Lecture II Rotational instability of minimal mass blow-up and backward-in-time construction of blow up

(CSS$_m$) $$\boxed{i\partial_t u - A_t[u]u + (\partial_r^2 + \tfrac{1}{r}\partial_r)u - \frac{(m+A_\theta[u])^2}{r^2}u + |u|^2 u = 0.}$$

$$A_\theta(r) = A_\theta[u] := -\frac{1}{2}\int_0^r |u|^2 r' dr'$$

$$A_t(r) = A_t[u] := -\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}$$

Solitons: $$\boxed{Q_{\lambda,\gamma} := e^{i\gamma}\frac{1}{\lambda}Q\left(\frac{r}{\lambda}\right)}$$

where $$\boxed{Q := \sqrt{8}(m+1)\frac{r^m}{1+r^{2(m+1)}} \quad \text{if} \quad m \geq 0}$$

Exact pseudoconformal blow up

$$\boxed{S(t,x) = \frac{1}{|t|}Q\left(\frac{r}{|t|}\right)e^{-i\frac{r^2}{4|t|}} \qquad (t<0.)}$$

Q. Dynamics near Q? ($\longleftrightarrow$ dynamics near S by $\mathcal{C}$)

First, linearized eqn!

Linearization of (CSS$_m$) near Q [Lawrie–O.–Shahshahani, Kim–Kwon I]

Goal: To arrive at "super-symmetric" form. (cf. [Raphaël–Rodnianski])

$$(B_m) \qquad D_u u = 0, \qquad D_u = \partial_r - \frac{(m + A_\theta[u])}{r}$$

Linearization of (B_m) (from Lecture I)

$$D_{w+\epsilon}(w+\epsilon) = D_w w + L_w \epsilon + (\cdots)$$

$$\rightsquigarrow \quad \boxed{\begin{aligned} L_w \epsilon &= D_w \epsilon - \frac{2}{r} A_\theta[w,\epsilon]\, w, \\ A_\theta[v,w] &= -\frac{1}{2}\int_0^r \mathrm{Re}(\bar{v} w)\, r' dr' \end{aligned}}$$

Dual: $\langle L_w u, v \rangle = \langle u, L_w^* v \rangle$

$$\boxed{\begin{aligned} L_w^* v &= D_w^* \epsilon \\ &= \left(-\partial_r - \frac{1}{r}\right)\epsilon - \frac{(m + A_\theta[w])}{r}\epsilon \\ &\qquad + \int_r^\infty \mathrm{Re}(\bar{w}\epsilon)\, w \, dr' \end{aligned}}$$

$$(CSS_m) \qquad \boxed{\begin{aligned} &\partial_t u + i\nabla E[u] = 0 \\ &E[u] = \tfrac{1}{2}\|D_u u\|_{L^2}^2 \end{aligned}} \qquad \text{(Hamiltonian form)}$$

Note: $\langle \nabla E[u], v\rangle = \frac{d}{ds}E[u+sv]\big|_{s=0} = \langle D_u u, L_u v\rangle$

$= \langle L_u^* D_u u, v\rangle$

$$\rightsquigarrow \quad \boxed{\nabla E[u] = L_u^* D_u u}$$

Linearization of ∇E

Also $\nabla^2 E[w]$, which is symm!

$$\nabla E[w+\varepsilon] = \nabla E[w] + \underline{\mathcal{L}_w \epsilon} + (\cdots)$$

$$\rightsquigarrow \quad \boxed{\mathcal{L}_w \epsilon = L_w^* L_w \epsilon + (L_{w+\epsilon}^* - L_w^*) D_w w}$$

For $w = Q$ solving $D_Q Q = 0$,

$$\boxed{\mathcal{L}_Q \epsilon = L_Q^* L_Q \epsilon.} \qquad \text{(supersymm factorization)}$$

Lin-CSS_m around Q

(ℓ-CSS_m) $\boxed{\partial_t \epsilon + i\mathcal{L}_Q \epsilon = 0.}$ $\qquad \mathcal{L}_Q = L_Q^* L_Q.$

Want :

