Lecture I Soliton resolution for equivariant self-dual Chern-Simons-Schrödinger

Recall: Cubic NLS on $\mathbb{R}^2$:

(NLS) $\boxed{i\partial_t \psi + \Delta\psi + |\psi|^2\psi = 0}$ $(\psi : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C})$

Chern-Simons-Schrödinger is a gauged version of (NLS):

$$\text{(CSS)} \quad \boxed{\begin{cases} iD_t\phi + \Delta_A\phi + g|\phi|^2\phi = 0. \\ \text{CS} \rightarrow \quad F_{\mu\nu} = \epsilon_{\mu\nu\lambda} J^\lambda \quad \text{charge current vec} \\ J^0 = -\frac{1}{2}|\phi|^2 \\ J^j = -\operatorname{Im}(\bar{\phi} D_j \phi) \end{cases}}$$

$\phi : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}$, $g \in \mathbb{R}$

covariant D.

$D_\mu = \partial_\mu + iA_\mu$ $(\mu = \overset{t}{\overset{\shortparallel}{0}}, 1, 2)$

A_μ : $\mathbb{R}$-valued 1-form on $\mathbb{R}_t \times \mathbb{R}^2_x$

$\Delta_A = D_1^2 + D_2^2$

$F_{\mu\nu} = (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(Action: $\int_{\mathbb{R}^{1+2}} \left[\frac{1}{2}\operatorname{Im}(\bar{\phi} D_t \phi) + \frac{1}{2}|D_x\phi|^2 - \frac{g}{4}|\phi|^4\right] dt\,dx + \frac{1}{2}\int_{\mathbb{R}^{1+2}} A \wedge dA$)

Exercise: Derive mass consv, energy consv, virial identity, phase rotation symm, time translation symm, scaling symm, pseudoconformal symmetry (or whatever else you have for (NLS)) for (CSS)

[Jackiw-Pi]: Noted special structure of (CSS) when $\boxed{g=1}$ $\rightsquigarrow$ self-duality

$$E[\phi, A](t) = \int\!\!\int_{\mathbb{R}^2} \left[\frac{1}{2}\sum_{j=1}^{2} |D_j\phi|^2 - \frac{g}{4}|\phi|^4\right] dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} |(D_1 + iD_2)\phi|^2 \, dx \quad \text{if } g = 1.$$

↳ covariant Cauchy-Riemann operator.

From now on, assume $g=1$ (self-duality).

In this case, $\boxed{Q(x) \text{ minimizes } E}$

$\Longleftrightarrow$ Bogomolnyi' eqn

$$\boxed{(B) \quad \begin{cases} (D_1 + iD_2)Q = 0 \\ F_{12} = -\frac{1}{2}|Q|^2 \end{cases}}$$

(If so, $(Q(x), A_t = \frac{1}{2}|Q|^2, A_x)$ solves (CSS))

Moreover, if so, $(Q(x), A_t = \frac{1}{2}|Q|^2, A_x)$ is a (time-independent) sol'n to (CSS).

Similar to:

- harmonic maps (or fns) $\mathbb{R}^2 \to (N^2, h)$

$$E[\Phi] = \int_{\mathbb{R}^2} \frac{1}{2} \sum_{j=1}^{2} \langle \partial_j \Phi, \partial_j \Phi \rangle_h \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 \Phi + J \partial_2 \Phi|_h^2 \, dx$$

($\Phi : \mathbb{R}^2 \to \mathbb{C}$

$$\int_{\mathbb{R}^2} \frac{1}{2} \sum_{j=1}^{2} |\partial_j \Phi|^2 \, dx$$

$$= \frac{1}{2} \int |(\partial_1 \pm i \partial_2) \Phi|^2 dx$$)

Φ minimizes E $\Longleftrightarrow$ Φ is holomorphic (or anti-holomorphic)

- (elliptic) Yang-Mills connection on $\mathbb{R}^4$

$$E[A] = \int_{\mathbb{R}^4} \frac{1}{4} \sum_{j,k=1}^{4} (-\mathrm{tr}\, F_{jk} F_{jk})$$

($A_j : \mathbb{R}^4_x \to su(2)$ $(j=1,\cdots,4)$
$F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$)

$$= c \int_{\mathbb{R}^4} -\mathrm{tr} (F_{jk} \pm (*F)_{jk})^2$$

A minimizes E $\Longleftrightarrow$ A is anti self dual (or self dual)

We are interested in the dynamics of (CSS).

