SINGULARITY FORMATION IN EVOLUTIONARY PDES

SUNG-JIN OH

ABSTRACT. This is an evolving set of lecture notes for Math 279 at UC Berkeley in Spring 2023.

1. WEEK 1: INTRODUCTION

Singularities are one of the most fundamental yet fascinating aspects of nonlinear PDEs. Take, for instance, one of the simplest nonlinear PDEs, the *inviscid Burgers equation* (also called Hopf's equation)

$$\partial_t u + u\partial_x u = 0, \qquad \text{where } u : I_t \times \mathbb{R}_x \to \mathbb{R}. \tag{1.1}$$

Being a (nonlinear) scalar first-order PDE, it can be shown – for instance, using the method of characteristics [**?**, Chapter 3]– that starting with regular initial data (e.g., $u_0 \in C^\infty(\mathbb{R};\mathbb{R})$) there exists a unique regular solution for a short time (i.e., $u \in C^\infty(I \times \mathbb{R};\mathbb{R})$ solving the PDE) that satisfies $u(0,x) = u_0(x)$. In general, however, such a regular solution cannot be extended for all times, as the following simple argument shows.

Proposition 1.1. *Let $u : [0,T)_t \times \mathbb{R}_x \to \mathbb{R}$ be a smooth solution to* (1.1)*. If $u_0(x) = u(0,x)$ is smooth and $u_0'(x_0) < 0$ at some $x_0 \in \mathbb{R}$, then it is impossible that $T = \infty$.*

Proof. Let us quickly reproduce the standard proof. One first notes that the solution u is constant on each straight line passing through $(0,X)$ with velocity $u_0(X) = u(0,X)$ (characteristic curve, given by $\{x = X + u_0(X)t\}$). Next, differentiating (1.1) in x, we obtain

$$(\partial_x u)_t + uu_{xx} + (\partial_x u)^2 = 0.$$

It follows that for $v(t,X) := \partial_x u(t, X + u_0(X)t)$, we have

$$\partial_t v(t,X) + v^2(t,X) = 0.$$

This is *Ricatti's equation*; it is well-known that $v(t,X) \to -\infty$ in some finite time T_X if $v(0,X) < 0$. In fact, separation of variables tells us that $v(t,X) = \frac{v(0,X)}{1+v(0,X)t}$ and thus $T_X = (-v(0,X))^{-1}$. In particular, $v(0,x_0) = \partial_x u_0(x_0) < 0$ by hypothesis, so it is impossible that $T > (-\partial_x u_0(x_0))^{-1}$. □

Under reasonable assumptions on the initial data, we are guaranteed to encounter singularity. For instance, if u_0 is smooth and compactly supported, then, by the mean value theorem, we are guaranteed to have a point where $\partial_x u_0(x_0) < 0$. Hence, singularities are generally *unavoidable* in this problem, and important mathematical questions for this PDE and become that of understanding singularities: how do they look like, and how do we reasonably continue the solution after singularity formation (the keywords here are *shocks* and *entropy solutions*; see [Hörmander, Lectures on nonlinear hyperbolic equations, Chapter 2][**?**, Ch. 2]). Singularities also have important ramification in the physics that the PDE is describing (shock waves in gas dynamics, in this case). The story is similar with many other nonlinear PDEs of interest.

The goal of this course is *to provide a survey of recent techniques and results pertaining to the topic of singularities for nonlinear evolutionary PDEs.* But before we proceed further, we should clarify the meaning of the terminology "singularity" that we will adopt in the remainder of this course.

Definition of singularity formation. In this course, we shall consider PDEs that admits an Cauchy (initial value) problem formulation (i.e., there exists a notion of time t, and solutions may be considered as t-dependent curves in a function space X), which is *(locally) well-posed.* In particular, given any initial data $u_0 \in X$, we may associate to it the unique well-posed solution $u \in C_t([0,T), X)$ on some $T = T(u_0) > 0$. The maximal time $T_+ > 0$ until which the well-posed solution is defined is called *the (future) lifespan* of u. For us, the working definition of singularity formation will be:

Definition 1.2. We say that u *forms singularity*, or *blows up in finite time*, if it cannot be extended indefinitely as a well-posed solution (i.e., $T_+ < +\infty$). In this case, we shall refer to T_+ as the *singular time.*

Later, we will give the precise statement of (locally) well-posed Cauchy problem for each PDE we study in detail. However, we shall not delve into their proofs; these belong to a different course. For those who are interested, we refer to [**?**] and [**?**] (for dispersive PDEs, but similar ideas often apply to other evolutionary PDEs).

Some examples. In this course, I will attempt to provide a broad overview of the topic of singularity formation, by demonstrating key ideas in some exemplar cases.

1. **Inviscid Burger's equation.**

$$\partial_t u + u \partial_x u = 0. \tag{Burgers}$$

2. **Euler equations.**

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla p &= 0. \end{aligned} \tag{Euler}$$

   i. *Compressible, isentropic.* $p = \rho^\gamma$.
   ii. *Incompressible.* $\rho \equiv 1$. Then $\nabla \cdot \mathbf{v} = 0$, and the pressure p has to be determined by the divergence free condition.

3. **Nonlinear Schrödinger/wave/heat equations.**

$$i\partial_t u - \Delta u + \mu |u|^{p-1} u = 0, \tag{NLS}$$

$$\partial_t^2 u - \Delta u + \mu |u|^{p-1} u = 0, \tag{NLW}$$

$$\partial_t u - \Delta u + \mu |u|^{p-1} u = 0. \tag{NLH}$$

4. **Other examples.** Later in the course, I will also discuss some geometric flows, such as harmonic map heat flow, wave maps, Yang–Mills and (self-dual) Chern–Simons–Schrödinger.

Basic questions. We now list some basic questions concerning singularities.

1. **Formation of singularity.** Given a nonlinear evolutionary PDE whose Cauchy problem is locally well-posed, the first question one should ask is *whether singularity formation is possible* (i.e., is it possible that $T_+ < +\infty$?). If not, one says that the Cauchy problem is *globally well-posed*.

2. **Description of singularity formation.** If singularity formation is possible, the natural follow-up question is whether we can describe the formation of singularity in more detail. I would like distinguish two different kinds of descriptions according to the typical tools that are involved:
   i. *Qualitative description of singularities.* One powerful way to analyze singularities is to use "soft" tools such as *compactness*, starting from *conservation laws* and *monotonicity formulas* for the PDE. A "model" statement that could be established with tools in this vein would be as follows:

      Suppose $T_+ < +\infty$. Then for t close to T_+, the solution u admits a decomposition of the form
      $$u(t,x) = \frac{1}{\lambda(t)^\beta} Q\left(\frac{x-X}{\lambda(t)}\right) + \epsilon(t,x)$$
      for some Q (*blow-up profile*), X (*blow-up locus*), $\lambda(t)$ (*blow-up* or *characteristic scale*) such that $\lambda(t) \to 0$ as $t \nearrow T_+$, and $\epsilon(t,x)$ that is "regular" near (T_+, X).

      This is a rather classical theme in parabolic PDEs ("bubbling" analysis). Analogous developments have been in achieved in hyperbolic and dispersive PDEs recently (e.g., threshold theorems, soliton resolution conjecture).

   ii. *Quantitative description of singularities.* Alternatively, one may restrict to a specific solution exhibiting singularity formation and bring "hard" analysis tools (i.e., linear stability analysis, multilinear estimates, etc.) to study that and nearby solutions. Some exemplary problems that can be answered by such an approach are:
      - precise description of $\lambda(t)$,
      - stability of the singularity formation under initial data perturbations (with estimates relating the perturbations with the solutions),
      - stability of the singularity formation under perturbations of the equation.

3. **Continuation beyond the first singularity.** By its very definition, the Cauchy problem fails to be well-posed when we encounter singularity. But a very interesting question to ask is if there is a meaningful way to continue the solution beyond the first singular time. The answer to this question would often require a precise understanding of singularity formation (i.e., precise answer to Question 2), as well as a consideration of the physical/geometric origin of the PDE. A textbook example of the study of this question would be the theory of entropy solution for the Burgers equation; see [**?**, Ch. 2].

In this course, we will mostly be concerned with *Question 1 (Formation of singularity)* and *Question 2.ii (Quantitative description of singularities)* for various evolutionary PDEs. In fact, these two questions are inherently related: *a powerful way to exhibit formation of* *singularity is to come up with a detailed quantitative blow-up scenario!* This is a viewpoint that will be emphasized throughout this course.

Classical, soft arguments for singularity formation. There are classical results that tell us the existence of singularity formation, which goes through a soft contradiction argument. Before we get to more quantitative approaches, let us have a quick review of the classical arguments.

For concreteness, let us consider (NLS).

Lemma 1.3. *Let $u : I_t \times \mathbb{R}^d \to \mathbb{C}$ be a smooth solution to*

$$i\partial_t u - \Delta u + \mu |u|^{p-1} u = 0, \tag{NLS}$$

satisfying $|x| u(t, x) \in L^2(\mathbb{R}^d)$ for all $t \in I_t$. Assume also that $\mu < 0$ (focusing).

(1) *For each $t \in I_t$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |x|^2 |u|^2 \, \mathrm{d}x = \int \mathbf{x} \cdot \mathrm{Im}(\overline{u} \nabla u) \, \mathrm{d}x.$$

(2) *For each $t \in I_t$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{Im}(\overline{u} \nabla u) \, \mathrm{d}x = 16E - \frac{d \cdot \frac{p-1}{2} - 2}{p+1} \int |u|^{p+1} \, \mathrm{d}x,$$

where

$$E = \int \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \, \mathrm{d}x.$$

Proposition 1.4. *Let $u : [0, T)_t \times \mathbb{R}^d_x \to \mathbb{C}$ be a smooth solution to*

$$i\partial_t u - \Delta u + \mu |u|^{p-1} u = 0. \tag{NLS}$$

where $\mu < 0$ (focusing) and $p \geq \frac{4}{d} + 1$ (L^2-supercritical).

Suppose that $u_0 \in H^1$, $|x| u_0 \in L^2$, and one of the following folds:

(1) *$E(u_0) < 0$,*
(2) *$E(u_0) = 0$ and $\int \mathbf{x} \cdot \mathrm{Im}(\overline{u} \nabla u) \, \mathrm{d}x < 0$, or*
(3) *$E(u_0) > 0$ and $\int \mathbf{x} \cdot \mathrm{Im}(\overline{u} \nabla u) \, \mathrm{d}x < -\sqrt{32E \int |x|^2 |u|^2 \, \mathrm{d}x}$,*

Then T cannot be infinite.

Proof. Suppose that $T = \infty$. Let

$$V(t) = \int |x|^2 |u|^2 (t, x) \, \mathrm{d}x, \quad F(t) = \int \mathbf{x} \cdot \mathrm{Im}(\overline{u} \nabla u) \, \mathrm{d}x.$$

Thanks to the assumption $p \geq \frac{4}{d} + 1$, we have $d \cdot \frac{p-1}{2} \geq 2$ and thus

$$V(t)'' \leq 16E,$$

where we note that $E = E(t)$ is conserved. Therefore,

$$V(t) \leq 8Et^2 + F(0)t + V(0).$$

The above conditions ensure that the RHS becomes negative in finite time, which leads to a contradiction. □

The advantage of this argument is its simplicity. The disadvantage, however, is that it gives us very little information as to how the singularity formation actually looks like.

Exercise: NLW (Levine), compressible Euler (Sideris).

Remark: Penrose's incompleteness theorem for Einstein's equation.

Remark 1.5. It must be

Self-similar singularities. Another basic strategy for exhibiting singularity formation, which is much more concrete, is to look for self-similar solutions.

$$x = \lambda(t) y, \quad u = \nu(t) U,$$

where we assume also that U depends only on y, i.e.,

$$U = U(y).$$

Often times, there are choices of $\lambda(t)$ and $\nu(t)$, that has to do with the scaling invariance properties of the PDE, that would lead to a simpler equation for U. We shall refer to the resulting scenario as *self-similar*. Roughly speaking, any singularity formation that exhibits the same $\lambda(t)$ as self-similar will be called *Type I*.

Let us consider some examples.

Inviscid Burgers equation. (direct approach)

$$u_t + u u_x = 0.$$

Consider the ansatz

$$x = \lambda(t) y, \quad u = \nu(t) Q$$

where we assume $Q = Q(y)$ (exact self-similarity). Then

$$-\frac{\dot{\lambda}\nu}{\lambda} y Q_y + \dot{\nu} Q + \frac{\nu^2}{\lambda} Q Q_y = 0$$

Assume that $\lambda(t) = (T-t)^b$ for some $b > 0$, where T is the singular time. In order to the same factor of $(T-t)$ in front of each term, we impose $\nu(t) = \frac{\lambda(t)}{T-t}$. We are led to

$$b y Q_y - (b-1) Q + Q Q_y = 0.$$

Inviscid Burgers equation. (dimensional analysis)

We may also deduce the above definitions of y and Q using *dimensional analysis*. We denote the dimensions of t, x and u by $[t]$, $[x]$ and $[u]$. Equating the dimension of all terms in Burgers equation, we see that $[u] = \frac{[x]}{[t]}$. We introduce y and Q to be dimensionless space and dependent variables, respectively, adapted to the (time-dependent) characteristic spatial scale $\lambda(t)$ and temporal scale $T - t$. Then by we are motivated set

$$x = \lambda(t) y, \quad u = \frac{\lambda(t)}{T-t} Q.$$

Nonlinear Schrödinger equation.

$$i \partial_t u + \Delta u - |u|^{p-1} u = 0.$$

Assume radiality. By dimensional analysis, we have

$$[t] = [x]^2, \quad [u] = [t]^{-\frac{1}{p-1}} = [x]^{-\frac{2}{p-1}}.$$

According to the first relation, we consider the characteristic scales $\lambda(t) = a(T-t)^{\frac{1}{2}}$. By the second relation, we are led to the change of variables

$$x = \lambda(t) y, \quad u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q.$$

Assuming that $Q = Q(y)$ (exactly self-similar), the equation for Q becomes

$$\partial_y^2 Q + \frac{d-1}{y}\partial_y Q - Q + ia\left(\frac{2}{p-1}Q + y\partial_y Q\right) + |Q|^{p-1}Q = 0.$$

Type II singularities. Any singularity formation that is not Type I will be referred to as *Type II*. For us, the important distinction is that for Type II singularities, it is *not possible* to rely on a reduced equation for their construction. Concretely, the determination of $\lambda(t)$ is now a part of the problem, which often has to be solved dynamically.