Invariant subspaces $\leftarrow$ generalized Kernel of $\mathcal{L}_Q$. $\left(\text{Note: } (i\mathcal{L}_Q)^* = \mathcal{L}_Q^*(-i) \neq -i\mathcal{L}_Q^* \right)$

We have ($\exists \rho$ smth s.t.) $\qquad \Lambda = r\partial_r + 1$

symm.

pseudoconf. symm

$$\boxed{\begin{aligned} &i\mathcal{L}_Q(\Lambda Q) = 0 && i\mathcal{L}_Q(iQ) = 0 \\ &i\mathcal{L}_Q\left(i\tfrac{r^2}{4}Q\right) = \Lambda Q && i\mathcal{L}_Q(\rho) = iQ \end{aligned}}$$

$iQ \in {}^{\perp}\ker(i\mathcal{L}_Q)^* = {}^{\perp}\ker(-\mathcal{L}_Q i) = {}^{\perp}(i\Lambda Q, Q)$

and

$$\boxed{\begin{aligned} &L_Q(\Lambda Q) = 0, && L_Q(iQ) = 0 \\ &L_Q^*\left(i\tfrac{r}{2}Q\right) = -i\Lambda Q, && L_Q^*\left(\tfrac{1}{2(m+1)} rQ\right) = Q \\ &L_Q\left(i\tfrac{r^2}{4}Q\right) = i\tfrac{r}{2}Q, && L_Q \rho = \tfrac{1}{2(m+1)} rQ \end{aligned}}$$

$\rightsquigarrow$ $\boxed{N_g(i\mathcal{L}_Q) = \left(\Lambda Q, iQ, \tfrac{i}{4}r^2 Q, \rho\right)}$

& $\boxed{\begin{aligned} N_g((i\mathcal{L}_Q)^*) &= N_g(-\mathcal{L}_Q^* i) \\ &= \left(i\Lambda Q, Q, \tfrac{1}{4}r^2 Q, i\rho\right) \end{aligned}}$

$\rightsquigarrow$ $\boxed{\begin{aligned} &\partial_t\langle \epsilon, i\Lambda Q\rangle = 0, \ \partial_t\langle \epsilon, Q\rangle = 0 \\ &\partial_t\langle \epsilon, \tfrac{1}{4}r^2 Q\rangle \qquad \partial_t\langle \epsilon, i\rho\rangle \\ &= -\langle \epsilon, i\Lambda Q\rangle \qquad = \langle \epsilon, Q\rangle \end{aligned}}$ under (ℓ-CSS_m)

under (ℓ-CSS_m)

Note : We obtain decomposition of L^2 into invariant spaces (under (ℓ-CSS_m)) (when $m \geq 2$)

$$\boxed{L^2 = N_g(i\mathcal{L}_Q) \oplus {}^{\perp}N_g((i\mathcal{L}_Q)^*)}$$

($\rho, -i\frac{y^2}{4}Q \notin L^2$ if $m = 0, 1$)

Dynamics near Q via generalized Kernel

What happens if we start with

$$u_0 = Q - ib_0\frac{r^2}{4}Q - (m+1)\eta_0\rho \quad ?$$

Then
$$\partial_t u\big|_{t=0} = -i\mathcal{L}_Q\left(-ib_0\frac{r^2}{4}Q - (m+1)\eta_0\rho\right) + \cdots$$
$$= b_0\Lambda Q + (m+1)\eta\, iQ + \cdots$$

$\rightsquigarrow$
$$\boxed{\begin{aligned} &u = \frac{e^{i\gamma(t)}}{\lambda(t)}Q\left(\frac{\cdot}{\lambda(t)}\right) + \cdots, \quad \text{where} \\ &\frac{\lambda_t}{\lambda}\Big|_{t=0} = -b_0, \quad \gamma_t\big|_{t=0} = (m+1)\eta_0 \end{aligned}}$$

Q. Can we construct exact sol'n $u = [P(b(s),\eta(s))]_{\lambda(t),\gamma(t)}$

with $P = Q - ib\frac{y^2}{4}Q - (m+1)\eta\rho + \cdots$

& ODEs for λ, γ, b, η ?