Assume equivariance (with index $m \in \mathbb{Z}$)

$$\boxed{\begin{aligned} \phi &= u(t,r)\, e^{im\theta}, \\ A &= A_t(r)\,dt + A_r(r)\,dr + A_\theta(r)\,d\theta \end{aligned}}$$

$$\left(\begin{array}{l} x^1 + i x^2 = r e^{i\theta} \\ u : \mathbb{R}_t \times (0,\infty)_r \to \mathbb{C} \end{array} \right)$$

To fix gauge invariance (i.e., $(\phi, A) \mapsto (e^{i\chi}\phi, A - d\chi)$ preserves sol'ns)
we impose Coulomb gauge χ : $\mathbb{R}$-valued ftn,

$$\boxed{\begin{aligned} &\mathrm{div}_x A = \tfrac{1}{r}\partial_r (r A_r) + \tfrac{1}{r^2}\partial_\theta A_\theta = 0. \\ &\& \quad A \to 0 \quad \text{as} \quad r \to \infty. \end{aligned}}$$

Under equivariance, Coulomb $\Longleftrightarrow$ $A_r = 0$.

$F_{\mu\nu} = \epsilon_{\mu\nu\lambda} J^\lambda$

$\Longrightarrow$

$$\boxed{\begin{aligned} A_\theta(r) &= A_\theta[u] := -\tfrac{1}{2}\int_0^r |u|^2 \, r' dr' \\ A_t(r) &= A_t[u] := -\int_r^\infty (m + A_\theta[u])\, |u|^2 \, \frac{dr'}{r'} \end{aligned}}$$

Eqn for u:

$$(\mathrm{CSS}_m) \quad \boxed{ i\partial_t u - A_t[u]\, u + \left(\partial_r^2 + \tfrac{1}{r}\partial_r\right) u - \frac{(m + A_\theta[u])^2}{r^2} u + |u|^2 u = 0. }$$

$\rightsquigarrow$ Cauchy problem is LWP in L^2_m [Liu-Smith]. $\boxed{T_+ : \text{fwd lifespan.}}$

Other well-posed ftn sp: $\boxed{H^s_m = H^s(\mathbb{R}^2) \cap \{ m\text{-equiv ftns} \}.}$

$\boxed{H^{s,s}_m = H^s_m(\mathbb{R}^2) \cap \{ r^s u \in L^2(\mathbb{R}^2) \}.}$

Bogomol'nyi eqn: $(D_1 + i D_2)(u e^{im\theta}) = e^{i(m+1)\theta}\left(\partial_r - \frac{(m + A_\theta[u])}{r}\right) u$

(B_m) $\boxed{D_Q Q = 0,}$ $\left(\boxed{D_u := \partial_r - \frac{(m + A_\theta[u])}{r}} \right)$

Some interesting sol'ns

Solitons Up to phase rotation & scaling symmetries

$u \mapsto ue^{i\theta}$ $\qquad$ $u \mapsto \frac{1}{\lambda}u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$

the non-zero H^1_m sol'ns to (Bm) are:

$$\boxed{Q := \sqrt{8}(m+1) \frac{r^m}{1+r^{2(m+1)}} \quad \text{if} \quad m \geq 0}$$

i.e.,
$$\boxed{Q_{\lambda,\gamma} := e^{i\gamma}\frac{1}{\lambda}Q(\frac{r}{\lambda})}$$

& no such sol'ns exist if $m<0$. $\qquad$ [Jackiw-Pi, ...]

Indeed, from $D_Q Q = 0$, we can derive $\underbrace{\Delta \log |Q|^2 = -|Q|^2}_{\text{Liouville eqn}}$, which can be solved

Minimal mass (or exactly pseudoconformal) blow up:

Applying pseudoconformal symmetry

$$u \mapsto \frac{1}{|t|}u(-\frac{1}{t}, \frac{r}{|t|})e^{i\frac{r^2}{4t}} \qquad t \neq 0$$

to Q, we obtain:

$$\boxed{S(t,x) = \frac{1}{|t|}Q(\frac{r}{|t|})e^{-i\frac{r^2}{4|t|}} \qquad (t<0)}$$

$\lambda(t) = |t| \ll \sqrt{|t|}$, $\qquad$ Type II blow up (as opposed to self-similar blow up)

Note: $\|S(t,\cdot)\|^2_{L^2} = \|Q\|^2_{L^2}$

"Minimal mass": Justified by threshold thm of [Liu-Smith], which says u: L^2 sol'n to (CSSm) is global & scattering if $\|u\|^2_{L^2} < \|Q\|^2_{L^2}$.