There are many problems where Type II singularities naturally occur. Consider again (NLS). In the L^2-*critical* case $p = \frac{4}{d}+1$, it can be shown that the radial self-similar equation does *not* admit an H^1 solution for any $a > 0$ [Sulem and Sulem, *The nonlinear Schrodinger equation. Self-focusing and wave collapse*, Chapter 8]. Nevertheless, the virial argument shows that singularity formation is possible, and in fact is generic (any initial data with negative energy). We refer to Perelmen [**?**] and Merle–Raphaël [**?**, **?**, **?**, **?**, **?**] for the study of Type II singularities in this problem.

To do:

- Summarize Perelman, Merle–Raphaël works.
- Other examples: Critical geometric wave/dispersive/heat equations. Here, there is no soft argument for the existence of singularity formation; the very first construction of singularity formation had to deal with Type II singularities (Chang–Ding–Ye [**?**] for HMHF; Krieger–Schlag–Tataru [**?**], Rodnianski–Sterbenz [**?**] and Raphaël–Rodnianski [**?**] for WM).

Outline of the course.

1. **Self-similar singularities and related topics.**
   i. *Perturbation of self-similar singularities.* We will follow my paper with Federico Pasqualotto on showing the existence of singularity formation for perturbations of the inviscid Burger's equation:

   $$\partial_t u + u\partial_x u + \Gamma u = 0$$

   as long as Γ is "of order less than 1". The proof will be based on a detailed, quantitative understanding of the self-similar blow-up solutions to (1.1) and its stability properties. These ideas that will be introduced here will be rudimentary (although in different forms) in the subsequent part of the course.

   ii. *Elgindi's proof of singularity formation for the incompressible Euler equations.* Next, we will follow the groundbreaking paper of T. Elgindi on the existence of (self-similar) singularity formation for the incompressible Euler equations.
2. **Type II singularities near a soliton.** (Model equation TBD)
   To construct Type II singularities, the laws for $\lambda(t)$ must be derived.
   i. *Backward construction.* TBD
   ii. *Forward construction.* TBD

2. Weeks 2–6. Self-similar solutions to the inviscid Burgers equation, their stability, and related topics

In the next few lectures, we shall focus on the *inviscid Burgers equation*

$$\partial_t u + u\partial_x u = 0, \qquad \text{(Burgers)}$$

and variants of it.

2.1. **Smooth self-similar solutions to Burgers equation.** We start by studying the exactly self-similar solutions to (Burgers). Recall that, with the change of variables

$$x = \lambda(t) y, \quad u = \frac{\lambda(t)}{(T-t)} Q,$$

where $\lambda(t) = (T-t)^b$ for some $b \in \mathbb{R}$ and $Q = Q(y)$, we arrive at the equation

$$by\frac{\mathrm{d}Q}{\mathrm{d}y} - (b-1)Q + Q\frac{\mathrm{d}Q}{\mathrm{d}y} = 0. \qquad (2.1)$$

Observe that (2.1) remains invariant under the transformation $Q \mapsto -Q(-y)$. In view of this symmetry, we shall look for *odd* solutions to (2.1).

The standard trick for solving (2.1) is to exchange the independent and dependent variables, which leads to an explicitly solvable linear equation. We remark that when performed in the setting of 2×2 systems, this trick is usually called the *hodograph transformation*; see [**?**, Chapter 4].

Assume that $Q = Q(y)$ is monotonic on an interval J. Inverting this relation, view y as a function of Q on $\mathcal{J} = Q(J)$. Since $\frac{\mathrm{d}y}{\mathrm{d}Q} = \left(\frac{\mathrm{d}Q}{\mathrm{d}y}\right)^{-1}$, we have

$$by\left(\frac{\mathrm{d}y}{\mathrm{d}Q}\right)^{-1} - (b-1)Q + Q\left(\frac{\mathrm{d}y}{\mathrm{d}Q}\right)^{-1} = 0.$$

Assuming $\frac{\mathrm{d}y}{\mathrm{d}Q} = (\frac{\mathrm{d}Q}{\mathrm{d}y})^{-1} \neq 0$, we arrive at

$$-(b-1)Q\frac{\mathrm{d}y}{\mathrm{d}Q} + by + Q = 0.$$

This is a first-order linear ODE, which we can explicitly solve.

When $b = 1$, we simply obtain

$$y = -Q.$$

For $b \neq 1$ and for $Q \neq 0$, this equation may be rewritten as

$$(b-1)Q^{1+\frac{b}{b-1}}\frac{\mathrm{d}}{\mathrm{d}Q}\left(Q^{-\frac{b}{b-1}}y\right) = Q,$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}Q}\left(Q^{-\frac{b}{b-1}}y\right) = \frac{1}{b-1}Q^{-\frac{b}{b-1}}.$$

Hence,

$$y = Q^{\frac{b}{b-1}}\left(-Q^{-\frac{1}{b-1}} - a\right) = -Q - aQ^{\frac{b}{b-1}},$$

where a is an integration constant.

We now observe that $Q \mapsto y$ is monotone and smooth (especially at $Q = 0$) when $a > 0$ and $\frac{b}{b-1} = 2k+1$ for some $k \in \mathbb{Z}_{\geq 0}$, or equivalently, $b = \frac{2k+1}{2k}$. It is useful to rewrite the defining relation of the odd function $Q_{k,a}$ as

$$\boxed{y = -Q_{k,a} - aQ_{k,a}^{2k+1}.} \tag{2.2}$$

From this relation, it is clear that $\frac{Q_{k,a}}{y} \geq -1$ for all y. Moreover, implicit differentiation gives $\partial_y Q_{k,a} = -\frac{1}{1+a(2k+1)Q_{k,a}^{2k}}$, which implies that $\partial_y Q_{k,a} \geq -1$.

Let us study the asymptotics of Q by looking at (2.2). When $k \geq 1$, the integration constant a is essentially the $(2k+1)$-th Taylor coefficient of $Q_{k,a}$ at 0. To wit, we (formally) expand $Q_{k,a} = \sum_{n \geq 0} \frac{Q_{k,a}^{(n)}(0)}{n!} y^n$, plug it in (2.2), and observe the following:

- all even order Taylor coefficients are zero (obvious due to the odd symmetry),
- $Q'_{k,a}(0) = -1$, and $Q_{k,a}^{(2j+1)}(0) = 0$ for all $0 \leq j < k$, and $Q_{k,a}^{(2k+1)}(0) = a(2k+1)!$

Remark 2.1. The appearance of the integration constant a is natural, in view of the possibility of modulating $\lambda(t) = a'(T-t)^b$ (**Exercise:** Show that a can be set equal to 1 by modifying $\lambda(t)$ into $\lambda(t) = a'(a)(T-t)^b$).

Let us also discuss the asymptotics of Q as $y \to \infty$. Assume again that $k \geq 1$ and $a > 0$. It is clear from (2.2) that $Q_{k,a} \to \infty$ as $y \to \infty$. This means that for large values of y, $aQ_{k,a}^{2k+1}$ would dominate Q. Thus, our initial approximation is

$$Q_{k,a} \approx -\left(\frac{y}{a}\right)^{\frac{1}{2k+1}} + \cdots$$

More rigorously, it is not difficult to show that

$$\lim_{y \to \infty} \frac{Q_{k,a}}{-\left(\frac{y}{a}\right)^{\frac{1}{2k+1}}} = 1.$$

Now plugging this approximation into (2.2), we can get the next order terms in the approximation as follows:

$$Q_{k,a} = -\left(\frac{y + Q_{k,a}}{a}\right)^{\frac{1}{2k+1}} = -\left(\frac{y}{a} - \frac{1}{a}\left(\frac{y + Q_{k,a}}{a}\right)^{\frac{1}{2k+1}}\right)^{\frac{1}{2k+1}} = \cdots \tag{2.3}$$

We summarize our observations so far as a proposition.

Proposition 2.2. *For every $k \geq 0$ and $a > 0$, the odd function $Q_{k,a}$ defined by the implicit relation*

$$y = -Q_{k,a} - aQ_{k,a}^{2k+1} \tag{2.2}$$

is a smooth solution to (2.1) *with $b = \frac{2k+1}{2k}$. The following global bounds hold:*

$$\partial_y Q \geq -1, \quad \frac{Q}{y} \geq -1. \tag{2.4}$$

The Taylor expansion of $Q_{k,a}$ near $y = 0$ takes the form

$$Q_{k,a} = -y - ay^{2k+1} + O(|y|^{2k+2}).$$

Higher order expansions can be obtained by studying (2.2). *As* $y \to \pm\infty$, *we have*

$$\lim_{y \to \pm\infty} \frac{Q_{k,a}}{\mp(\frac{y}{a})^{\frac{1}{2k+1}}} = 1.$$

More precise expansions can obtained through (2.3).

We now discuss how $Q_{k,a}$ looks like in the original variables t, x, and u. Let us start with the case when $k = 0$ or $a = 0$, which is instructive but degenerate. In this case, $Q_{k,a} = -y$, so we simply have

$$u = -\frac{x}{T-t},$$

For every fixed $x \neq 0$, this solution blows up as $t \nearrow$. This behavior is not so interesting since it is very far from what we expect in shock formation; if we start with a bounded smooth initial data, then by the conservation of the L^∞ norm (or maximum principle), the solution must be bounded all the way up to the singular time.

When $k \geq 1$ and $a > 0$, we get something more interesting. We have

$$u(t,x) = (T-t)^{b-1} Q_{k,a}(\tfrac{x}{(T-t)^b}), \quad \partial_x u(t,x) = (T-t)^{-1} Q'_{k,a}(\tfrac{x}{(T-t)^b}).$$

Fix any $x \in (0,1)$ and take $t \nearrow T$. Then $y = \frac{x}{(T-t)^b} \nearrow \infty$, so

$$|u(t,x)| = |(T-t)^{b-1} Q_{k,a}(\tfrac{x}{(T-t)^b})| \lesssim (T-t)^{b-1} \left(\frac{x}{a(T-t)^b}\right)^{\frac{1}{2k+1}} = \left(\frac{x}{a}\right)^{\frac{1}{2k+1}},$$

and we see that $u(t,x)$ stays bounded as $t \nearrow T$. However, if we track the value of $\partial_x u(t,0)$, then

$$\partial_x u(t,0) = (T-t)^{-1} Q'_{k,a}(0) = -(T-t)^{-1} \to -\infty \text{ as } t \nearrow T,$$

which is consistent with what one expects for the Burgers equation (recall Proposition 1.1)!

At this point, one might complain that the initial data for this $u(t,x)$ is still not bounded. However, this issue can be easily fixed by a cut-off argument, along with some use of the method of characteristics.

Exercise: Cut off the initial data for $Q_{k,a}$ with $k \geq 1$, $a > 0$ to produce an $H^\infty(\mathbb{R})(= \cap_{m \geq 0} H^m(\mathbb{R}))$ initial data leading to a finite time blow-up.

Exercise: Explain why the same procedure does *not* work for $Q_0 = -y$.

2.2. **Perturbation of self-similar solutions to Burgers I: Introduction.** Having studied exact self-similar blow up, now we wish to understand the dynamics around these solutions.

2.2.1. *Renormalized time variable, $\sharp/\flat$-notation.* To study the dynamics, we want to introduce a new set of variables with respect to which the exact self-similar solution looks simpler. To this end, in addition to the renormalized space and dependent variables y and U, we introduce a *renormalized time (or self-similar time) variable* s as follows: take $t = t(s)$, where

$$\mathrm{d}t = (T-t)\mathrm{d}s.$$

The simple motivation behind this relation is that it leads to an s-derivative term without extra factors of the characteristic scale in the resulting equation; see (2.5) below. Note that it is also consistent with dimensional analysis (s is the dimensionless time, $T-t$ has the dimension of time).

Normalizing $s|_{t=T} = \infty$, we have

$$s = -\log(T - t).$$

Hence, $\lambda = (T - t)^b = e^{-bs}$. As before, we introduce

$$x = e^{-bs} y, \quad u(t, x) = e^{-(b-1)s} U(s, y).$$

After a straightforward computation, we obtain *the self-similar Burgers equation*

$$\partial_s U - (b - 1)\, U + by \partial_y U + U \partial_y U = 0. \tag{2.5}$$

To facilitate the transformation of the dependent variable, we introduce the $\sharp/\flat$-notation of Kim and Kwon.

Definition 2.3. Given $U = U(s, y)$, we will write $U^\sharp(t, x) = e^{-(b-1)s} U(s, y)\big|_{(s=s(t), y=y(t,x))}$ ("going up to (t, x, u) variables"), and given $u = u(t, x)$, we define $u^\flat(s, y)$ by the relation $u(t, x) = e^{-(b-1)s} u^\flat(s, y)\big|_{(s=s(t), y=y(t,x))}$ ("going down to (s, y, U) variables").

