Notes for Beale-Kato-Majda Blowup Criterion and some Applications

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Preliminaries

For reference, in these notes, we will largely follow Tao's Math 254a notes as well as chapter 4 of [3].

Recall that we are studying the incompressible Euler equations on $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= -\nabla p \\ \nabla \cdot u &= 0 \\ u(0,x) &= u_0(x) \end{aligned} \tag{1}$$

Let us also recall the well-posedness result discussed last time.

Theorem 1. Let $s > \frac{d}{2}+1$ and let $u_0 \in H^2_x(\mathbb{R}^d \to \mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$. Then there exists a unique $T_* > 0$ and a unique $u \in C^0_{t,loc}H^s([0,T_*) \times \mathbb{R}^d)$ and $p \in C^0_{t,loc}H^2([0,T_*) \times \mathbb{R}^d)$ such that

- If $0 < T < T_*$ and $u_0^n \in H^\infty_x(\mathbb{R}^d)$ is divergence free with $u_0^n \to u_0$ in H^s, then for n large enough, there is an H^∞ solution (u^n, p^n) to (1) with initial data u_0^n on $[0,T] \times \mathbb{R}^d$ such that u^n and p^n converges to u, p on $C^0_t H^s_x([0,T] \times \mathbb{R}^d)$.

- If $T_* < \infty$ then $\lim_{t \uparrow T_*} \|u(t)\|_{H^s} = \infty$ and $\lim_{t \uparrow T_*} \|\nabla u\|_{L^1_t L^\infty_x([0,T_*) \times \mathbb{R}^d)} = \infty$

In particular, we have a blowup criterion involving the L^∞ norm of the velocity gradient. One of the main goals in these notes will be to refine this to a blowup criterion involving the vorticity. Before getting into the main result, let us start with a brief discussion of vorticity.

As a quick remark, in the below discussion, **we will assume s is an integer**. This is the direction taken in Tao's blog and is done for simplicity of presentation.

Recall that the vorticity, ω is the anti-symmetric tensor field given by

$$\omega_{ij} = \partial_i u_j - \partial_j u_i \tag{2}$$

As a quick remark, in three dimensions it is common to write $\omega = \nabla \times u = (\omega_{23}, \omega_{31}, \omega_{12})$, which is perhaps more familiar.

What is the point in studying vorticity instead of the velocity gradient? One good reason is that the vorticity for (regular enough) solutions to (1) obeys a relatively simple equation in two and three dimensions. Indeed, suppose (u,p) is a smooth solution to (1). Writing everything out component-wise gives

$$\partial_t u_j + u_k \partial_k u_j = -\partial_j p \tag{3}$$

Applying ∂_i to both sides gives

$$\partial_t \partial_i u_j + \partial_i u_k \partial_k u_j + u_k \partial_i \partial_k u_j = -\partial_i \partial_j p \tag{4}$$

We can eliminate pressure by swapping the indices i and j and subtracting the two equations. Some simple algebraic manipulations then lead to the equation

$$D_t\omega_{ij} + \omega_{ik}\partial_j u_k - \omega_{jk}\partial_i u_k = 0 \tag{5}$$

Where $D_t := \partial_t + u_k\partial_k$. Now, we make some observations about the vorticity in two and three dimensions. We have

- $D_t\omega_{12} = 0$ when $d = 2$
- $D_t\overline{\omega} = (\omega \cdot \nabla)u$ when $d = 3$ where we write $\overline{\omega} := (\omega_{23}, \omega_{31}, \omega_{12})$

To see why this is the case, note from (5), we have (where I'm summing over repeated indices),

$$\begin{aligned} D_t\omega_{12} &= \omega_{2k}\partial_1 u_k - \omega_{1k}\partial_2 u_k \\ &= \omega_{21}\partial_1 u_1 - \omega_{12}\partial_2 u_2 \\ &= \omega_{21}\nabla \cdot u = 0 \end{aligned} \tag{6}$$

for $d = 2$.

For $d = 3$, we have (for e.g. the first component of $\overline{\omega}$),

$$\begin{aligned} D_t\overline{\omega_1} &= -\omega_{2k}\partial_3 u_k + \omega_{3k}\partial_2 u_k \\ &= -\omega_{21}\partial_3 u_1 - \omega_{23}\partial_3 u_3 + \omega_{31}\partial_2 u_1 + \omega_{32}\partial_2 u_2 \\ &= -\omega_{32}\partial_1 u_1 + \omega_{31}\partial_2 u_1 - \omega_{21}\partial_3 u_1 \\ &= \omega_{23}\partial_1 u_1 + \omega_{31}\partial_2 u_1 + \omega_{12}\partial_3 u_1 \\ &= (\overline{\omega} \cdot \nabla)u_1 \end{aligned} \tag{7}$$

where in the third line we used incompressibility of u. The other components are handled in an identical fashion. With a few basic tools in hand, now we turn our attention to the Beale-Kato-Majda Theorem.