Thm (Soliton resolution for (CSS_m) [Kim–Kwon–O.2])

Let $m \geq 0$. (GWP & scattering holds for $m<0$ by [Liu-Smith])

- (finite time blow up)

If u: H^1_m-sol'n to (CSS_m) with $T_+<+\infty$, then $z^* \in H^1_m$

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} - z^*(\cdot) \xrightarrow{t\nearrow T_+} 0 \quad \text{in } L^2$$

where

$$\lambda(t) \lesssim_{\underset{\text{mass}}{M[u]}} \begin{cases} \overset{\text{energy}}{\sqrt{E}}\,(T_+-t) & \text{if } m \geq 1 \\ \sqrt{E}\,\dfrac{T_+-t}{|\log(T_+-t)|^{1/2}} & \text{if } m=0. \end{cases}$$

- (global sol'n)

If u: $H^{1,1}_m$ sol'n to (CSS_m) with $T_+=+\infty$ but does not scatter, then

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} - e^{it\Delta^{(-m-2)}}u^* \xrightarrow{t\nearrow+\infty} 0 \quad \text{in } L^2$$

where

$$\lambda(t) \lesssim_{M[u]} \begin{cases} \sqrt{E[\mathcal{C}u]} & \text{if } m \geq 1 \\ \sqrt{E[\mathcal{C}u]}\,\dfrac{1}{|\log t|^{1/2}} & \text{if } m=0. \end{cases}$$

(Here, $M[u] = \int_0^\infty |u|^2 r\,dr$, $E[u] = \int_0^\infty \left[\frac{1}{2}(\partial_r u)^2 + \frac{1}{2}\frac{(m+A_\theta[u])^2}{r^2}u^2 - \frac{1}{4}u^4\right] r\,dr$

$\mathcal{C}u := \frac{1}{|t|}u\left(-\frac{1}{t}, \frac{r}{|t|}\right)e^{i\frac{r^2}{4t}}$)

Rmk ① Note that at most one soliton can appear in the resolution!

This is a distinct feature of (CSS_m). For WM & YM when $T=+\infty$, two-bubbles appear [Jendrej, Jendrej–Lawrie, Rodrigues] but for $T_+<+\infty$, it is an open question whether multi-bubble is possible.

② $m=0$ vs. $m \geq 1$: $S \in H^1_m$ ($\Leftrightarrow$ $Q \in H^{1,1}_m$)

$\Leftrightarrow$ $m \geq 1$.

Literature on CSS

(CSS): Bergé–de Bouard–Saut, Huh, Lim, O.–Pusateri
Liu–Smith–Tataru $(A_t = \mathrm{div}_x A)$, ...

(CSS_m): Liu–Smith, Li–Liu, Dodson, ...

pf of thm (Will omit details on nonlinearity)

Ingredients L_Q: linearized Bogomol'nyi operator,

$$D_{Q+\epsilon}^{Q+\epsilon} = L_Q \epsilon + \cdots$$

$$\longrightarrow \boxed{\begin{aligned} L_Q\epsilon &= D_Q\epsilon - \frac{2}{r} A_\theta[Q,\epsilon]Q, \\ A_\theta[v,w] &= -\frac{1}{2}\int_0^r \mathrm{Re}(\bar{v} w)\, r' dr' \end{aligned}}$$

From symm & uniqueness, $\boxed{\mathrm{Ker}\, L_Q = (\Lambda Q, iQ)}$ $\quad \Lambda = r\partial_r + 1$

$z_1, z_2 \in (C_c^\infty)_m$ satisfy transversality w/ $\mathrm{Ker}\, L_Q = (\Lambda Q, iQ)$

$$\det\begin{pmatrix} \langle \Lambda Q, z_1\rangle & \langle iQ, z_1\rangle \\ \langle \Lambda Q, z_2\rangle & \langle iQ, z_2\rangle \end{pmatrix} \neq 0.$$

Lemma (see [Kim-Kwon 1, Kim-Kwon-O. 1])

$m \geq 0$. $\forall f \in \dot{\mathcal{H}}^1_m$ with $\langle f, z_1\rangle = \langle f, z_2\rangle = 0$,

$$\boxed{\| L_Q f\|_{L^2} \sim \|f\|_{\dot{\mathcal{H}}^1_m} := \begin{cases} \|\partial_r f\|_{L^2} + \|\langle \log_- r\rangle^{-1} r^{-1} f\|_{L^2} & m=0 \\ \|f\|_{\dot{H}^1_m} & m \geq 1 \end{cases}}$$

Step 0. By C, suffices to consider $T_+ < +\infty$ & $u : \dot{H}^1_m$ sol'n

Step 1 $u : \dot{H}^1_m$-sol'n, $T_+ < +\infty$.