2.2.2. *Perturbation equation.* Write $U = Q + \epsilon$, where Q is one of the exact self-similar solutions. Then

$$\mathcal{L}_Q \epsilon = -\epsilon \partial_y \epsilon,$$

where $\mathcal{L}_Q$ is the *linearized operator*

$$\mathcal{L}_Q \epsilon := \partial_s \epsilon - (b - 1)\, \epsilon + by \partial_y \epsilon + \epsilon \partial_y Q + Q \partial_y \epsilon$$

2.2.3. *Linear stability analysis.* We now study the linear equation

$$\mathcal{L}_Q \epsilon = f. \tag{2.6}$$

The first question we might ask is whether $\mathcal{L}_Q$ is stable, i.e., if solutions to (2.6) with suitable initial data and f (e.g., $f = 0$) decay as $s \to \infty$. Unfortunately, the answer turns out to be no. To wit, consider a smooth solution ϵ to $\mathcal{L}_Q \epsilon = 0$, and evaluate the equation at $y = 0$. The value of $\epsilon(s, 0)$ satisfies the ODE

$$0 = \partial_s \epsilon(s, 0) - (b - 1)\epsilon(s, 0) - \epsilon(s, 0) = \partial_s \epsilon(s, 0) - b\epsilon(s, 0),$$

and since $b > 0$, we see that $\epsilon(s, 0)$ grows exponentially as $s \to \infty$.

Remark 2.4. One should not be surprised by the existence of an instability. In fact, some degree of instability is naturally expected in this problem due to the symmetries of the underlying equation! To wit, continuous symmetries of (Burgers) (e.g., space translation, time translation, Galilean transformation) applied to $Q^\sharp$ give rise to continuous deformations of $Q^\sharp$ as solutions to (Burgers); differentiation along such a deformation gives rise to a solution to the linearized equation $\mathcal{L}_Q \varphi = 0$, which may exhibit growth. Indeed, later we will see that the growth of $\epsilon(s, 0)$ has to do with the Galilean invariance of (Burgers).

However, it must also be noted that there are instabilities that do *not* seem to come from any symmetries in this problem as well. We will return to the discussion of instabilities coming from symmetries later.

How do we then find parts of solutions that decay (as $s \to \infty$)? For this purpose we will introduce a basic, but important, idea that will be useful throughout the course, namely, that *differentiating the equation with respect to y improves the decay as* $s \to \infty$. The heuristic reason behind this phenomenon is the natural *separation of scales* (or *red-shift effect*) that occurs in finite time blow-up problems.

An explanation of this intuition in the concrete case of (2.6) is as follows. Note that $\mathcal{L}_Q$ resembles $\partial_s - (b-1) + by\partial_y$ as $|y| \to \infty$ (since $|\partial_y Q| \ll |b-1|$ and $|Q| \ll |y|$). But observe that $(\partial_s - (b-1) + by\partial_y)\epsilon = 0$ is precisely $\partial_t \epsilon^\sharp = 0$ for $t < T$. Hence, the solution $\epsilon^\sharp$ does not change in time in the original (t,x)-variables. On the other hand, if seen with respect to the characteristic scale λ in the (s,y)-variables, then ϵ gets rescaled to have bigger and bigger support as $s \to \infty$ (separation of scales 1 and λ); hence, $\partial_y^m \epsilon$ decays better. Correspondingly, in the language of Fourier transform, we see that the y-frequency support of ϵ shrinks to 0 as $s \to \infty$ (red-shift effect).

Having motivated the idea of taking ∂_y's, let us carry it out concretely. We begin with a computation.

Lemma 2.5. *Let ϵ be a smooth solution to* (2.6). *Then $\partial_y^m \epsilon$ satisfies the equation*

$$\mathcal{L}_{Q;m} \partial_y^m \epsilon = \sum_{m' : 0 \le m' < m} \alpha^m_{m'} \partial_y^{m+1-m'} Q \partial_y^{m'} \epsilon + \partial_y^m f, \tag{2.7}$$

where $\alpha^m_{m'}$ are some constants and

$$\mathcal{L}_{Q;m}\psi := \partial_s \psi + ((m-1)b + 1 + (m+1)\partial_y Q)\, \psi + (by + Q)\partial_y \psi. \tag{2.8}$$

Proof. Let us take ∂_y^m of the both sides of $\mathcal{L}_Q \epsilon = f$, and keep track of all terms involving ϵ of order m and higher. We have $[\partial_y^m, y] = my\partial_y^{m-1}$ and $[\partial_y^{m+1}, Q] = (m+1)\partial_y Q \partial_y^m - \sum_{m'=0}^{m-1} \alpha^m_{m'} \partial_y^{m+1-m'} Q \partial_y^{m'}$ for some (combinatorial) constants $\alpha^m_{m'}$. Thus,

$$\begin{aligned}\partial_y^m f &= \partial_y^m (\partial_s \epsilon - (b-1)\epsilon + by\partial_y \epsilon + \partial_y (Q\epsilon)) \\ &= \partial_s \psi + ((m-1)b + 1 + (m+1)\partial_y Q)\, \psi + (by+Q)\partial_y \psi \\ &\quad - \sum_{m'=0}^{m-1} \alpha^m_{m'} \partial_y^{m+1-m'} Q \partial_y^{m'} \epsilon,\end{aligned}$$

which proves the lemma. □

An easy way to see the positive effect of taking ∂_y is to study the higher Taylor coefficients of ϵ at $y = 0$. Introduce $c_m(s) = \partial_y^m \epsilon(s, 0)$. By the preceding lemma, we have:

Lemma 2.6. *Let ϵ be a smooth solution to* (2.6). *Then $c_m(s) = \partial_y^m \epsilon(s,0)$ satisfies the equation*

$$\partial_s c_m + ((m-1)(b-1) - 1)\, c_m = \sum_{m' : 0 \le m' < m} \alpha^m_{m'} \partial_y^{m+1-m'} Q(0) c_{m'} + \partial_y^m f(s,0) \tag{2.9}$$

Note that $((m-1)(b-1) - 1)$ turns negative as soon as $m \ge 1$!

Taking ∂_y also improves the behavior of energy (i.e., L^2)-type quantities. This phenomenon can be most clearly seen at the level of $\mathcal{L}_{Q;m}$:

Lemma 2.7. *For any $\psi \in C_t^\infty \mathcal{S}_y$, we have*

$$\frac{1}{2} \partial_s \|\psi\|_{L^2}^2 \le -\beta_m \|\psi\|_{L^2}^2 + \langle \psi, \mathcal{L}_{Q;m} \psi \rangle \tag{2.10}$$

where $\langle \varphi, \psi \rangle = \int \varphi \psi \,\mathrm{d}y$ and

$$\beta_m = (m - \tfrac{3}{2})(b-1) - 1.$$

Proof. We start with $\langle \mathcal{L}_{Q;m}\psi, \psi\rangle$ and proceed as follows:

$$\begin{aligned}\langle \mathcal{L}_{Q;m}\psi, \psi\rangle &= \langle \partial_s \psi + ((m-1)b + 1 + (m+1)\partial_y Q)\,\psi + (by+Q)\partial_y\psi, \psi\rangle \\ &= \frac{1}{2}\partial_s\|\psi\|^2 + \langle ((m-1)b - m + (m+1)(\partial_y Q + 1))\psi, \psi\rangle + \langle (by+Q)\partial_y\psi, \psi\rangle \\ &= \frac{1}{2}\partial_s\|\psi\|^2 + \langle ((m-1)b - m + (m+1)(\partial_y Q + 1))\psi, \psi\rangle + \langle -(\tfrac{1}{2}b + \tfrac{1}{2}\partial_y Q)\psi, \psi\rangle \\ &= \frac{1}{2}\partial_s\|\psi\|^2 + \langle ((m-\tfrac{3}{2})(b-1) - 1 + (m+\tfrac{1}{2})(\partial_y Q + 1))\psi, \psi\rangle.\end{aligned}$$

In particular, in the third equality, we used integration by parts to write

$$\langle (by+Q)\partial_y\psi, \psi\rangle = \int (by+Q)\frac{1}{2}\partial_y\psi^2 \, \mathrm{d}y = -\int \frac{1}{2}\partial_y(by+Q)\psi^2 \, \mathrm{d}y.$$

Recall from Proposition 2.2 that $\partial_y Q \geq -1$. Thus,

$$\begin{aligned}\frac{1}{2}\partial_s\|\psi\|^2 &= -\langle ((m-\tfrac{3}{2})(b-1) - 1 + (m+\tfrac{1}{2})(\partial_y Q + 1))\psi, \psi\rangle + \langle \mathcal{L}_{Q;m}\psi, \psi\rangle \\ &\leq -\langle ((m-\tfrac{3}{2})(b-1) - 1)\psi, \psi\rangle + \langle \mathcal{L}_{Q;m}\psi, \psi\rangle,\end{aligned}$$

as desired. □

We want to prove the following result.

Theorem 2.8. *Let $Q = Q_{k,a}$ for some $k \in \mathbb{Z}_{\geq 1}$ and $a > 0$. The space*

$$X^{2k+2} = \{\psi \in \dot{H}^{2k+2}(\mathbb{R}) : \partial_y^m \psi(y=0) = 0 \text{ for } m = 0, 1, \dots, 2k+1\}$$

is an "invariant subspace" for $\mathcal{L}_Q \epsilon = f$, in the sense that if $\epsilon(0,y) = \epsilon_0(y) \in X^{2k+2}$ and $f(s,y) \in X^{2k+2}$ for every s, then $\epsilon(s,y) \in X^{2k+2}$ for all s. Moreover, we have the following "stability properties":

$$\|\partial_y^{2k+2}\epsilon(s)\|_{L^2} \lesssim e^{-\frac{1}{4k}s}\|\partial_y^{2k+2}\epsilon_0\|_{L^2} + \int_0^s e^{-\frac{1}{4k}(s-s')}\|\partial_y^{2k+2} f(s')\|_{L^2} \, \mathrm{d}s'.$$

Indeed, if $f = 0$, then the above theorem tells us that ϵ with $\epsilon(s = s_0) \in X^{2k+2}$ decay to zero as $s \to \infty$. Furthermore, ϵ decays if $\|\partial_y^{2k+2} f\|_{L^2}$ decays as well; to see this, the following computation is useful:

Lemma 2.9. *For any $\alpha, \beta \in \mathbb{R}$ and $s > 0$, we have*

$$\int_0^s e^{-\alpha(s-s')}e^{-\beta s'} \, \mathrm{d}s' \simeq \begin{cases} e^{-\beta s} & \text{if } \alpha > \beta, \\ e^{-\alpha s} & \text{if } \alpha < \beta, \\ s e^{-\alpha s} & \text{if } \alpha = \beta. \end{cases}$$

Proof. We compute

$$\int_0^{s'} e^{-\alpha(s-s')}e^{-\beta s'} \, \mathrm{d}s' = e^{-\alpha s}\int_0^{s'} e^{(\alpha-\beta)s'} \, \mathrm{d}s' \simeq e^{-\alpha s}\begin{cases} e^{(\alpha-\beta)s} & \alpha > \beta, \\ s & \alpha = \beta \\ 1 & \beta < \alpha. \end{cases}$$

□

Now we turn to the proof. What we have is almost enough, except that we need to consider the contribution of $\alpha^m_{m'}\partial_y^{m+1-m'}Q\partial_y^{m'}\epsilon$ for the $\dot{H}^{2k+2}$-norm bound. I will use an elegant approach due to Ely Sandine, which consists of commuting also with negative powers of y and using induction. We need the following computation:

Lemma 2.10. *Let ϵ be a smooth solution to (2.6) satisfying $\partial_y^{m'}\epsilon(s,0) = 0$ for $m' = 0,\dots,m-1$. Then $y^{-\ell}\partial_y^{m-\ell}\epsilon$ satisfies the equation*

$$\mathcal{L}_{Q;m,\ell}(y^{-\ell}\partial_y^{m-\ell}\epsilon) = \sum_{m'=0}^{m-\ell-1} \boldsymbol{\alpha}^{m,\ell}_{m'}(y)\partial_y^{m'}\epsilon + y^{-\ell}\partial_y^{m-\ell}f, \tag{2.11}$$

where

$$\mathcal{L}_{Q;m,\ell}\psi = \partial_s\psi + \big((m-1)b + 1 + \ell y^{-1}Q + (m-\ell+1)\partial_y Q\big)\,\psi + (b + y^{-1}Q)y\partial_y\psi,$$

$\boldsymbol{\alpha}^{m,\ell}_{m'}(y)$ is smooth and

$$|\boldsymbol{\alpha}^{m,\ell}_{m'}(y)| \lesssim \langle y\rangle^{\frac{1}{2k+1}-m-1+m'}.$$

Moreover, for any $\ell = 0,\dots,m$,

$$\frac{1}{2}\partial_s\|\psi\|^2_{L^2} \le -\beta_m\|\psi\|^2_{L^2} + \langle\psi, \mathcal{L}_{Q;m,\ell}\psi\rangle \tag{2.12}$$

where β_m is as before.

Proof. We compute

$$\begin{aligned} y^{-m}f &= y^{-m}\left(\partial_s\epsilon - (b-1)\epsilon + by\partial_y\epsilon + \epsilon\partial_y Q + Q\partial_y\epsilon\right) \\ &= \partial_s(y^{-m}\epsilon) - (b-1)(y^{-m}\epsilon) + by\partial_y(y^{-m}\epsilon) + mb(y^{-m}\epsilon) \\ &\quad + y^{-m}\epsilon\partial_y Q + \frac{Q}{y}y\partial_y(y^{-m}\epsilon) + m\frac{Q}{y}(y^{-m}\epsilon). \end{aligned}$$

For the second part, use that $y^{-1}Q \ge -1$ from Proposition 2.2. □

We also need Hardy's inequality.