Beale-Kato-Majda Theorem

The above simple identities motivate one to try to find a blowup criterion similar to (1) that involves just the vorticity. since this quantity appears to be much simpler to work (in fact it is transported by the Euler flow) with than ∇u. If the time of existence, T_* as in Theorem 1 is finite, then it seems reasonable to conjecture that $\|\omega\|_{L^1L^\infty([0,T_*)\times\mathbb{R}^d)} = \infty$. Let us make a few observations.

First, notice that taking the divergence of the vorticity gives

$$\partial_i\omega_{ij} = \partial_{ii}u_j - \partial_j\partial_i u_i \tag{8}$$

Thanks to incompressibility, this reduces to

$$\partial_i\omega_{ij} = \Delta u_j \tag{9}$$

and so, we have

$$\partial_k u_j = \Delta^{-1}\partial_i\partial_k\omega_{ij} \tag{10}$$

where $\Delta^{-1}\partial_i\partial_k$ is defined in the usual way by the Fourier transform. As a brief remark, this can be compared with the pressure formula

$$p = R_i R_j u_i u_j \tag{11}$$

where R_i is the Riesz transform. The naive thing to hope for is that $\Delta^{-1}\partial_i\partial_k$ is bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ since this would give us

$$\|\nabla u\|_{L^\infty} \lesssim_d \|\omega\|_{L^\infty} \tag{12}$$

which would immediately yield the desired blowup criterion. Of course, as it turns out, we do not have boundedness from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. I will omit an example showing this, for brevity (but counterexamples can be found in most standard harmonic analysis textbooks or on Tao's blog). Fortunately, this failure of boundedness is quite mild. In fact, it essentially only fails by a logarithmic factor for our purposes. This turns out to be totally fine since Gronwall type arguments are flexible enough to deal with such a factor. To be precise, we have the following:

Lemma 2. For $s > \frac{d}{2} + 1$, we have

$$\|\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \lesssim_{s,d} \|\omega\|_{L^\infty} \log(2 + \|u\|_{H^s_x}) + \|u\|_{L^2} + 1 \tag{13}$$

Before proving this, it is useful to note that the L^2 norm of u that appears here will end up being rather inconsequential when dealing with solutions (u, p) to the Euler equation from Theorem 1, since at this regularity, this is a conserved quantity. With these remarks out of the way, let us outline the proof.

We will start by working with Schwartz functions. The general case follows by an approximation argument. The idea is to perform a Littlewood-Paley decomposition on the vorticity. That is,

$$\omega = P_{\leq 1}\omega + \sum_{N>1} P_N\omega \tag{14}$$

and so, it is enough to control

$$\|P_{\leq 1}\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} + \sum_{N>1} \|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \tag{15}$$

As usual, we are summing over dyadic numbers. In light of the above discussion, it is favorable to estimate the low frequency part (i.e. $P_{\leq 1}\omega$) in (14) by the L^2 norm of u. A simple application of Bernstein's inequality, Plancherel, and the fact that $P_{\leq 1}\omega$ has Fourier transform supported on a ball of radius approximately 1 (which allows us to throw away the derivative in the vorticity) gives

$$\begin{aligned} \|P_{\leq 1}\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} &\lesssim_d \|P_{\leq 1}\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^2} \\ &\lesssim_d \|P_{\leq 1}\omega\|_{L^2} \\ &\lesssim_d \|u\|_{L^2} \end{aligned} \tag{16}$$

For high frequency terms, it is favourable to control the vorticity by the H^s norm of u since $s > 1 + \frac{d}{2}$. We have for instance, by Bernstein and Plancherel again, and using the fact that the

Fourier transform of $P_N\omega$ is supported at frequencies of approximately size N, we have

$$\begin{aligned}\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} &\lesssim_d N^{\frac{d}{2}}\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^2} \\ &\lesssim_d N^{\frac{d}{2}}\|P_N\omega\|_{L^2} \\ &\lesssim_d N^{\frac{d}{2}+1-s}\|u\|_{H^s_x}\end{aligned} \tag{17}$$