By (subcrit.) LWP, $\|u\|_{\dot{H}^1} \xrightarrow{t \nearrow T_+} +\infty$.

On the other hand, $E[u]$ is conserved, and so is $M[u]$.

Note: $v(t,r) = \hat{\lambda}(t)\, u(t, \hat{\lambda}(t)\, r)$ with $\hat{\lambda}(t) = \dfrac{\|Q\|_{\dot{H}^1}}{\|u(t)\|_{\dot{H}^1}}$ has

$\|v(t)\|_{\dot{H}^1} = \|Q\|_{\dot{H}^1}$ & $E[v] \xrightarrow{t \nearrow T_+} 0$, so we expect

" $v \xrightarrow{t \nearrow T_+} Q$ " in some sense.

Claim (variational) $\forall M > 1, \forall \delta > 0, \exists \alpha^* > 0$ s.t.

if $u \in \dot{H}^1_m \setminus \{0\}$, $\|u\|_{L^2}^2 \le M$, $\sqrt{E[u]} \le \alpha^* \|u\|_{\dot{H}^1_m}$

then
$$\| e^{-i\hat{\gamma}} \hat{\lambda}\, u(\hat{\lambda}(\cdot)) - Q \|_{\dot{H}^1_m} < \delta$$
for $\hat{\lambda} = \dfrac{\|Q\|_{\dot{H}^1}}{\|u\|_{\dot{H}^1}}$ and for some $\hat{\gamma} \in [0, 2\pi)$.

Idea: Contradiction: Given w_n s.t. $d_{\dot{H}^1_m}(w_n, \{e^{i\gamma}Q\}) \ge \eta$

but $E[w_n] \searrow 0$, $\|w_n\|_{L^2}^2 \le M$, $\|w_n\|_{\dot{H}^1} = \|Q\|_{\dot{H}^1}$,

It can be shown $w_n \rightharpoonup w_\infty$ in $\dot{H}^1_m$, $w_\infty = Q_{\lambda,\gamma}$.

We claim that, in fact, $w_n \to w_\infty$ in $\dot{H}^1_m$.

$$w_n = [Q + \tilde{w}_n]_{\lambda,\gamma}$$

lin. of $D_u u$ around Q

$$\underset{\substack{\downarrow \\ 0}}{E[Q + \tilde{w}_n]} = \frac{1}{2} \|L_Q \tilde{w}_n\|_{L^2}^2 + \underset{\substack{\downarrow \\ 0}}{(\text{nonlin})}$$

($\because$ strong L^p conv)

Then by strong L^p conv. of $\tilde{w}_n$, $\tilde{w}_n \to 0$ in $\dot{H}^1_m$.

This is a contradiction.

Step 2 (Modulation)

Want to select $\lambda(t), \gamma(t)$ with better regularity in t.

$$T_\delta := \{ u \in \dot{H}^1_m : d_{\dot{H}^1_m}(\{u_{\lambda^{-1},-\gamma}\}_{\substack{\lambda>0 \\ \gamma\in[0,2\pi)}}, Q) < \delta \}.$$

$z_1, z_2 \in (C_c^\infty)_m$ satisfy transversality w/ $\operatorname{Ker} L_Q = (\Lambda Q, iQ)$

Claim. $\forall\, 0 < \eta \ll 1$, $\exists\, \delta > 0$ s.t.

if $u \in T_\delta$, $\exists!\ \lambda > 0$, $\gamma \in [0, 2\pi)$, $\epsilon \in \dot{H}^1_m$ s.t.

$$\boxed{u = [Q + \epsilon]_{\lambda,\gamma}}$$

satisfying $\boxed{\langle \epsilon, z_1 \rangle = \langle \epsilon, z_2 \rangle = 0}$,

$$\boxed{\|\epsilon\|_{\dot{H}^1_m} < \eta} \quad \boxed{\left| \frac{\|u\|_{\dot{H}^1_m}}{\|Q\|_{\dot{H}^1_m}} \lambda - 1 \right| \lesssim \|\epsilon\|_{\dot{H}^1_m}.}$$

Idea: Implicit ftn thm to $F(\epsilon; \lambda, \gamma) = \begin{pmatrix} \langle u_{\lambda^{-1},-\gamma} - Q, z_1 \rangle \\ \langle u_{\lambda^{-1},-\gamma} - Q, z_2 \rangle \end{pmatrix}$

Step 3 This is the most important step.