Lemma 2.11 (Hardy's inequality)**.** *Let $m \ge 1$.*

(1) If f is smooth and $f(0) = \cdots = \partial_y^{m-1}f(0) = 0$, we have

$$\||y|^{-m}f\|_{L^2} \lesssim \|\partial_y^m f\|_{L^2}.$$

(2) More generally, if $f \in \dot{H}^m$ and $r > 0$, we have

$$\|(r+|y|)^{-m}f\|_{L^2} \lesssim r^{-m}\|f\|_{L^2(|y|\le r)} + \|\partial_y^m f\|_{L^2}.$$

Proof. We start with (1). It will be sufficient to work on the positive half-line. We first prove an intermediate bound. We have

$$\begin{aligned} 0 &\le \int_{y_0}^{y_1} (\partial_y f - cy^{-1}f)^2 y^{-d}\,\mathrm{d}y \\ &= \int_{y_0}^{y_1} \big((\partial_y f)^2 - 2cy^{-1}\partial_y f^2 + c^2y^{-2}f^2\big)\,y^{-d}\,\mathrm{d}y \\ &= \int_{y_0}^{y_1} \big((\partial_y f)^2 - c(d+1)y^{-2}f^2 + c^2y^{-2}f^2\big)\,\mathrm{d}y - cy^{-1}f^2(y)\big|_{y_0}^{y_1}, \end{aligned}$$

where we used integration by parts in the last equality. We choose c so that $c(d+1-c) > 0$ and $c > 0$, e.g., $c = \frac{d+1}{2}$. Then

$$c(d+1-c)\int_{y_0}^{y_1} y^{-d-2} f^2 \, \mathrm{d}y + c y_1^{-d-1} f^2(y_1) \leq \int_{y_0}^{y_1} y^{-d} (\partial_y f)^2 \, \mathrm{d}y + c y_0^{-d-1} f^2(y_0).$$

Provided that f vanishes to a sufficiently high order at $y = 0$ so that $\lim_{y_0 \to 0+} y_0^{-d-1} f^2(y_0). = 0$, we obtain

$$c(d+1-c)\int_0^\infty y^{-d-2} f^2 \, \mathrm{d}y \leq \int_0^\infty y^{-d} (\partial_y f)^2 \, \mathrm{d}y.$$

Indeed, when $m = 1$, the desired inequality follows by taking $d = 0$ and observing that $y_0^{-1} f^2(y_0) \to 0$ if f is smooth and $f(0) = 0$. For higher m, we concatenate the above inequality (with different choices of f and d) and estimate

$$\int_0^\infty y^{-2m} f^2 \, \mathrm{d}y \lesssim \int_0^\infty y^{-2m+2} (\partial_y f)^2 \, \mathrm{d}y \lesssim \int_0^\infty y^{-2m+4} (\partial_y^2 f)^2 \, \mathrm{d}y \lesssim \cdots \lesssim \int_0^\infty (\partial_y^m f)^2 \, \mathrm{d}y.$$

To prove (2), we first notice that, by scaling, we may set $r = 1$ without loss of generality. Assuming this, we simply split $f = \chi f + (1-\chi) f$, where χ is a smooth bump function supported in $\{|y| < 1\}$. We then apply (1) to $(1-\chi)f$, and use $\|f\|_{L^2(|y| \leq 1)}$ to bound the contribution of χf. □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. To close the $\|\cdot\|_{\dot{H}^{2k+2}}$ bound for ϵ, start with $\||y|^{-2k-2}\epsilon\|_{L^2}$, which goes through. Then work with $\|y^{-2k-1}\partial_y \epsilon\|_{L^2}$, where the extra term on the RHS can be dealt with using $\||y|^{-2k-2}\epsilon\|_{L^2}$. Continue inductively.

This procedure gives us the desired theorem with $\||y|^{-m} f(s')\|_{L^2} + \cdots \|\partial_y^m f(s')\|_{L^2}$ on the RHS. Since $f(s')$ satisfies the vanishing conditions at $y = 0$, we can use Hardy's inequality to bound this from above by $\|\partial_y^m f(s')\|_{L^2}$. □

Remark 2.12. Observe that the simple proof of (1) goes through since f is assumed to also obey the vanishing condition at $y = 0$ up to the $(2k+1)$-th derivative (i.e., $f(s, \cdot) \in X$ for all s). In the nonlinear analysis below, however, we would need to relax this assumption on f while assuming only that $\epsilon(s, \cdot) \in X$ for all s.

Remark 2.13. In fact, the eigenfunctions can be computed, and we can obtain a direct sum decomposition of the space $\dot{H}^{2k+2}(\mathbb{R})$ into invariant subspaces: unstable, center and stable; see the appendix below. However, as we shall see, such a refined decomposition is not needed for nonlinear stability results!

Appendix: Identification of unstable and center linear subspaces. Introduce

$$H_Q \psi := (b-1)\psi - b y \partial_y \psi - \partial_y Q \psi - Q \partial_y \psi.$$

We wish to solve the eigenvalue equation

$$H_Q \psi = \nu \psi.$$

For concreteness, fix $Q = Q_{k,1}$. The trick for solving this equation is to use $y = -Q - Q^{2k+1}$ and make the change of variables from y to Q [?]. Then

$$\psi\left((\tfrac{1}{2k} - \nu)(1 + (2k+1)Q^{2k}) + 1 \right) = \tfrac{\mathrm{d}\psi}{\mathrm{d}Q} \left(\tfrac{1}{2k} Q + \tfrac{2k+1}{2k} Q^{2k+1} \right).$$

Integrating this equation, we obtain (formally and up to normalization)

$$\psi = \frac{Q^{2k+1-2k\nu}}{1+(2k+1)Q^{2k}} = \frac{1}{2k+2-2k\nu}\partial_y Q^{2k+2-2k\nu}.$$

Observe that ψ is smooth at the origin if and only if $2k+1-2k\nu$ is a nonnegative integer. We therefore define the numbers

$$\nu_m = \frac{2k+1-m}{2k}, \quad m = 0, 1, 2, \dots.$$

and smooth eigenfunctions

$$\psi_m = \frac{Q^m}{1+(2k+1)Q^{2k}}.$$

Note that these numbers agree with the rates for c_m that we computed. We may define the unstable and center linear invariant subspaces of $\dot{H}^s$ as

$$X_u^s = \mathrm{span}(\psi_0, \dots, \psi_{2k}), \quad X_c^s = \mathrm{span}(\psi_{2k+1}).$$

Exercise: Prove the direct sum decomposition

$$\dot{H}^{2k+2} = X_u^{2k+2} \oplus X_c^{2k+2} \oplus X^{2k+2},$$

where X^{2k+2} is as before.

Remark 2.14. It is also instructive to carry out a similar computation for the adjoint H_Q^*. We have

$$H_Q^*\varphi = (b-1)\varphi + b\partial_y(y\varphi) - \partial_y Q\varphi + \partial_y(Q\varphi) = (2b-1)\varphi + by\partial_y\varphi + Q\partial_y\varphi.$$

If φ solves $H_Q^*\varphi = \overline{\nu}\varphi$, we have

$$\begin{aligned}(-2b+1+\overline{\nu})\varphi &= (by+Q)\partial_y\varphi = (by+Q)\partial_y Q\partial_Q\varphi \\ &= \frac{(b-1)Q+bQ^{2k+1}}{1+(2k+1)Q^{2k}}\partial_Q\varphi = \frac{Q}{2k}\partial_Q\varphi.\end{aligned}$$

Thus, formally and up to normalization,

$$\varphi = Q^{-2k-2+2k\overline{\nu}}.$$

2.3. **Perturbation of self-similar solutions to Burgers II: Stability modulo finitely many unstable directions.**

2.3.1. *Formulation of stability results.* Explain further:

- Recall: Stable manifold theorem from ODEs. We wish to do something similar.
- What do we need to worry about when adding nonlinearity? That our function space admits product estimate. This will be okay. The key is to control $\partial_y\epsilon \in L^\infty$.
- What do we need to worry about when modifying the equation (so that Q is not the self-similar solution anymore)? Contribution of $\Gamma Q^\sharp$, where $Q^\sharp = \frac{\lambda(t)}{T-t}Q(\frac{x}{\lambda(t)})$. Compare strengths of each term in the equation:

$$\begin{aligned}\partial_t Q^\sharp &\sim (T-t)^{-1}(T-t)^{b-1} = (T-t)^{b-2}, \\ Q^\sharp\partial_x Q^\sharp &\sim (T-t)^{2(b-1)}(T-t)^{-b} = (T-t)^{b-2}, \\ \Gamma Q^\sharp &\sim (T-t)^{b-1}(T-t)^{-\alpha b} = (T-t)^{(1-\alpha)b-1}.\end{aligned}$$

We expect the last term to be smaller than the first two if $(1-\alpha)b - 1 > b - 2$, or equivalently, $1 > \alpha b$. Indeed, this turns out to be a correct sufficient condition; see the theorem below.

- References:
  - (compressible Euler) Buckmaster–Shkoller–Vicol ($k = 1$), Buckmaster–Iyer ($k > 1$)
  - (Whitham equation, fKdV, Burgers–Hilbert) $\Gamma = \sqrt{\tanh |\partial_x| |\partial_x|}\partial_x$: V. Hur–L. Tao, V. Hur (different methods); $\Gamma = |\partial_x|^{-1}\partial_x$: R. Yang ($k = 1, 2$)
  - (fractal Burgers) Alibaud–Drouniot–Vovelle (very different method), Chickering–Morano-Vasquez–Pandya (similar method, more L^∞-based, $k = 1$)
  - (Burgers with transverse viscosity) Collot–Ghoul–Masmoudi.

Let χ be a smooth cutoff adapted to $(-1, 1)$; assume also that χ is even, $0 \leq \chi \leq 1$, and $\operatorname{supp} \chi \subseteq (-2, 2)$. We also use the shorthand $Q_k = Q_{k,1}$ (i.e., $y = -Q_k - Q_k^{2k+1}$).

Theorem 2.15. *Consider*

$$u_t + uu_x + \Gamma u = 0, \tag{2.13}$$

where either $\widehat{\Gamma f} = |\xi|^{\alpha-1} i\xi f$ *(fKdV) or* $|\xi|^\alpha f$ *(fractal Burgers) with* $0 \leq \alpha < 1$. *Given* $k \in \mathbb{Z}_{\geq 1}$, *if*

$$\alpha < \frac{2k}{2k+1} \quad \textit{(equivalently, } b\alpha < 1\textit{)},$$

then there exists $\delta_0 > 0$ *and* $\gamma_0 > 0$ *such that the following holds. For any* $s_0 > \delta_0^{-1}$ *and* $\epsilon_0 \in H^{2k+1}(\mathbb{R})$ *with* $\|\epsilon_0\|_{H^{2k+3}} < e^{-4\gamma_0 s_0}$ *and* $\epsilon_0(0) = \cdots = \partial_x^{2k}\epsilon_0 = 0$, *there exists* $\mathbf{c}_0 = (c_{0;0}, \ldots, c_{0;2k}) \in \{\mathbf{c} \in \mathbb{R}^{2k+1} : |\mathbf{c}| < e^{-\gamma_0 s_0}\}$ *such that the initial data*

$$u_0 = e^{-(b-1)s_0} \left(\chi(x) Q_k(s_0, e^{bs_0}x) + \chi(e^{bs_0}x) \sum_{m=0}^{2k} c_{0;m} e^{mbs_0} x^m + \epsilon_0(e^{bs}x) \right)$$

lead to a well-posed H^{2k+3} *solution* u *that blows up at time* $T = e^{-s_0}$. *In self-similar coordinates* (s, y, U) *with* $(t = T - e^{-s}, x = (T-t)^b y, u = (T-t)^{b-1}U)$, *we have the asymptotics*

$$\left\|\partial_y (U - \chi((T-t)^b y) Q_k)\right\|_{L^2_y} + \left\|\partial_y^{2k+3} \left(U - \chi((T-t)^b y) Q_k\right)\right\|_{L^2_y} \to 0 \textit{ as } s \to \infty.$$

Remark 2.16. In the original (t, x) variables, the asymptotics become

$$\begin{aligned}
&(T-t)^{-\frac{1}{2}b+1} \left\|\partial_x (u - \chi(x)(T-t)^{b-1} Q_k((T-t)^{-b}x))\right\|_{L^2_x} \\
&+ (T-t)^{(2k+3)b - \frac{3}{2}b + 1} \left\|\partial_x^{2k+3} \left(u - \chi(x)(T-t)^{b-1} Q_k((T-t)^{-b}x)\right)\right\|_{L^2_x} \to 0
\end{aligned}$$

as $t \to T$. This control is sufficient to establish blow-up. However, the above control does not give a uniform control on $\|u\|_{L^\infty}$ as $t \to T$.

On the other hand, we know a lot more about the regularity of u near the singular time in the base case of the inviscid Burgers equation. Indeed, in that case we expect u to stay bounded for all times until T by the maximum principle. Moreover, an explicit computation shows that the exact self-similar solution $(T-t)^{b-1}Q((T-t)^{-b}x)$ stays uniformly bounded in $C^{\frac{1}{2k+1}}$.

In fact, these conclusions can be extended to (2.13) by using an adequate weighted L^2-Sobolev space; we shall discuss this in an appendix.

Aside: Singularity formation for defocusing energy supercritical NLS. TO DO: Fill in details

$$i\partial_t u + \Delta u - |u|^{p-1}u = 0.$$

Madelung transform:

$$u(t,x) = \rho(t,x)e^{i\phi(t,x)}$$

and define

$$\mathbf{u} = \nabla\phi.$$

Then

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,$$
$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla(\rho^{p-1} + P_Q),$$

where $P_Q = -\frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$. If we omit P_Q (quantum pressure), then in view of

$$\nabla\rho^{p-1} = \frac{p-1}{p}\rho^{-1}\nabla\rho^p,$$

the resulting equation is precisely the (irrotational) compressible Euler equations with equation of state $P = \frac{p-1}{p}\rho^p$ (ideal polytropic gas)!