A. Probably no for many models (e.g. Wave maps, Yang-Mills, Schrödinger maps)

but YES for (CSS_m) (also (NLS), [Merle-Raphaël-Szeftel])

This will reveal an interesting nonlinear dynamics called rotational instability of blow up, which is (likely) relevant to WM, YM, SM, ...

Key structure: Existence of S.

$$S(t,x) = \frac{1}{|t|} Q\left(\frac{r}{|t|}\right) e^{-i\frac{r^2}{4|t|}}$$
$$= \left[Q e^{-ib\frac{y^2}{4}}\right]_{\lambda,\gamma} \quad \text{with} \quad \lambda = |t| \quad \gamma = 0 \quad \text{“} b = |t|, \ \eta = 0 \text{”}$$

Claim: Defining $P(y; b, 0) = Q(y) e^{-ib\frac{y^2}{4}}$, and considering

$$u(t,r) = [P(\cdot, b(t), 0)]_{\lambda(t),\gamma(t)},$$

we can "derive" S.

Adapted coordinates (s, y, U) given λ, γ

$$\boxed{dt = \lambda^2 ds, \quad r = \lambda y, \quad u = \frac{e^{i\gamma}}{\lambda} U}$$

$$(\text{i.e.,} \quad u = [U(s(t), \cdot)]_{\lambda,\gamma} \ )$$

Notation: $u^\flat = U$, $U^\# = u$

equ for u $\longleftrightarrow$

$$\boxed{\underbrace{\partial_s U - \frac{\lambda_s}{\lambda} \Lambda U + \gamma_s i U}_{\text{from } \partial_t u} + \underbrace{i L_U^* D_U U}_{\text{from } i L_u^* D_u u} = 0.}$$

Plug in $U \to P = Q(y) e^{-ib\frac{y^2}{4}}$, $b = b(s)$.

$$\boxed{b_s \overbrace{\partial_b P}^{-i\frac{y^2}{4}P} - \frac{\lambda_s}{\lambda} \Lambda P + \gamma_s i P + i L_P^* D_P P = 0.}$$

Let us compute $L_P^* D_P P = 0$.

$$D_P P = \left(\partial_y - \frac{m + \overbrace{A_\theta[Q e^{-ib\frac{y^2}{4}}]}^{A_\theta[Q] \ (\because \text{ phase rotation symm})}}{y}\right)(Q e^{-ib\frac{y^2}{4}})$$
$$= \cancel{(D_Q Q)} e^{-ib\frac{y^2}{4}} + ib\frac{y}{2} Q e^{-ib\frac{y^2}{4}}$$
$$= -ib\frac{y}{2} P$$

$$L_P^* D_P P = - L_P^* \left(i b \tfrac{y}{2} P \right)$$

$$\left(\begin{aligned} \text{Recall: } L_P^* w &= -\partial_y w - \frac{1+m+A_\theta[P]}{y} w - \int_y^\infty \mathrm{Re}(\bar{P} w)\, dy' \; P \\ \& \quad D_P P &= \partial_y w - \frac{m + A_\theta[P]}{y} w \end{aligned} \right)$$

$$-L_P^*\left(i b \tfrac{y}{2} P\right) = + i \tfrac{b}{2} P - b \tfrac{y}{2} L_P^*(iP)$$

$$= + i \tfrac{b}{2} P - i b \tfrac{y}{2} \left(-2\partial_y - \tfrac{1}{y} \right) P - i b \tfrac{y}{2} D_P P + \int_y^\infty \overset{\overset{0}{\shortparallel}}{\mathrm{Re}(\bar{P}\, i P)}\, dy' \; P$$

$$= + i b \Lambda P - b^2 \tfrac{y^2}{4} P$$

$$\rightsquigarrow \quad \boxed{0 = -\tfrac{\lambda_s}{\lambda} \Lambda P + \gamma_s i P - i b_s \tfrac{y^2}{4} P - b \Lambda P - i b^2 \tfrac{y^2}{4} P}$$

$$\therefore \quad u = \left[Q e^{-ib\frac{y^2}{4}} \right]_{\lambda,\gamma} \text{ sol'n if}$$

$$\begin{cases} \frac{\lambda_s}{\lambda} + b = 0, & \gamma_s = 0, \\ b_s + b^2 = 0 & \end{cases}$$

