L_Q = H \longleftarrow$$

linearized operator for harmonic maps around $(m+1)$-equivariant soliton.

C. Bootstrap & Shooting argument.

$\longrightarrow$ Involved, but similar to other stability pf's we have seen.

[Schorkhuber, Munoz, Shahshahani's lectures]