TO DO: Cite the rates b of self-similar solutions to compressible Euler from [?]. Make the heuristic computation in the present case, and see in which case the quantum pressure P_Q is weaker; compare with [?]. Also look at [?] (dissipative perturbation of compressible Euler).

2.3.2. *Preparations.* We now collect some preparations for the proof of Theorem 2.15.
Perturbed equation in self-similar variables. We compute

$$\partial_s U - (b-1)U + by\partial_y U + U\partial_y U + e^{-(1-b\alpha)s}\Gamma U = 0. \tag{2.14}$$

Profile. We shall write $Q = Q_k$. We begin by defining the profile.

$$P(s,y) = \chi(e^{-bs}y)Q(y)$$

Note that $(\partial_s + by\partial_y)\chi(e^{-bs}y) = 0$. Hence,

$$\partial_s P - (b-1)P + (by+P)\partial_y P = -\chi(e^{-bs}y)Q\partial_y\big((1-\chi(e^{-bs}y))Q\big) =: \Psi_B$$

Indeed,

$$\begin{aligned}\partial_s P - (b-1)P + by\partial_y P + P\partial_y P &= \chi(e^{-bs}y)(\partial_s Q - (b-1)Q + by\partial_y Q) + P\partial_y P\\ &= -\chi(e^{-bs}y)Q\partial_y Q + \chi(e^{-bs}y)Q\partial_y P.\end{aligned}$$

Lemma 2.17. *With the profile P and the Burgers profile error Ψ_B defined as above, we have*

$$\Psi_B = 0 \textit{ for all } s>0 \textit{ in a neighborhood of } y=0,$$
$$\|\partial_y^m\Psi_B\|_{L^2} \lesssim_m e^{(-m-\frac{1}{2}+\frac{2}{2k+1})bs}.$$

Proof. The first property is clear from the support property of Ψ. The second property follows from the asymptotics $\partial_y^m Q \approx -y^{\frac{1}{2k+1}-m}$ for $y \gg 1$. □

With respect to the perturbed equation (2.13), we defined the full profile error Ψ as

$$\partial_s P - (b-1)P + (by+P)\partial_y P + e^{-(1-b\alpha)s}\Gamma P = \Psi_B + e^{-(1-b\alpha)s}\Gamma P =: \Psi.$$

Lemma 2.18. *Let $0 \leq \alpha < \frac{1}{b}$. With the profile P defined as above, we have*

$$e^{-(1-b\alpha)s}\|\partial_y^m \Gamma P\|_{L^2} \lesssim_{m,\delta} e^{-(1-b\alpha)s} \max\{e^{(\frac{1}{2}+\frac{1}{2k+1}-m-\alpha+\delta)bs}, 1\},$$
$$e^{-(1-b\alpha)s}|\partial_y^m \Gamma P(0)| \lesssim_{m,\delta} e^{-(1-b\alpha)s} \max\{e^{(\frac{1}{2k+1}-m-\alpha+\delta)bs}, 1\}.$$

In the first estimate, we may take $\delta = 0$ if $\frac{1}{2} + \frac{1}{2k+1} - m - \alpha$ is not an integer. Similarly, in the second estimate, we may take $\delta = 0$ if $\frac{1}{2k+1} - m - \alpha$ is not an integer.

Proof. Given $R \in 2^{\mathbb{Z}_{\geq 0}}$, let $\chi_1(y) = \chi(y)$ and $\chi_R(y) = \chi(R^{-1}y) - \chi(2R^{-1}y)$, so that $\{\chi_R\}$ forms a smooth partition of unity. By the asymptotics of Q, we have

$$\|\partial_y^m(\chi_R(y)P)\|_{L^2} \lesssim R^{-m+\frac{1}{2}+\frac{1}{2k+1}},$$

By interpolation and Sobolev,

$$\|\partial_y^m \Gamma(\chi_R(y)P)\|_{L^2} \lesssim R^{-m-\alpha+\frac{1}{2}+\frac{1}{2k+1}}, \quad \|\partial_y^m \Gamma(\chi_R(y)P)\|_{L^\infty} \lesssim R^{-m-\alpha+\frac{1}{2k+1}}.$$

Note moreover that $\chi_R P = 0$ if $R \gg 2^{bs}$ by the support properties of P. The desired bounds follow by summation in $R \in 2^{\mathbb{Z}_{\geq 0}}$. □

Corollary 2.19. *For $0 \leq \alpha < \frac{1}{b}$ and $M > 0$, there exists $\mu > 0$ such that, for any $0 \leq m \leq M$,*

$$e^{-(1-b\alpha)s}\|\partial_y^m \Gamma P\|_{L^2} \lesssim e^{-\mu s},$$
$$e^{-(1-b\alpha)s}|\partial_y^m \Gamma P(0)| \lesssim e^{-\mu s},$$
$$\|\langle y\rangle^m \partial_y^{1+m}(P - Q)\|_{L^\infty} \lesssim e^{-\mu s},$$
$$\|\partial_y^{1+m}\Psi_B\|_{L^2} \lesssim_m e^{-\mu s}.$$

Furthermore,

$$\|\Psi_B\|_{L^2} \lesssim_m e^{\frac{3-2k}{4k}s}.$$

Initial data and decomposition of the solution. In the self-similar variables, we look for a solution $U(s, y)$ of the form

$$U = P(s, y) + M(s, y) + \epsilon(s, y)$$

where $M(s, y)$ is given as

$$M(s, y) := \chi(y) \sum_{m=0}^{2k+1} c_m(s) y^m,$$

the remainder ϵ satisfies the vanishing conditions

$$\epsilon(s, 0) = \cdots = \partial_y^{2k+1}\epsilon(s, 0) = 0,$$

with the following initial data at $s = s_0$:

$$U(s_0, y) = P(s_0, y) + \sum_{m=0}^{2k+1} c_{0;m} y^m \chi(y) + \epsilon_0(y),$$
$$c_m(s_0) = c_{0;m} \quad \text{for } m = 0, 1, \ldots, 2k+1.$$

Indeed, the RHS is exactly the initial data in Theorem 2.15 expressed in the self-similar coordinates. We will also use the notation

$$W(s, y) = U(s, y) - P(s, y) = M(s, y) + \epsilon(s, y).$$

2.3.3. *Main bootstrap argument.*

Proposition 2.20 (Main bootstrap argument)**.** *There exist* $\gamma_0, \gamma_1, \gamma_2 > 0$ *and* $\delta_0 > 0$ *such that*

$$0 < \gamma_2 < \gamma_1 < \gamma_0 < \tfrac{1}{2} \min\{2 - b, 1 - b\alpha, \mu\}, \tag{2.15}$$

where $\mu > 0$ *is as in Corollary 2.19, and the following holds. Assume that* U *exists as a* H^{2k+3} *well-posed solution on* $[s_0, s_1]$*, satisfies the* trapping assumption

$$\|\mathbf{c}(s)\| \leq e^{-\gamma_0 s} \text{ for } s \in [s_0, s_1], \tag{2.16}$$

and the bootstrap assumptions

$$\|\partial_y^{2k+2} \epsilon(s)\|_{L^2} \leq 2e^{-\gamma_1 s} \text{ for } s \in [s_0, s_1], \tag{2.17}$$

$$\|\partial_y^{2k+3} \epsilon(s)\|_{L^2} + \|\partial_y \epsilon(s)\|_{L^2} \leq 2e^{-\gamma_2 s} \text{ for } s \in [s_0, s_1], \tag{2.18}$$

$$e^{-(1-b\alpha)s} \|\Gamma \epsilon(s)\|_{L^\infty(|y| \leq 1)} \leq 2e^{-2\gamma_0 s} \text{ for } s \in [s_0, s_1]. \tag{2.19}$$

For $s_0 > \delta_0^{-1}$, $\|\epsilon_0\|_{H^{2k+3}} < e^{-4\gamma_0 s_0}$*, we have the following improvement of the bootstrap assumptions:*

$$\|\partial_y^{2k+2} \epsilon(s)\|_{L^2} \leq e^{-\gamma_1 s} \text{ for } s \in [s_0, s_1], \tag{2.20}$$

$$\|\partial_y^{2k+3} \epsilon(s)\|_{L^2} + \|\partial_y \epsilon(s)\|_{L^2} \leq e^{-\gamma_2 s} \text{ for } s \in [s_0, s_1], \tag{2.21}$$

$$e^{-(1-b\alpha)s} \|\Gamma \epsilon(s)\|_{L^\infty(|y| \leq 1)} \leq e^{-2\gamma_0 (s - s_0)} e^{-2\gamma_0 s_0} \text{ for } s \in [s_0, s_1]. \tag{2.22}$$

We emphasize the distinction between the bootstrap assumptions – which are improved – and the trapping assumption – which is not improved! As we will see soon, the role of Proposition 2.20 is to reduce the proof of global-in-s existence of U (along with global bounds) to just verifying (2.16) concerning a finite-dimensional vector $\mathbf{c}$. We will then employ a topological argument to find an adequate initial data for $\mathbf{c}$ that guarantees (2.16) for all times.

2.3.4. *Proof of Theorem 2.15 assuming Proposition 2.20.* To deduce Theorem 2.15 assuming Proposition 2.20, we will follow the standard *shooting argument*, which goes back at least to [**?**]. We first introduce two definitions.

Definition 2.21. A solution U is *trapped* on $[s_0, s_1]$ if all hypotheses of Proposition 2.20 hold.

Definition 2.22. Given an initial data U_0 of the form as above, we define the (first) *exit time* $s_{\text{exit}} = s_{\text{exit}}(U_0)$ to be the smallest s_1 such that $|\mathbf{c}(s_1)| = e^{-\gamma_0 s_1}$ but $|\mathbf{c}(s)| < e^{-\gamma_0 s}$ for all $s \in [s_0, s_1)$ (here, $\mathbf{c}(s)$ is defined from the solution U arising from U_0). If $|\mathbf{c}(s_0)| = e^{-\gamma_0 s_0}$, then we just set $s_{\text{exit}} = s_0$.

As a consequence of the main bootstrap argument (Proposition 2.20 and continuous induction), it follows that if $s_0 > \delta_0^{-1}$ and $\|\epsilon_0\|_{H^{2k+3}} < e^{-4\gamma_0 s_0}$, then $s_{\text{exit}}(U_0)$ is well-defined and U is trapped on $[s_0, s_{\text{exit}}]$. (**Exercise:** Prove this assertion by a continuous induction on s.) In particular, the improved bootstrap bounds (2.20)–(2.21) hold for all $s \in [s_0, s_{\text{exit}}]$. If we could show that $s_{\text{exit}} = +\infty$ for some choice of $\mathbf{c}_0 \in B(0; e^{-\gamma_0 s_0})$, then the proof of Theorem 2.15 would be complete.

In order to prove the existence of $\mathbf{c}_0$ such that $s_{\text{exit}} = +\infty$, we will assume otherwise and derive a contradiction using a topological argument (shooting argument).

The key information we need on the unstable coefficients $\mathbf{c}(s)$ is the following "outgoing property" near the boundary.

Lemma 2.23 (Outgoing property of unstable ODEs)**.** *There exists $c_0 > 0$ such that, for s_0 sufficiently large, the following holds. Let U be a trapped solution on $[s_0, s_1]$. For any $s \in [s_0, s_1]$ such that*

$$\frac{1}{2}e^{-\gamma_0 s} < |\mathbf{c}(s)| \leq e^{-\gamma_0 s},$$

we have

$$\partial_s(e^{2\gamma_0 s}|\mathbf{c}(s)|^2) \geq 2c_0 e^{2\gamma_0 s}|\mathbf{c}(s)|^2$$

In words, if U is near the boundary of the trapped region, then it definitely will exit soon (with also a quantitative bound on the rate). We defer the proof until next time. For now, let us proceed with the shooting argument.

Proof of Theorem 2.15 assuming Proposition 2.20 (shooting argument). For each $|\mathbf{c}_0| \leq e^{-\gamma_0 s_0}$, denote by $U_{\mathbf{c}_0}(s,y)$ the solution with initial data at $s = s_0$ induced by $\mathbf{c}_0$ and ϵ_0. For the purpose of contradiction, suppose that for all $\mathbf{c}_0 \in B(0; e^{-\gamma_0 s_0})$, $s_{\mathrm{exit}}(U_{\mathbf{c}_0}) < +\infty$.

Claim: The map $H : B(0; e^{-\gamma_0 s_0}) \to \partial B(0;1)$, $\mathbf{c}_0 \mapsto e^{-\gamma_0 s_{\mathrm{exit}}(\mathbf{c}_0)}\mathbf{c}_{\mathbf{c}_0}(s_{\mathrm{exit}}(\mathbf{c}_0))$ is continuous.

Assuming the claim, the proof is easy to conclude. Note that the map $\mathbf{x} \mapsto H(e^{-\gamma_0 s_0}\mathbf{x})$ defines a continuous map from $B(0;1)$ to $\partial B(0;1)$, which is also the identity map on the boundary (obvious by definition). But such a map (retraction of $B(0;1)$ to the boundary) does not exist (cf. the proof of Brouwer's fixed point theorem), which completes the proof.

We now turn to the proof of the claim. We divide the proof into two subclaims:

Subclaim 1: $(s, \mathbf{c}_0) \mapsto \mathbf{c}_{\mathbf{c}_0}(s)$ is continuous.

Idea: Transfer bootstrap assumptions on $[s, s_{\mathrm{exit}}]$ to a uniform H^{2k+3} bound on $[t, t(s_{\mathrm{exit}})]$ in the (t,x) coordinates. Using standard local well-posedness theory in the (t,x)-coordinates (but transforming it back to the (s,y) coordinates), we will have the Lipschitz continuity of $(s, \mathbf{c}_0) \mapsto U_{\mathbf{c}_0}(s) \in H^{2k+2}$ for $s \in [s_0, s_{\mathrm{exit}}]$. In particular, by the Sobolev embedding, $(s, \mathbf{c}_0) \mapsto \mathbf{c}_{\mathbf{c}_0}(s)$ is continuous for $s \in [s_0, s_{\mathrm{exit}}(U_0)]$.