In t,
$$\begin{cases} \partial_t \lambda + \frac{b}{\lambda} = 0, & \gamma_t = 0, \\ b_t + \frac{b^2}{\lambda^2} = 0. & \end{cases}$$

Exercise: $\lambda = |t|$, $b = |t|$, $\gamma = 0$ is a sol'n.

Q. How about $P(\cdot\,; b, \eta)$ with $\eta \neq 0$?

(Recall: $\quad P = Q - ib\frac{y^2}{4}Q - (m+1)\eta\rho + \cdots$)

equ for $U = P(y; b(s), \eta(s))$

$\longrightarrow$

$$\boxed{b_s \partial_b P + \eta_s \partial_\eta P - \frac{\lambda_s}{\lambda}\Lambda P + \gamma_s iP + i L_P^* D_P P = 0.}$$

Q. Can we construct family $P(\cdot\,; b, \eta)$ so that $i L_P^* D_P P$ is a linear combination of $\partial_b P, \partial_\eta P, \Lambda P, iP$?

A. (Amazingly) YES! $\exists P$ that solves

can be motivated by consideration of solvability.

$$\begin{cases} i L_P^* D_P P = i\left(ib\Lambda P - (m+1)\eta P - (b^2 + \eta^2)\frac{y^2}{4}P\right) \\ \& \quad \partial_b P = -i\frac{y^2}{4}P \end{cases}$$

$\left(\Leftrightarrow \; P(\cdot\,; b, \eta) = P(\cdot\,; 0, \eta) e^{-ib\frac{y^2}{4}} \right)$

cf. NLS for $b = 0$, P is real &

note difference!

$$-\Delta P + P + |P|^2 P = -\eta\frac{y^2}{4}P, \quad P = Q + \eta\rho + \cdots$$

is easier to solve since $\mathcal{L}_Q$ is invertible for real-valued ftns.

But for (CSS_m), $\mathcal{L}_Q$ is not invertible even for real-valued ftns, and it is difficult to solve the eqn.

(after imposing $-(m+1)\eta P$, $-\eta^2$ in front of $\frac{y^2}{4}P$ is motivated by solvability consideration.)

Idea of [Kim-Kwon]: For $b = 0$, $\eta > 0$, solve

$$D_P P = -(m+1)\eta\frac{y}{2}P \qquad \text{(which is easier)}$$

then

$$P(\cdot\,; b, \eta) = P(\cdot\,; 0, \eta) e^{-ib\frac{y^2}{4}} \text{ solves the above.}$$

$$-ib_s\frac{y^2}{4}P + y_s\partial_y P - \frac{\lambda_s}{\lambda}\Lambda P + \gamma_s iP + iL_p^* D_p P = 0.$$

$$iL_p^* D_p P = -b\Lambda P - i(m+1)\eta P - i(b^2+y^2)\frac{y^2}{4}P$$

$\rightarrow$ u is a sol'n if

$$\begin{cases} \frac{\lambda_s}{\lambda} + b = 0, & \gamma_s - (m+1)\eta = 0 \\ b_s + b^2 + \eta^2 = 0, & \eta_s = 0 \end{cases}$$

Integration of this ODE.

$$\begin{cases} \lambda_t + \frac{b}{\lambda} = 0, & \gamma_t - (m+1)\frac{\eta}{\lambda^2} = 0 \\ b_t + \frac{b^2+\eta^2}{\lambda^2} = 0, & \eta_t = 0. \end{cases}$$

Observe:

$E[P]$.