Lemma (nonlinear coercivity of E)

$\forall M > 0$, $\exists \eta > 0$ s.t.

if $\epsilon \in H^1_m$, $\|\epsilon\|_{L^2}^2 \le M$, $\leftarrow$ could be arbitrarily big

$\langle \epsilon, \mathcal{Z}_1 \rangle = \langle \epsilon, \mathcal{Z}_2 \rangle = 0$, $\|\epsilon\|_{\dot H^1_m} < \eta$

then $$\boxed{E[Q+\epsilon] \gtrsim_M \|\epsilon\|_{\dot H^1_m}^2}$$

Cor. $\boxed{\|\epsilon\|_{\dot H^1_m} \sim_M \lambda \sqrt{E[u]}}$ for t close to T_+.

pf idea :

$$2E[Q+\epsilon] = \|D_{Q+\epsilon}(Q+\epsilon)\|_{L^2}^2 \qquad \parallel \nearrow \quad \parallel \lesssim_M \|\epsilon\|_{\dot H^1_m}^{3/2}$$

$$= \Big\| L_Q\epsilon - \underline{\frac{2A_\theta[Q,\epsilon]}{r}}\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon \Big\|_{L^2}^2$$

$$= \Big\| L_Q\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon \Big\|_{L^2}^2 + o_{\eta\to 0}(\|\epsilon\|_{\dot H^1_m}^2)$$

$$= \underline{\| L_Q(\chi_{\{r\le R\}}\epsilon)\|_{L^2}^2} \leftarrow \text{coercivity!}$$

$$+ \underline{\Big\| \Big(D_Q - \frac{A_\theta[\epsilon]}{r}\Big)(\chi_{\{r\gtrsim R\}}\epsilon)\Big\|_{L^2}^2} \leftarrow \text{see below}$$

$$+ \underline{o_{R\to\infty}(\|\epsilon\|_{\dot H^1_m}^2) + o_{\eta\to 0}(\|\epsilon\|_{\dot H^1_m}^2)} \leftarrow \text{acceptable}$$

$$+ \underline{O_M\Big(\Big\|1_{[R,2R]}\Big(|\partial_r\epsilon| + \tfrac{1}{r}|\epsilon|\Big)\Big\|_{L^2}^2\Big)}$$

Coercivity

$\langle \chi_{\{r \lesssim R\}} \epsilon, \mathcal{Z}_j \rangle = 0$ for R large

$$\Rightarrow \boxed{\| L_Q (\chi_{\{r \lesssim R\}} \epsilon) \|_{L^2}^2 \gtrsim \| \chi_{\{r \lesssim R\}} \epsilon \|_{\dot{\mathcal{H}}^1_m}^2}$$

Nonlinear Hardy

$$\boxed{\| (D_Q - \frac{A_\theta[\epsilon]}{r}) f \|_{L^2(r \geq R)}^2 \gtrsim_M \| \frac{1}{r} f \|_{L^2(r \geq R)}^2}$$

(then the RHS can be easily changed to $\| \partial_r f \|_{L^2(r \geq R)}^2 + \| \frac{1}{r} f \|_{L^2(r \geq R)}^2$)

$$\| (D_Q - \frac{A_\theta[\epsilon]}{r}) f \|_{L^2(r \geq R)}^2$$

$$= \int_R^\infty | (\partial_r - \frac{m + A_\theta[Q] + A_\theta[\epsilon]}{r}) \epsilon |^2 r \, dr$$

< 0 if R is large ↙ always < 0

$$= \int_R^\infty \Big[|\partial_r \epsilon|^2 + \frac{(m + A_\theta[Q] + A_\theta[\epsilon])^2}{r^2} |\epsilon|^2 \Big] r \, dr$$

$$+ \int_R^\infty - \frac{A_\theta[\epsilon]}{r} 2 \mathrm{Re} (\bar{\epsilon} \partial_r \epsilon) \, r \, dr \qquad \geq c \, |\partial_r \epsilon|^2 \text{ by Cauchy-Schwarz}$$

$$+ \int_R^\infty \Big[- \frac{m + A_\theta[Q]}{\cancel{r}} \overbrace{2 \mathrm{Re} (\bar{\epsilon} \partial_r \epsilon)}^{\partial_r |\epsilon|^2} \Big] \cancel{r} \, dr \longleftarrow \text{integrate by parts}$$

$$\parallel$$

$$- \frac{1}{2} \int_R^\infty |Q|^2 |\epsilon|^2 r \, dr \, , \quad \text{acceptable for } R \text{ large}$$