Subclaim 2: $\mathbf{c}_0 \mapsto s_{\mathrm{exit}}(\mathbf{c}_0)$ is continuous.

It remains to show that $\mathbf{c}_0 \mapsto s_{\mathrm{exit}}(\mathbf{c}_0)$ is continuous; this part is the crux of the proof. By Lemma 2.23, for any $\epsilon > 0$, if $U_{\mathbf{c}_0}$ is trapped on $[s_0, s_1]$ and $e^{\gamma_0 s_1}|\mathbf{c}_{\mathbf{c}_0}(s_1)| > e^{-c_0\epsilon}$, then $|s_{\mathrm{exit}}(\mathbf{c}_0) - s_1| < \epsilon$.

When $\mathbf{c}_0 \notin \partial B(0, e^{-\gamma_0 s_0})$, $s_{\mathrm{exit}}(\mathbf{c}_0) > s_0$. Clearly, there exists s_1 such that $s_0 < s_1 < s_{\mathrm{exit}}(\mathbf{c}_0)$ such that $e^{-c_0\epsilon} < e^{\gamma_0 s_1}|\mathbf{c}_{\mathbf{c}_0}(s_1)| < 1$. By Subclaim 1, for $\mathbf{c}_0'$ sufficiently close to $\mathbf{c}_0$, we will have $e^{-c_0\epsilon} < e^{\gamma_0 s_1}|\mathbf{c}_{\mathbf{c}_0'}(s_1)| < 1$ as well, which implies that $|s_{\mathrm{exit}}(\mathbf{c}_0) - s_{\mathrm{exit}}(\mathbf{c}_0')| < |s_{\mathrm{exit}}(\mathbf{c}_0) - s_1| + |s_{\mathrm{exit}}(\mathbf{c}_0') - s_1| < 2\epsilon$; this establishes the desired continuity.

Now the claim follows, by composition, from the two subclaims. □

2.3.5. *Closing the bootstrap assumptions.* We now prove Proposition 2.20 (Main Bootstrap) and Lemma 2.23 (Outgoing Property).

Equations of motion. We begin by deriving the equations of motion. Using the decomposition $U = P + W$, we have

$$\boxed{\mathcal{L}_P W + e^{-(1-b\alpha)}\Gamma W = -\Psi - W\partial_y W.} \tag{2.23}$$

where $\mathcal{L}_P$ is the linearized (inviscid) Burgers operator around P:

$$\mathcal{L}_P = \partial_s - (b-1) + by\partial_y + (\partial_y P) + P\partial_y = \mathcal{L}_Q + (\partial_y(P-Q)) + (P-Q)\partial_y.$$

Further decomposing $W = \chi \sum_{m=0}^{2k+1} c_m(s) y^m + \epsilon$, and putting all the terms involving $P - Q$ and ϵ on the RHS, we arrive at the following equation for ϵ:

$$\boxed{\begin{aligned}\mathcal{L}_Q\epsilon = &-\Psi - \mathcal{L}_P M - M\partial_y M - e^{-(1-b\alpha)s}\Gamma M \\ &-(\partial_y(P-Q))\epsilon - (P-Q)\partial_y\epsilon - e^{-(1-b\alpha)s}\Gamma\epsilon - M\partial_y\epsilon - \epsilon\partial_y M - \epsilon\partial_y\epsilon.\end{aligned}} \tag{2.24}$$

At this point, we remind the reader the key condition for ϵ (i.e., ϵ is in the linear stable subspace for $\mathcal{L}_Q$):

$$\boxed{\epsilon(s,0) = \cdots = \partial_y^{2k+1}\epsilon(s,0) = 0.} \tag{2.25}$$

Next, to derive the equations for $\mathbf{c}$, we evaluate (2.23) and its derivatives up to order $2k+1$ at $y = 0$.

$$\boxed{\begin{aligned}\partial_s c_m + ((m-1)(b-1)-1)\, c_m = &\sum_{m':0\leq m'<m} \alpha^m_{m'} \partial_y^{m+1-m'} Q(0) c_{m'} \\ &-(m!)e^{-(1-b\alpha)s}\partial_y^m \Gamma(P+M+\epsilon)(0) \\ &-\frac{m!}{2}\partial_y^{m+1}((M+\epsilon)^2)(0).\end{aligned}} \tag{2.26}$$

Indeed, this equation follows by applying Lemma 2.6 to

$$\mathcal{L}_Q W + e^{-(1-b\alpha)}\Gamma W = -\Psi - (\partial_y(P-Q))W - (P-Q)\partial_y W - W\partial_y W,$$

where $-(\partial_y(P-Q))W - (P-Q)\partial_y W$ vanish near 0 and $\Psi(0) = e^{-(1-b\alpha)s}\Gamma P(0)$. At this point, it is useful to make the following observation:

Lemma 2.24. *We have*[1]

$$\begin{aligned}|\partial_y^n(\mathcal{L}_P M + M\partial_y M)| \lesssim &\left(|\mathbf{c}| + |\mathbf{c}|^2\right)\min\{|y|^{2k+2-n}, 1\}\mathbb{1}_{|y|\leq 2} \\ &+ e^{-(1-b\alpha)s}\sum_{m=0}^{2k+1} |\partial_y^m \Gamma(P+\epsilon)(0)| \min\{|y|^{m-n}, 1\}\mathbb{1}_{|y|\leq 2}.\end{aligned}$$

Proof. The key observation is that (2.26) ensures that

$$\begin{aligned}&\mathcal{L}_Q\left(\sum_{m=0}^{2k+1} c_m(s)y^m\right) + \left(\sum_{m=0}^{2k+1} c_m(s)y^m\right)\partial_y\left(\sum_{m=0}^{2k+1} c_m(s)y^m\right) \\ &= O\left(\left(|\mathbf{c}| + |\mathbf{c}|^2\right)\right)y^{2k+2} + e^{-(1-b\alpha)s}\sum_{m=0}^{2k+1} O\left(e^{-(1-b\alpha)s}|\partial_y^m\Gamma(P+\epsilon)(0)|\right)y^m,\end{aligned}$$

where the factors of the form $O(\cdot)$ are all functions of s. To complete the proof, also incorporate the cutoff χ. □

---

[1]We remark that, in fact, the weaker version with the min's below by 1 follows easily by simply using (2.26) to eliminate $\partial_s c_m$, and it is only this weaker version that is needed in our argument.

A robust $\dot{H}^{2k+2}$ energy estimate. We begin with $\mathcal{L}_Q \epsilon = f$. Our main result is the following more robust version of the $\dot{H}^{2k+2}$-energy estimate (Theorem 2.8), where the dependence on the vanishing conditions is less rigid.

Proposition 2.25. *Let ϵ, f solve $\mathcal{L}_Q \epsilon = f$, where $\epsilon(s,0) = \cdots = \partial_y^{2k+1}\epsilon(s,0) = 0$ for all $s \in [s_0, s_1]$. For $c_1 > 0$ sufficiently small, we have*

$$\|\partial_y^{2k+2}\epsilon(s)\|_{L^2}^2 \lesssim e^{-2c_1(s-s_0)} \left(\|\partial_y^{2k+2}\epsilon(s_0)\|_{L^2}^2 + \|\langle y\rangle^{-2k-2}\epsilon(s_0)\|_{L^2}^2\right) + \int_{s_0}^{s} e^{-2c_1(s-s')} \left(|\langle \partial_y^{2k+2} f, \partial_y^{2k+2}\epsilon\rangle(s')| + \|\langle y\rangle^{-2k-2} f(s')\|_{L^2}^2\right) \,\mathrm{d}s'.$$

As we shall see, this energy estimate is at the heart of our proof of Proposition 2.20 (Main Bootstrap). It is a technical improvement over Theorem 2.8, where (i) the dependence on $\partial_y^{2k+2} f$ is made more explicit so that we can perform integration by parts, and (ii) we do not require f to satisfy the vanishing conditions. The basic ideas are similar to those of Theorem 2.8, namely, commuting with ∂_y^{2k+2} and using an appropriate weight to take care of the lower order terms. Since we do *not* want to impose the vanishing conditions for f at $y = 0$, however, we need to be a bit more careful in the designing of the weight.

We shall divide the proof into smaller pieces. We begin by recording an energy inequality that we will use for $\partial_y^{2k+2}\epsilon$, with a little more attention paid to the lower order terms:

Lemma 2.26. *Let $\psi \in C_t^\infty \mathcal{S}_x$. Then for any $m \geq 0$ and $\eta > 0$, there exist $C = C(m,\eta)$ and $R = R(m,\eta)$ such that*

$$\frac{1}{2}\partial_s \|\partial_y^m \psi\|_{L^2}^2 \leq -(\beta_m - \eta)\|\partial_y^m \psi\|_{L^2}^2 + C\|\langle y\rangle^{-m-1+\frac{1}{2k+1}}\psi\|_{L^2(|y|\leq R)}^2 + \langle \partial_y^m \mathcal{L}_Q \psi, \partial_y^m \psi\rangle.$$

where $\beta_m = (m - \frac{3}{2})(b-1) - 1$.

An ingredient for the proof is the following interpolation lemma:

Lemma 2.27. *For any $0 \leq m' \leq m$ and $a \in \mathbb{R}$,*

$$\|\langle y\rangle^{a-j}\partial_y^j f\|_{L^2} \lesssim \|\langle y\rangle^a f\|_{L^2}^{1-\frac{j}{2k+2}} \|\partial_y^{2k+2} f\|_{L^2}^{\frac{j}{2k+2}}$$

Proof. We decompose the LHS into $\|\cdots\|_{L^2(|y|\leq 1)} + \sum_{\ell=0}^\infty \|\cdots\|_{L^2(2^\ell \leq |y| \leq 2^{\ell+1})}$. Then by rescaling, the estimate reduces to the unit scale bounds

$$\|\partial_y^j f\|_{L^2(I)} \leq \|f\|_{L^2(I)}^{1-\frac{j}{2k+2}} \|\partial_y^{2k+2} f\|_{L^2(I)}^{\frac{j}{2k+2}}$$

for $I = \{|y| \leq 1\}$ or $\{1 \leq |y| \leq 2\}$. This follows from the standard Gagliardo–Nirenberg inequality. □

We are now ready to prove Lemma 2.26.

Proof of Lemma 2.26. By Lemma 2.5,

$$\partial_y^m \mathcal{L}_Q \psi = \mathcal{L}_{Q,m}\partial_y^m \psi - \sum_{m'=0}^{m-1} \alpha_{m'}^m \partial_y^{m+1-m'} Q \partial_y^{m'}\psi.$$

By Lemma 2.7, it follows that

$$\frac{1}{2}\partial_s \|\partial_y^m \psi\|_{L^2}^2 \leq -\beta_m \|\partial_y^m \psi\|_{L^2}^2 + \langle \psi, \partial_y^m \mathcal{L}_Q \psi\rangle + \sum_{m'=0}^{m-1} |\alpha_{m'}^m| \left|\partial_y^{m+1-m'} Q \partial_y^{m'}\psi, \partial_y^m \psi\right|$$

$$\leq -\beta_m \|\partial_y^m \psi\|_{L^2}^2 + \langle \psi, \partial_y^m \mathcal{L}_Q \psi \rangle + \sum_{m'=0}^{m-1} |\alpha_{m'}^m| \|\langle y \rangle^{-m-1+m'+\frac{1}{2k+1}} \partial_y^{m'} \psi\|_{L^2} \|\partial_y^m \psi\|_{L^2},$$

where we used $|\partial_y^{m+1-m'} Q| \lesssim \langle y \rangle^{-m-1+m'+\frac{1}{2k+1}}$ on the last line. By the interpolation lemma (Lemma 2.27) and Young's inequality,

$$\begin{aligned}|\alpha_{m'}^m| \|\langle y \rangle^{-m-1+m'+\frac{1}{2k+1}} \partial_y^{m'} \psi\|_{L^2} \|\partial_y^m \psi\|_{L^2} &\leq C \|\langle y \rangle^{-m-1+\frac{1}{2k+1}} \psi\|_{L^2}^{1-\frac{m'}{m}} \|\partial_y^m \psi\|_{L^2}^{1+\frac{m'}{m}} \\ &\leq \frac{1}{2m} \eta \|\partial_y^m \psi\|_{L^2}^2 + C_\eta \|\langle y \rangle^{-m-1+\frac{1}{2k+1}} \psi\|_{L^2}^2,\end{aligned}$$

by which we arrive at

$$\tfrac{1}{2} \partial_s \|\partial_y^m \psi\|_{L^2}^2 \leq -(\beta_m - \tfrac{1}{2}\eta) \|\partial_y^m \psi\|_{L^2}^2 + C_\eta \|\langle y \rangle^{-m-1+\frac{1}{2k+1}} \psi\|_{L^2}^2 + \langle \partial_y^m \mathcal{L}_Q \psi, \partial_y^m \psi \rangle.$$

To proceed further, by Hardy's inequality (Lemma 2.11.(2)), for any $R > 1$ we may write

$$\begin{aligned}\|\langle y \rangle^{-m-1+\frac{1}{2k+1}} \psi\|_{L^2(|y| \geq R)}^2 &\leq C R^{\frac{1}{2k+1}-1} \|\langle y \rangle^{-m} \psi\|_{L^2(|y| \geq R)}^2 \\ &\leq C R^{\frac{1}{2k+1}-1} \left(\|\partial_y^m \psi\|_{L^2} + \|\psi\|_{L^2(|y| \leq 1)} \right)^2.\end{aligned}$$

Choosing $R = R(m, \eta)$ large enough so that $C_\eta C R^{\frac{1}{2k+1}-1} < \frac{1}{2}\eta$, and trivially estimating $\|\psi\|_{L^2(|y| \leq 1)} \leq \|\langle y \rangle^{-m-1+\frac{1}{2k+1}} \psi\|_{L^2(|y| \geq R)}$, we arrive at the desired conclusion. □

Next, we record an energy inequality for $w^{\frac{1}{2}} \psi$, where w is an s-independent weight.