$$\partial_t\left(\frac{b^2+\eta^2}{\lambda^2}\right) = \frac{2b\overbrace{b_t}^{-\frac{b^2+\eta^2}{\lambda^2}}}{\lambda^2} - \frac{2(b^2+\eta^2)\overbrace{\lambda_t}^{-\frac{b}{\lambda}}}{\lambda^3} = 0 \ !$$

$$\ell_0^2 := \frac{b_0^2+\eta_0^2}{\lambda_0^2} > 0 \quad \rightarrow \quad \boxed{\begin{aligned} & b = \ell_0^2(T-t), \qquad \eta = \eta_0 \\ & \lambda = \sqrt{\frac{b^2+\eta^2}{\ell_0^2}} = \sqrt{\ell_0^2(T-t)^2 + \left(\frac{\eta_0}{\ell_0}\right)^2} \end{aligned}}$$

$$\gamma_t = (m+1)\frac{\ell_0^2\eta_0}{b^2+\eta_0^2} = (m+1)\frac{\ell_0^2\eta_0}{(\ell_0^2(T-t))^2+\eta_0^2}$$

$$\rightarrow \boxed{\gamma(t) = (m+1)\arctan\left(\frac{\ell_0^2}{\eta_0}(t-T)\right)}$$

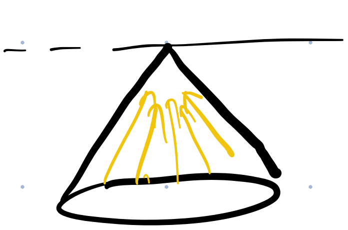
Nonlinear Rotational Instability of $\mathcal{S}$

$$\boxed{b = \ell_0^2 (T-t), \qquad \eta = \eta_0 \\ \lambda = \sqrt{\frac{b^2+\eta^2}{\ell_0^2}} = \sqrt{\ell_0^2 (T-t)^2 + \left(\frac{\eta_0}{\ell_0}\right)^2}}$$

$$\boxed{\gamma(t) = (m+1) \arctan\left(\frac{\ell_0^2}{\eta_0}(t-T)\right)}$$

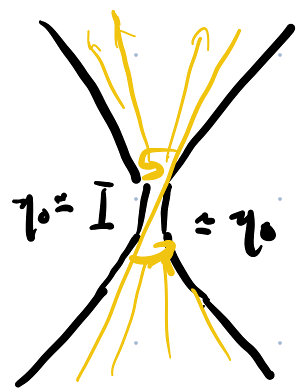
$\eta_0 = 0$

$\rightsquigarrow \lambda = \ell_0 |T-t|$



$t = T$

$\eta_0 \neq 0.$



$$\boxed{\gamma(T+|\eta_0|^{1-}) - \gamma(T-|\eta_0|^{1-}) \longrightarrow \pm(m+1)\pi \quad \text{as} \quad \pm\eta_0 \searrow 0.}$$

In fact, a similar phenomenon has been predicted for smth blow up for: harmonic map heat flow, Schrödinger maps, Wave maps, Yang-Mills in the energy critical case [Van den Berg - Williams, Merle - Raphaël - Rodnianski]

for non-smth blow up sol'ns, the story is different! See [Krieger - Miao - Schlag]

For S : rotational instability :

Q. Other blow up sol'ns with similar dynamics?

Often times, "backword" construction is a good first try. (cf. [Bourgain–Wang] [Merle–Raphaël–Szeftel] for (NLS))

Thm. [Kim–Kwon] Let $m \geq 1$. $\exists \alpha^* > 0$ s.t. the following holds.

z^* : $\partial_r^{m+4} z^*$, $r^{-(m+3)} z^* \in L$

① (backword construction) $\exists u$: H^1_m-sol'n to (CSS$_m$) on $(-\infty, 0)$ with

$$u - S(t,r) - z^* \xrightarrow{t \nearrow 0} 0 \quad \text{in } H^1_m.$$

② ($\exists$ rotational instability of u) $\exists$ 1-parameter family $u^{(\eta)}$: H^1_m-sol'ns to (CSS$_m$) where $\eta \in [0, \eta^*]$, with