Step 4 Bound on λ

We claim: $\lambda|\lambda_t| + \lambda^2|\gamma_t| \lesssim \|\epsilon\|_{\dot{\mathcal{H}}^1_m}$

then $|\lambda_t| \lesssim \lambda^{-1}\|\epsilon\|_{\dot{\mathcal{H}}^1_m} \lesssim_M \sqrt{E[u]}$ ✓

To prove the claim, introduce renormalized coords (s, y)

$$\boxed{dt = \lambda^2 ds, \qquad r = \lambda y.}$$

View $\epsilon = \epsilon(s,y)$.

$$\rightsquigarrow \boxed{\partial_s \epsilon + \frac{\lambda_s}{\lambda}\Lambda(Q+\epsilon) - \gamma_s i(Q+\epsilon) + i\mathcal{L}_Q\epsilon + iR_Q\epsilon = 0.}$$

Test against $\mathcal{Z}_j$:

$$\frac{\lambda_s}{\lambda}\langle \Lambda(Q+\overset{\nearrow 0}{\underline{\epsilon}}), \mathcal{Z}_j\rangle - \gamma_s\langle i(Q+\overset{\nearrow 0}{\underline{\epsilon}}), \mathcal{Z}_j\rangle$$

$$= -\underbrace{\langle i\mathcal{L}_Q\epsilon, \mathcal{Z}_j\rangle}_{\lesssim \|\epsilon\|_{\dot{\mathcal{H}}^1_m} \text{ by duality}} - \underbrace{\langle iR_Q\epsilon, \mathcal{Z}_j\rangle}_{\nwarrow \text{ nonlinearity, } \lesssim_M \|\epsilon\|^2_{\dot{\mathcal{H}}^1_m}}.$$

$$\Longrightarrow \quad \left|\frac{\lambda_s}{\lambda}\right| + |\gamma_s| \lesssim \|\epsilon\|_{\dot{\mathcal{H}}^1_m} \quad ✓$$

If $m=0$, then we can improve this bound (coming from slow decay of ΛQ & $\epsilon \in \dot{\mathcal{H}}^1_m$). See [Kim-Kwon-O. 2]

Step 5 $\exists z^*$ & convergence

Recall: $\|\epsilon^*\|_{H^1_{-m-2}} = \| u - Q_{\lambda(t),\gamma(t)} \|_{H^1_{-m-2}} \lesssim 1$ up to T_+.

Claim: (outer conv) $\exists z^* \in L^2$ s.t. $\forall R > 0$,

$$1_{\{r>R\}} \epsilon^* \xrightarrow{t \nearrow T_+} 1_{\{r>R\}} z^* \quad \text{in } L^2$$

The desired statement $u - Q_{\lambda(t),\gamma(t)} \xrightarrow{t \nearrow T_+} z^*$ in L^2 follows by Rellich-Kondrachov.

(Note: $z^* \in H^1_{-m-2}$ as well. By the claim,
if u : $H^{1,1}_m$ sol'n, then $r z^* \in L^2$ as well.)

Pf of the claim Truncate to $\{r > R\}$.

Main linear error: $(i\partial_t + \Delta^{(m)})(\chi_{r \gtrsim R} u)$

$$= -[\Delta^{(m)}, \chi_{\{r \gtrsim R\}}] u + \cdots$$

$$-[\Delta^{(m)}, \chi_{\{r \gtrsim R\}}] u \sim \frac{1}{R} \chi_{\{r \simeq R\}} \partial_r u + \cdots$$

t close to T_+

$$\sim \frac{1}{R} \chi_{\{r \simeq R\}} \partial_r (u - Q_{\lambda,\gamma}) + \cdots$$

$$\left\| \frac{1}{R} \chi_{\{r \simeq R\}} \partial_r (u - Q_{\lambda,\gamma}) \right\|_{L^2} \lesssim \frac{1}{\lambda} \| \partial_y \epsilon \|_{L^2} \lesssim \sqrt{E[u]}$$

In fact, $\| (i\partial_t + \Delta^{(m)})(\chi_{r \gtrsim R} u) \|_{L^2} \lesssim 1$ for t close to T_+,
which implies the claim.