Lemma 2.28. *Let $\psi \in C_t^\infty \mathcal{S}_x$, and let $w = w(y)$ be a positive C^1 function such that (i) $y \partial_y w \leq 0$ everywhere and (ii) w grows at most polynomially as $|y| \to \infty$. Then for every $\eta > 0$, there exists $C = C(w, \eta)$ such that*

$$\frac{1}{2} \partial_s \|w \psi\|_{L^2}^2 \leq -\langle ((-\tfrac{y \partial_y w}{w} - \tfrac{3}{2})(b-1) - 1) w \psi, w \psi \rangle + \|w \mathcal{L}_Q \psi\|_{L^2}^2.$$

In particular, if there exist $m_w \in \mathbb{R}$, $C_w > 0$ and $y_0 > 0$ such that

$$\frac{y \partial_y w}{w} \leq -m_w \text{ for } |y| \geq y_0,$$

then we have

$$\frac{1}{2} \partial_s \|w \psi\|_{L^2}^2 \leq -((m_w - \tfrac{3}{2})(b-1) - 1) \|w \psi\|_{L^2}^2 + |m_w| \|w \psi\|_{L^2(|y| \leq y_0)}^2 + \|w \mathcal{L}_Q \psi\|_{L^2}^2.$$

Proof. We compute

$$\begin{aligned}&\langle \mathcal{L}_Q \psi, w^2 \psi \rangle \\ &= \langle \partial_s \psi + (-(b-1) + \partial_y Q) \psi + (by + Q) \partial_y \psi, w^2 \psi \rangle \\ &= \tfrac{1}{2} \partial_s \langle w \psi, w \psi \rangle + \langle (-(b-1) + \partial_y Q) w \psi, w \psi \rangle + \langle (by + Q) \partial_y \psi, w^2 \psi \rangle \\ &= \tfrac{1}{2} \partial_s \langle w \psi, w \psi \rangle + \langle (-(b-1) + \partial_y Q) w \psi, w \psi \rangle - \tfrac{1}{2} \langle (b + \partial_y Q + 2(b + y^{-1} Q) \tfrac{y \partial_y w}{w}) w \psi, w \psi \rangle \\ &= \tfrac{1}{2} \partial_s \langle w \psi, w \psi \rangle + \langle ((-\tfrac{y \partial_y w}{w} - \tfrac{3}{2})(b-1) - 1 + \tfrac{1}{2}(\partial_y Q + 1) - (y^{-1} Q + 1) \tfrac{y \partial_y w}{w}) w \psi, w \psi \rangle \\ &\geq \tfrac{1}{2} \partial_s \langle w \psi, w \psi \rangle + \langle ((-\tfrac{y \partial_y w}{w} - \tfrac{3}{2})(b-1) - 1) w \psi, w \psi \rangle,\end{aligned}$$

where we used $\partial_y Q \geq -1$, $y^{-1} Q \geq -1$ and $y \partial_y w \leq 0$ on the last line. The second statement follows immediately from the first. □

We are now ready to prove Proposition 2.25.

Proof of Proposition 2.25. Applying Lemma 2.26 to ϵ with $m = 2k+2$, we have

$$\frac{1}{2}\partial_s \|\partial_y^{2k+2}\epsilon\|_{L^2}^2 \leq -(\beta_{2k+2} - \eta_1)\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 + C_1 \|\langle y\rangle^{-2k-3+\frac{1}{2k+1}}\epsilon\|_{L^2(|y|\leq R)}^2 + \langle \partial_y^{2k+2} f, \partial_y^{2k+2}\epsilon\rangle.$$

where we observe that $\beta_{2k+2} > 0$.

Next, given $y_0 > 0$ and $\eta_2 > 0$, we shall construct a positive weight w that it even ($w(y) = w(-y)$) and normalized so that $w(1) = 1$. Introduce $m_w = 2k + \frac{3}{2} + 2k\eta_2$; note that $(m_w - \frac{3}{2})(b-1) - 1 = \eta_2$. For $y_0 < y < 1$, we choose $\frac{y\partial_y w}{w} = -m_w$ and then transition (so that $y\partial_y w \leq 0$ everywhere) to $y\partial_y w = 0$ for $0 < y < \frac{y_0}{2}$. For $y > 1$, choose $\frac{y\partial_y w}{w} = -2k-2$ (which is less than $-m_w$ for η_2 small). By Lemma 2.28, we obtain

$$\frac{1}{2}\partial_s \|w\epsilon\|_{L^2}^2 \leq -\eta_2 \|w\epsilon\|_{L^2}^2 + C|m_w| \|y_0^{-2k-\frac{3}{2}-2k\eta_2}\epsilon\|_{L^2(|y|\leq y_0)}^2 + \|wf\|_{L^2}^2,$$

where we used the fact that $w \simeq y_0^{-m_w}$ for $|y| \leq y_0$. Introducing a positive number $a > 0$ and adding these two inequalities up, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_s \left(a\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 + \|w\epsilon\|_{L^2}^2\right) \leq & -a(\beta_{2k+2} - \eta_1)\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 - \frac{1}{2}\eta_2\|w\epsilon\|_{L^2}^2 \\
& + aC_1\|\langle y\rangle^{-2k-3+\frac{1}{2k+1}}\epsilon\|_{L^2(|y|\leq R}^2 + CC|m_w| \|y_0^{-2k-\frac{3}{2}-2k\eta_2}\epsilon\|_{L^2(|y|\leq y_0)}^2 \\
& + a\langle \partial_y^{2k+2} f, \partial_y^{2k+2}\epsilon\rangle + \|wf\|_{L^2}^2 \\
\leq & -a(\beta_{2k+2} - \eta_1)\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 - (\tfrac{1}{2}\eta_2 - aC_1C_2)\|w\epsilon\|_{L^2}^2 \\
& + a\langle \partial_y^{2k+2} f, \partial_y^{2k+2}\epsilon\rangle + \|wf\|_{L^2}^2 \\
& + C|m_w||y_0|^{\frac{1}{2}-2k\eta_2}\|y_0^{-2k-2}\epsilon\|_{L^2(|y|\leq y_0)}^2.
\end{aligned}$$

where we used $C_2 w \geq \langle y\rangle^{-2k-3+\frac{1}{2k+1}}$ for some $C_2 > 0$. Fixing the small constants η_1, η_2 and a in order, we may find $c_1, c_2 > 0$ and $C > 0$ such that

$$\begin{aligned}
\frac{1}{2}\partial_s \left(a\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 + \|w\epsilon\|_{L^2}^2\right) \leq & -c_1\left(a\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 + \|w\epsilon\|_{L^2}^2\right) + a\langle \partial_y^{2k+2} f, \partial_y^{2k+2}\epsilon\rangle + \|wf\|_{L^2}^2 \\
& - ac_1\|\partial_y^{2k+2}\epsilon\|_{L^2}^2 + C|y_0|^{c_2}\|y_0^{-2k-2}\epsilon\|_{L^2(|y|\leq y_0)}^2.
\end{aligned}$$

At this point, we invoke the vanishing conditions on ϵ and Hardy's inequality (Lemma 2.11.(1)), which implies that the last line is nonpositive if $|y_0|$ is chosen small enough. □

We are ready to close the first bootstrap assumption. It will be convenient to recall (2.24):

$$\begin{aligned}
\mathcal{L}_Q\epsilon = & -\Psi - \mathcal{L}_P M - M\partial_y M - e^{-(1-b\alpha)s}\Gamma M \\
& -(\partial_y(P-Q))\epsilon - (P-Q)\partial_y\epsilon - e^{-(1-b\alpha)s}\Gamma\epsilon - M\partial_y\epsilon - \epsilon\partial_y M - \epsilon\partial_y\epsilon.
\end{aligned}$$

Proof of (2.20). In what follows, we will estimate each term on the RHS of (2.24) in either

(1) $\|\partial_y^{2k+2}(\cdot)\|_{L^2} + \|\cdot\|_{L^2(|y|\leq 1)} \lesssim e^{-cs}$, or
(2) $|\langle \partial_y^{2k+2}(\cdot), \partial_y^{2k+2}\epsilon\rangle| + \|\langle y\rangle^{-2k-2}(\cdot)\|_{L^2}^2 \lesssim e^{-(c+\gamma_1)s}$

for some c. Moreover, we will often use Hölder to bound $\|\cdot\|_{L^2(|y|\leq 1)} \leq \|\cdot\|_{L^\infty}$ and control the L^∞ norm. In view of Proposition 2.25 (as well as Lemma 2.9), we would succeed in improving the bootstrap (for s_0 large enough and $\|\epsilon_0\|_{H^{2k+3}}$ small enough) if we can achieve the above for every term with

$$c > \gamma_1.$$

We treat each term on the RHS of (2.24) as follows.

- $-\Psi$: Using Corollary 2.19,

$$\|\partial_y^{2k+2}\Psi\|_{L^2} + \|\Psi\|_{L^\infty} \lesssim e^{-\mu s},$$

which is acceptable by (2.15).

- $\mathcal{L}_P M + M\partial_y M$: By Lemma 2.24,

$$|\partial_y^{2k+2}(\mathcal{L}_P M + M\partial_y M)| \lesssim \left(|\mathbf{c}| + |\mathbf{c}|^2\right) \mathbb{1}_{|y|\le 2} + e^{-(1-b\alpha)s} \sum_{m=0}^{2k+1} |\partial_y^m \Gamma(P+\epsilon)(0)| \mathbb{1}_{|y|\le 2}.$$

Hence,

$$\begin{aligned}\|\partial_y^{2k+2}(\mathcal{L}_P M + M\partial_y M)\|_{L^2} &\lesssim \left(|\mathbf{c}| + |\mathbf{c}|^2\right) + e^{-(1-b\alpha)s} \sum_{m=0}^{2k+1} \left(|\partial_y^m \Gamma P(0)| + |\partial_y^m \Gamma\epsilon(0)|\right) \\ &\lesssim e^{-\gamma_0 s} + e^{-(1-b\alpha)s}\left(e^{-\mu s} + e^{-2\gamma_0 s} + e^{-\gamma_2 s}\right),\end{aligned}$$

which is acceptable. The $L^2(|y|\le 1)$ norm is controlled similarly.

- $e^{-(1-b\alpha)s}\Gamma M$: We have

$$e^{-(1-b\alpha)s}\left(\|\partial_y^{2k+2}\Gamma M\|_{L^2} + \|\Gamma M\|_{L^2(|y|\le 1)}\right) \lesssim e^{-(1-b\alpha)s}|\mathbf{c}| \lesssim e^{-(1-b\alpha+\gamma_0)s}.$$

- $e^{-(1-b\alpha)s}\Gamma\epsilon$: We have

$$e^{-(1-b\alpha)s}\|\partial_y^{2k+2}\Gamma\epsilon\|_{L^2} \lesssim e^{-(1-b\alpha)s}e^{-\gamma_2 s},$$

as well as

$$e^{-(1-b\alpha)s}\|\Gamma\epsilon\|_{L^2(|y|\le 1)} \lesssim e^{-(1-b\alpha)s}\|\Gamma\epsilon\|_{L^\infty(|y|\le 1)} \le 2e^{-2\gamma_0 s}.$$

- $(\partial_y(P-Q) + \partial_y M)\epsilon$: By Gagliardo–Nirenberg interpolation and Hardy, we have

$$\|\partial_y^{2k+2}((\partial_y(P-Q) + \partial_y M)\epsilon)\|_{L^2} + \||y|^{-2k-2}((\partial_y(P-Q) + \partial_y M)\epsilon)\|_{L^2} \lesssim (e^{-\mu s} + e^{-\gamma_0 s})e^{-\gamma_1 s}$$

- $((P-Q) + M + \epsilon)\partial_y\epsilon$

  In $\partial_y^{2k+2}(((P-Q) + M + \epsilon)\partial_y\epsilon)$, the only new case to consider is

$$(P-Q) + M + \epsilon)\partial_y\partial_y^{2k+2}\epsilon,$$

since all the other terms are treated like in the previous case, with bounds $(e^{-\mu s} + e^{-\gamma_0 s} + e^{-\gamma_2 s})e^{-\gamma_1 s}$ (here, $e^{-\gamma_2 s}e^{-\gamma_1 s}$ arises from $\epsilon\partial_y\epsilon$. For this term, we integrate by parts:

$$\begin{aligned}&|\langle (P-Q) + M + \epsilon)\partial_y\partial_y^{2k+2}\epsilon, \partial_y^{2k+2}\epsilon\rangle| \\ &= |\langle \partial_y(P-Q) + \partial_y M + \partial_y\epsilon)\partial_y^{2k+2}\epsilon, \partial_y^{2k+2}\epsilon\rangle| \\ &\lesssim \left(\|\partial_y(P-Q)\|_{L^\infty} + \|\partial_y M\|_{L^\infty} + \|\partial_y\epsilon\|_{L^\infty}\right)\|\partial_y^{2k+2}\epsilon\|_{L^2} \\ &\lesssim (e^{-\mu s} + e^{-\gamma_0 s} + e^{-\gamma_2 s})e^{-\gamma_1 s}.\end{aligned}$$

Next, we estimate

$$\begin{aligned}\|\langle y\rangle^{-2k-2}(P-Q) + M + \epsilon)\partial_y\epsilon\|_{L^2} &\lesssim \|\langle y\rangle^{-1}((P-Q) + M + \epsilon)\langle y\rangle^{-2k-1}\partial_y\epsilon\|_{L^2} \\ &\lesssim \|\langle y\rangle^{-1}((P-Q) + M + \epsilon)\|_{L^\infty}\||y|^{-2k-1}\partial_y\epsilon\|_{L^2} \\ &\lesssim (e^{-\mu s} + e^{-\gamma_0 s} + e^{-\gamma_2 s})e^{-\gamma_1 s},\end{aligned}$$

where we used Hardy's inequality for $|y|^{-2k-1}\partial_y \epsilon$ and

$$\|\langle y\rangle^{-1}\epsilon\|_{L^\infty} \lesssim \|\langle y\rangle^{-1}\epsilon\|_{L^2}^{\frac{1}{2}} \|\partial_y(\langle y\rangle^{-1}\epsilon)\|^{\frac{1}{2}} \lesssim \|\partial_y \epsilon\|_{L^2}^{\frac{1}{2}} \|\partial_y^2 \epsilon\|^{\frac{1}{2}},$$

by Gagliardo–Nirenberg interpolation and Hardy. □

$\dot{H}^{2k+3}$ and $\dot{H}^1$ energy estimates. Next, we establish control on $\|\partial_y^{2k+3} u\|_{L^2}$ and $\|\partial_y u\|_{L^2}$. Our argument will be similar to the proof of (2.20), with the important distinction being that in order to close the top-order energy estimate, we will use the fact that $\partial_t + \Gamma$ is either dispersive or dissipative.