i) $u^{(0)} = u$

ii) $\eta \mapsto u^{(\eta)}$ is continuous in H^{1-}_m

iii) for $\eta \neq 0$, $u^{(\eta)}$ scatters both forward & backward in time

iv)
$$u^{(\eta)} = [P(\cdot\,; b^{(\eta)}, \eta)]_{\lambda^{(\eta)}, \gamma^{(\eta)}} + O_{H^1_m}(\alpha^*)$$

with
$$\begin{cases} b^{(\eta)} = |t|, \\ \lambda^{(\eta)} = (|t|^2 + \eta^2)^{\frac{1}{2}}, \\ \gamma^{(0)}(-\tau) = O(\alpha^* \tau), \\ \lim_{\eta \searrow 0} \sup \left| \gamma^{(\eta)}(\tau) - \gamma^{(\eta)}(-\tau) - (m+1)\pi \right| = O(\alpha^* \tau) \end{cases}$$

Taking $\mathcal{C}$, we obtain :

Cor. Let $m \geq 1$. Then $\mathcal{C}u^{(0)} - Q - e^{i\Delta_{-m-2} t} u^* \to 0$ in L^2

but $\mathcal{C}u^{(\eta)}$ scatters forward in time for $\eta > 0$.

Backward construction (cf. [Bourgain–Wang], [Merle–Raphaël–Szeftel])

1. Main decomposition

$$u^{(\eta)} = \left[P(\cdot\,; b(t), \eta) + \epsilon\right]_{\lambda(t), \gamma(t)} + z$$

$$\left(\Leftrightarrow \; U^{(\eta)} = P(y; b, \eta) + \epsilon + z^{b}\right)$$

$$\boxed{\text{Will specify } \lambda, \gamma, b, z \text{ \& solve for } \epsilon \text{ from } t=0.}$$

2. Construction of z

Need to take the nonlocal interaction btw $P_{\lambda,\gamma}$ & z. (cf. Lecture I)

solve $\begin{cases} \partial_t z + \tilde{L}_z^* \tilde{D}_z z = 0 \\ z(t=0) = z^* \end{cases}$ $\left(\tilde{D}_z, \tilde{L}_z^* \text{ defined with } \tilde{A}_\theta, \text{ where } \tilde{A}_\theta[z] = -\frac{1}{2}\int_0^\infty |Q|^2 r' dr' + A_\theta[z] \right)$

Since z^* is vanishing rapidly as $r \to 0$ & is regular, so does z:

$$\sup_{t \in [-1, 0]} \left(|\partial_r^{\ell} z| + |\tfrac{1}{r} \partial_r z| \right) \lesssim \alpha^* \min\{r^{m+2-\ell}, 1\}$$

3. Equation for ϵ

$$W = Q + z^{b}$$

$$\rightsquigarrow \; \partial_s \epsilon + i \mathcal{L}_W \epsilon + b \Lambda \epsilon + i(m+1)\eta\epsilon$$

$$= \left(\frac{\lambda_s}{\lambda} + b\right) \Lambda (P + \epsilon) + (b_s + b^2 + \eta^2) \frac{|y|^2}{4} P$$

$$- i\left(\gamma_s + \Theta_{z^b \to P} - (m+1)\eta\right) P - i(\gamma_s - (m+1)\eta)\epsilon$$

$$- i \tilde{R}_{P,z} \quad + \text{(better nonlinear terms)}$$

nonlinear interaction btw P & z; vanishing of z at $t=0$, $r=0$ makes this error term small

(there is also $\tilde{L}_z \tilde{D}_z z - L_z D_z z$ in (zCSS))

When $\eta = 0$,

$$\partial_s \epsilon + i \mathcal{L}_W \epsilon + b \Lambda \epsilon + i(m+1)\eta\epsilon$$

$$= \left(-2\left(\frac{\lambda_s}{\lambda}+b\right)b + (b_s + b^2 + \eta^2)\right)\frac{|y|^2}{4} Q e^{-ib\frac{y^2}{4}}$$

$$+ \left(\frac{\lambda_s}{\lambda}+b\right)(\Lambda Q e^{-ib\frac{y^2}{4}} + \epsilon)$$

$$- i(\gamma_s + \Theta_{\substack{b\\ z\to P}} - (m+1)\eta) Q e^{-ib\frac{y^2}{4}} - i(\gamma_s - (m+1)\eta)\epsilon$$

$$- i \tilde{R}_{P,z}$$

4. Equations for λ, γ, b & energy-virial fun'l.

We can specify three equs (use implicit fn thm).

