Topics in incompressible fluids: instabilities and singularity formation

Federico Pasqualotto

May 22, 2024

In the course of these lectures, we will present three instances in which physical instabilities in fluids generate small-scale effects of different nature. Instabilities are often detected at the *linear level*, and in this spirit we will focus on three different results that use insights from *linear analysis* to produce *nonlinear* solutions with interesting properties.

1. In the first talk, we will focus on the paper of Grenier [3]. We will describe Grenier's iterative scheme to obtain a (nonlinearly) unstable solution to the 2d incompressible Euler equations from a linearly unstable solution. We will also comment on applications to the vortex sheet problem in two space dimensions.

2. The second talk will concern the work by Kiselev and Šverák on double-exponential growth of the vorticity gradient for the two dimensional, incompressible Euler equations on a bounded domain [4], and the work of Zlatoš on exponential growth in the case of periodic boundary conditions [5]. In this case, a linear analysis of the Bahouri–Chemin patch reveals a solution whose vorticity gradient grows double exponentially. The scheme of [4] starts from an approximation of the Biot–Savart law, and a specific choice of the vorticity configuration. These allow the authors of [4] to deduce long-time properties of the vorticity configuration which can in turn be used to show that some Lagrangian particles are advected exactly as in the Bahouri–Chemin patch scenario. We will also describe a result by Zlatoš [5] which applies to solutions of the 2d Euler equations on the torus $\mathbb{T}^2$ in the regularity class $C^{1,\alpha}$ for the vorticity, with $\alpha > 0$.

3. The third talk will focus on singularity formation. We will describe a recent result, obtained in collaboration with Tarek Elgindi, in which we construct finite-energy solutions to the incompressible 3d Euler equations on $\mathbb{R}^3$ which become singular in finite time [1, 2]. Our solutions are axisymmetric *with swirl* (and the swirl drives the singularity formation mechanism). We will directly focus on the 2d Boussinesq equations, which are a good approximate model for the 3d Euler equations in axisymmetry away from the symmetry axis. In this context, we will adopt the viewpoint of looking for solutions in the $C^{1,\alpha}(\mathbb{R}^3)$ regularity class for the velocity.

   We will describe how, by reducing the initial conditions to a special geometric configuration, a linear instability effect appears in the (radially) zero-homogeneous system. This is a manifestation of what in the literature is known as the *Rayleigh–Bénard instability*. We will then describe how a singularity formation scenario can be constructed as a second order effect (in the parameter α) arising on top of the physical instability.

References

[1] Tarek M Elgindi and Federico Pasqualotto. From instability to singularity formation in incompressible fluids. *ArXiv preprint: 2310.19780*, 2023.

[2] Tarek M Elgindi and Federico Pasqualotto. Invertibility of a linearized Boussinesq flow: a symbolic approach. ArXiv preprint:2310.19781, 2023.

[3] Emmanuel Grenier. On the nonlinear instability of Euler and Prandtl equations. *Communications on Pure and Applied Mathematics*, 53(9):1067–1091, 2000.

[4] Alexander Kiselev and Vladimir Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Annals of mathematics*, 180(3):1205–1220, 2014.

[5] Andrej Zlatoš. Exponential growth of the vorticity gradient for the Euler equation on the torus. *Advances in Mathematics*, 268:396–403, 2015.