Proof of (2.21) *for* $\partial_y^{2k+3}\epsilon$. By Lemma 2.26 with $m = 2k+3$ and $0 < \eta \ll \beta_{2k+3}$ fixed, we have

$$\frac{1}{2}\partial_s \|\partial_y^{2k+3}\epsilon\|_{L^2}^2 \leq -(\beta_{2k+3} - \eta)\|\partial_y^{2k+3}\epsilon\|_{L^2}^2 + C\|\epsilon\|_{L^2(|y|\leq R)}^2 + \langle \partial_y^{2k+3}\mathcal{L}_Q \epsilon, \partial_y^{2k+3}\epsilon\rangle,$$

where we simply used $\langle y\rangle^{-2k-4+\frac{1}{2k+1}} \lesssim_R 1$.

For the second term on the right-hand side, we use Hardy's inequality and (2.17) to bound

$$\|\epsilon\|_{L^2(|y|\leq R)}^2 \lesssim e^{-\gamma_1 s},$$

which is acceptable since $\gamma_1 < \gamma_2$.

We treat $\langle \partial_y^{2k+3}\mathcal{L}_Q\epsilon, \partial_y^{2k+3}\epsilon\rangle$ as in the proof of (2.21). The only significant difference is that we need to use

$$\langle -e^{-(1-b\alpha)s}\Gamma\partial_y^{2k+3}\epsilon, \partial_y^{2k+3}\epsilon\rangle \begin{cases} = 0 & \text{if } \Gamma = |\partial_y|^{\alpha-1}\partial_y, \\ \leq 0 & \text{if } \Gamma = |\partial_y|^\alpha. \end{cases}$$

□

Proof of (2.21) *for* $\partial_y \epsilon$. Here, the observation is that we need to only close $\chi_{|y|\gtrsim R}\epsilon$, since we may use (2.17) for the rest. Then the energy estimate for $\partial_y \chi_{|y|\gtrsim R}\epsilon$ goes though, essentially because $\mathcal{L}_Q \approx \partial_s - (b-1) + by\partial_y$. □

Estimate for $\Gamma\epsilon$. We finally turn to (2.19) concerning $\Gamma\epsilon$ – we remark that if $\Gamma = 0$, then our proof will be complete at this point! The key point in the proof here is to retrieve the exponentially decay $e^{-2\gamma_0 s}$, which is stronger than the two bootstrap assumptions that we closed (i.e., $2\gamma_0 > \gamma_1, \gamma_2$).

We begin by using the integral kernel of Γ to write $\Gamma f(y)$ in terms of f.

Lemma 2.29. *For $y \in [-1, 1]$, we have*

$$|\Gamma f(y)| \lesssim \int_{|y'|\leq 4} \frac{1}{|y-y'|^\alpha}|\partial_y f(y')|\,\mathrm{d}y' + \int_{|y'|\geq 2} \frac{1}{|y'|^{1+\alpha}}|f(y')|\,\mathrm{d}y'.$$

Proof. With either option for Γ, we have

$$\Gamma\epsilon(y) = \int K(y,y')\partial_y \epsilon(y')\,\mathrm{d}y',$$

where

$$|K(y,y')| \lesssim \frac{1}{|y-y'|^\alpha}, \quad |\partial_{y'}K(y,y')| \lesssim \frac{1}{|y-y'|^{1+\alpha}}.$$

In fact, when $\Gamma = |\partial_y|^{\alpha-1}\partial_y$, $K(y,y') = c|y-y'|^{-\alpha}$ (integral kernel for the fractional integration of order $1-\alpha$). The desired lemma follows by splitting $\partial_y f(y') = \chi(y')\partial_y f + (1 -$

$\chi(y'))\partial_y f$, integrating ∂_y by parts away from f for the contribution of $(1-\chi(y'))\partial_y f$, and observing that $|y'-y| \simeq |y'|$ if $|y'| \geq 2$. □

Consider $e^{-(1-b\alpha)s}\Gamma\epsilon(y)$, written out using the above decomposition. The contribution of $\partial_y \epsilon$ is already good, since it will decay and we have additional factor $e^{-(1-b\alpha)s}$. We would need to look at the equation to derive estimates for ϵ. Here, an important technical observation is that we can establish almost optimal control on ϵ itself (without any derivatives and possibly with slowly decaying weight), as long as the expectation is worse than $e^{-\gamma_2 s}$ (which is the case).

Lemma 2.30. *Under the hypotheses of Proposition 2.20, we have, for all $s \in [s_0, s_1]$,*

$$\|\langle y\rangle^{-\frac{3}{2}+\frac{1}{b}-\frac{\gamma_2}{b}}\epsilon\|_{L^2} + e^{(1-\frac{3}{2}b-\gamma_2)s}\|\epsilon\|_{L^2} \lesssim 1.$$

This bound is almost optimal in the sense that, if $(\partial_s - (b-1) + by\partial_y)\epsilon = 0$, then the above estimate with $\gamma_2 = 0$ would be sharp.

Proof. Carry out! Again, it suffices to consider $\epsilon = \chi_{|y|\gtrsim R}\epsilon$. Use the usual energy inequality and weighted energy inequality. □

We are now ready to improve the last bootstrap assumption.

Proof of (2.22). Fix $s \in [s_0, s_1]$. Using Lemma 2.29, for $|y| \leq 1$, we will estimate

$$\begin{aligned}
|\Gamma\epsilon(y)| &\lesssim \int_{|y'|\leq 4} \frac{1}{|y-y'|^{\alpha}}|\partial_y \epsilon(y')|\,\mathrm{d}y' + \int_{2\leq|y'|\leq e^{bs}} \frac{1}{|y'|^{1+\alpha}}|\epsilon(y')|\,\mathrm{d}y' \\
&\quad + \int_{|y'|\geq e^{bs}} \frac{1}{|y'|^{1+\alpha}}|\epsilon(y')|\,\mathrm{d}y' \\
&\lesssim \|\partial_y \epsilon\|_{L^\infty} + (1 + e^{(-1-\alpha+\frac{3}{2}-\frac{1}{b}+\frac{\gamma_2}{b}+\frac{1}{2})bs})\|\langle y'\rangle^{-\frac{3}{2}+\frac{1}{b}-\frac{\gamma_2}{b}}\epsilon\|_{L^2} + e^{-(\frac{1}{2}+\alpha)bs}\|\epsilon\|_{L^2} \\
&\lesssim 1 + e^{((1-\alpha)b-1+\gamma_2)s},
\end{aligned}$$

where we used $\|\partial_y \epsilon\|_{L^\infty} \lesssim e^{-\gamma_2 s} \lesssim 1$ and Lemma 2.30 on the last line. Thus,

$$e^{-(1-b\alpha)s}|\Gamma\epsilon(y)| \lesssim e^{-(1-b\alpha)s} + e^{-(2-b-\gamma_2)s},$$

so since $1-b\alpha > \gamma_0$ and $(2-b)-\gamma_2 > 2\gamma_0$ (which follows from (2.15)), we obtain (2.22) by taking s_0 large enough. □

Remark 2.31. Conceptually, the basic reason why this proof works is because $e^{-(1-b\alpha)s}|\Gamma\epsilon(y)|$ is expected to decay; indeed, if $(\partial_s - (b-1) + by\partial_y)\epsilon = 0$, then $e^{-(1-b\alpha)s}|\Gamma\epsilon(y)| \lesssim e^{-(2-b)s}$.

Analysis of the unstable ODEs.

Proof of Lemma 2.23. Sketch:

- Write the ODEs as upper triangular system.
- Diagonalize the ODE.
- Using the lower bound, absorb everything.

□

Appendix: Sharp pointwise bounds. Weighted H^k bounds from O.–Pasqualotto.

2.4. Perturbation of self-similar solutions to Burgers III: Modulation theory.

- Fix $k = 1$.
- Identify instabilities with symmetries. Space translation, time translation, Galilean transformation, a (rescaling)
- State: Asymptotic stability result. Key: Modulation
- Introduce

$$t = \tau - e^{-s}, \quad x = \lambda y + \xi, \quad u = \frac{\lambda}{\tau - t} U + \kappa.$$

(We expect κ to be related to ξ_s in view of Galilean transformation. In fact, we shall naturally derive this relation when we derive the evolutionary equations for the modulation parameters.)

- $$\begin{aligned} 0 = &\partial_s U - (b-1)U + by\partial_y U + \left(-e^{bs}\xi_s + (1 + e^s \tau_s) e^{(b-1)s}\kappa\right)\partial_y U + e^{(b-1)s}\kappa_s \\ &+(1 + e^s\tau_s) U\partial_y U + (1 + e^s\tau_s) e^{-s(1-b\alpha)}\Gamma U. \end{aligned}$$

- Also add in Q_a and vary in a. Derive equations for ϵ.
- Now set $\epsilon(s,0) = \partial_y \epsilon(s,0) = \partial_y^2 \epsilon(s,0) = \partial_y^3 \epsilon(s,0) = 0$. Get ODEs for ξ ($e^{bs}\xi_s - (1 + e^s\tau_s)e^{(b-1)s}\kappa$), τ ($e^s\tau_s$), κ ($e^{(b-1)s}\kappa_s$) and a (a_s), estimate the RHS's.
  (For $k > 1$, use $W(s,0) = \partial_y W(s,0) = \partial_y^{2k} W(s,0) = \partial_y^{2k+1} W(s,0) = 0$. For $c_2, \dots, c_{2k-1}$, we need to use a shooting argument)
- Redo the bootstrap argument. When $k = 1$, there are no trapped assumptions!

Appendix: ODE blow-up. Consider the ODE

$$\partial_t u = |u|^{p-1} u.$$

Assume that u depends also on x, i.e., $u = u(t,x)$. We may consider the self-similar variables (s, y, U) adapted to the characteristic scale $\lambda(t) = (T-t)^b$:

$$\mathrm{d}t = (T-t)\mathrm{d}s, \quad x = \lambda(t) y, \quad u = \frac{1}{(T-t)^{\frac{1}{p-1}}} U.$$

Then U solves

$$\partial_s U + \frac{1}{p-1} U + by\partial_y U = |U|^{p-1} U. \tag{2.27}$$

Consider a positive stationary solution to (2.28), which solves

$$\frac{1}{p-1} Q + by\partial_y Q = Q^p. \tag{2.28}$$

This is an explicitly solvable ODE (Bernoulli's ODE)! We have

Lemma 2.32. *For $k = 0, 1, 2, \dots$, $p > 1$ and $a > 0$, the function*

$$Q(y) = \left((p-1) + a y^{2k}\right)^{-\frac{1}{p-1}}.$$

is a smooth solution to (2.28) *with*

$$b = \frac{1}{2k}.$$

Furthermore, all smooth positive solutions are of the above form.

Proof. We follow the standard procedure for solving Bernoulli's ODE. We rewrite the equation as

$$\partial_y Q + \frac{1}{p-1}\frac{1}{by}Q = \frac{1}{by}Q^p,$$

and introduce

$$Z = Q^{1-p}.$$

Then

$$\partial_y Z = (1-p)Q^{-p}\partial_y Q = \frac{1}{by}Z - \frac{p-1}{by}.$$

We now perform the method of integrating factors. We may write

$$y^{\frac{1}{b}}\partial_y(y^{-\frac{1}{b}}Z) = -\frac{p-1}{b}y^{-1}$$

Hence,

$$\partial_y(y^{-\frac{1}{b}}Z) = -\frac{p-1}{b}y^{-\frac{1}{b}-1}.$$

Upon integration, we immediately get

$$Z = (p-1) + ay^{\frac{1}{b}},$$

where a is an integration constant. Changing back to $Q = Z^{-\frac{1}{p-1}}$, and introducing $2k = \frac{1}{b}$, we obtain the above result. □

Exercise: Develop linear stability theory when p is an odd integer.

Exercise [**?**]**:** Prove the existence of smooth blow-up for $\partial_t u = \Delta u + u^p$ with p an odd integer such that $p > 2$ (Hint: use $k > 1$). For the stable case ($k = 1$), see [**?**].

3. Weeks 7–8. Singularity formation for the incompressible Euler equation

References

Department of Mathematics, UC Berkeley, Berkeley, CA 94720
Email address: sjoh@math.berkeley.edu