Use: $\langle \epsilon, z_1 \rangle = \langle \epsilon, z_2 \rangle = 0$ (Lecture I)

& the coefficient in front of least decaying term on the RHS (which is important when $\eta = 0$)

$$-2\left(\frac{\lambda_s}{\lambda}+b\right)b + (b_s + b^2 + \eta^2) = 0$$

Test against z_1, z_2 to derive equs for λ_s, γ_s (Lecture I)

For ϵ: energy method starts with

$$\langle i\partial_s \epsilon + \mathcal{L}_W \epsilon - ib\Lambda\epsilon + (m+1)\eta\epsilon, \partial_s \epsilon \rangle$$

linearized energy, virial, mass

$$= \lambda^2 \partial_s \left[\lambda^{-2}\left(E_W^{(qd)}[\epsilon] + b\tilde{\Phi}[\epsilon] + \frac{(m+1)}{2}\eta M[\epsilon]\right)\right] + (\text{better})$$

$E_W^{(qd)}[\epsilon] = E[W+\epsilon] - E[W] - \langle \nabla E[W], \epsilon \rangle$ | $\tilde{\Phi}[\epsilon] = \frac{b}{2}\int \operatorname{Im}(\bar{\epsilon}\, r\partial_r \epsilon)$

In reality, $\tilde{\Phi}$ is not well-defined for $\epsilon \in H^1_m$, so perform truncation $\rightsquigarrow \Phi$.

([Raphaël-Szeftel] cf. Muñoz's Lecture)

Then $\mathcal{I}(s) = \lambda^{-2}\left(E_W^{(qd)}[\epsilon] + b\,\tilde{\Phi}[\epsilon] + \frac{(m+1)}{2}\eta M[\epsilon]\right)$

$\gtrsim \|\epsilon\|_{\dot{\mathcal{H}}^1_m}$ if $\langle \epsilon, \mathcal{Z}_1\rangle = \langle \epsilon, \mathcal{Z}_2\rangle = 0$

& b small & $\eta \geq 0$.

& $-\lambda^2 \partial_s \mathcal{I}(s) \lesssim \|\tilde{R}_{p,z}\|_{\dot{H}^1_m}$ + (acceptable)

Now bootstrap pb !! Thanks to

$\|\tilde{R}_{p,z}\|_{\dot{H}^1_m} \lesssim \lambda^M$ for M large

& (expect) λ is smallest at $t=0$

& we are integrating backwards,

we can establish decay of $\mathcal{I}$ & close. ✓

$m=0$? z^* exerts nontrivial effect on the ODEs.

Thm [Kim–Kwon–O., forthcoming] (& WM, YM: [Krieger–Schlag–Tataru]; SM: [Perelman]; WM: [Jendrej–Lawrie–Rodriguez])

$m=0$. Take

$$z^* = q\, r^{\nu} \chi_{\lesssim 1}(r). \quad \nu \in \mathbb{C},\ \mathrm{Re}\,\nu > 0. \quad (\text{Note: } z^* \in H^{1+\nu-})$$

Then $\exists u$: H^1_m-sol'n to (CSS_m) with $T_+ = 0$ s.t.

$$u - Q_{\lambda,\gamma} - z^* \xrightarrow{t \nearrow T_+} 0$$

where

$$\boxed{\lambda(t) e^{i\gamma(t)} = -\frac{\sqrt{2}}{4} \frac{\Gamma(\frac{\nu}{2})}{\mathrm{Re}\,\nu + 1}\, q\, \frac{(4it)^{\frac{\nu}{2}+1}}{|\log|t||}}$$

Idea: Run backward construction but $\hat{R}_{P,z^b}$ does NOT vanish very fast, instead forces modulation equns.

(cf. [Jendrej], [Jendrej–Lawrie–Rodriguez])