As long as we sum over dyadics $N > N_0$ for some appropriate N_0 to be chosen, we then obtain

$$\sum_{N>N_0}\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \lesssim_{s,d} N_0^{\frac{d}{2}+1-s}\|u\|_{H^s} \tag{18}$$

We will not want our bound to depend more than logarithmically on the H^s norm of u, and so we choose $N_0 = (2+\|u\|_{H^s})^{\frac{1}{s-\frac{d}{2}-1}}$. This will give

$$\sum_{N>N_0}\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \lesssim_{s,d} 1 \tag{19}$$

Now, let us handle the dyadics $1 < N < N_0$. We will show that we have the bound

$$\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \lesssim_d \|\omega\|_{L^\infty} \tag{20}$$

By rescaling, we may assume WLOG that $N = 2$. Note that

$$\mathcal{F}[P_2\Delta^{-1}\partial_i\partial_k\omega_{ij}](\xi) = \phi(\xi)\frac{\xi_i\xi_j}{|\xi|^2}\widehat{\omega_{ij}} \tag{21}$$

where ϕ is a smooth function with compact support, which also vanishes on the unit ball. Now, taking the inverse Fourier transform of the above identity and using Young's convolution inequality gives

$$\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} \lesssim_d \|\omega\|_{L^\infty} \tag{22}$$

And, so we finally have

$$\begin{aligned}\sum_{1<N\leq N_0}\|P_N\Delta^{-1}\partial_i\partial_k\omega_{ij}\|_{L^\infty} &\lesssim_{s,d} \sum_{1<N\leq N_0}\|\omega\|_{L^\infty} \\ &\lesssim_{s,d} \log(2+\|u\|_{H^s})\|\omega\|_{L^\infty}\end{aligned} \tag{23}$$

where the last line just follows from counting the dyadics between 1 and N_0. Combining all of the above estimates completes the proof $\square$.

Now, let us actually put this lemma to use to prove the so-called Beale-Kato-Majda theorem.

Theorem 3. Let $s > \frac{d}{2}+1$, and $u_0 \in H^s(\mathbb{R}^d)$ be divergence free. Let $u, p \in C^0_{t,loc}H^s([0,T_*)\times\mathbb{R}^d)$ be the solution from Theorem 1. Then we have

- For $0 \leq t < T_*$

$$\|u(t)\|_{H^s_x} \lesssim_{s,d} \exp(C_{s,d}(t(1+\|u_0\|_{L^2}))) \cdot \exp(\exp(C_{s,d}(\int_0^t \|\omega(s)\|_{L^\infty}ds)))\|u_0\|_{H^s_x} \tag{24}$$

- If T_* is finite, then
$$\|\omega\|_{L^1_t L^\infty_x([0,T_*)\times\mathbb{R}^d)} = \infty \tag{25}$$

One sees clearly that the second part of the theorem is an immediate consequence of the first part and Theorem 1. As another remark, the double exponential may seem somewhat odd, but this is essentially the cost of trying to apply Gronwall's inequality with the extra logarithmic term from Lemma 2 as we will see.

Let us now prove the theorem. I will omit some technical estimates that are similar in spirit to what was done in the last lecture. There are more details on Tao's Math 254a. Let us introduce the higher energy,
$$E(t) = \sum_{0\le m\le s} \|\nabla^k u\|_{L^2}^2 \tag{26}$$
We obtain (after some easy estimates)
$$\partial_t E(t) \lesssim_{s,d} E(t)\|\nabla u(t)\|_{L^\infty} \tag{27}$$
and so, applying Lemma 2 and L^2 conservation for u, we obtain
$$\partial_t E(t) \lesssim_{s,d} E(t)(\|\omega(t)\|_\infty \log(2+E(T)) + \|u_0\|_{L^2} + 1) \tag{28}$$
which gives
$$\partial_t (E(t)+2) \lesssim_{s,d} (E(t)+2)(\|\omega(t)\|_\infty \log(2+E(T)) + \|u_0\|_{L^2} + 1) \tag{29}$$
so that
$$\partial_t \log(E(t)+2) \lesssim_{s,d} \|\omega(t)\|_\infty \log(2+E(T)) + \|u_0\|_{L^2} + 1 \tag{30}$$

and so, the theorem follows now from Gronwall's inequality. $\square$.

Now, let us discuss some applications of this blowup criterion.

Applications of the BKM Theorem

Let us discuss a few applications of Theorem 3. First, we will prove global H^s well-posedness for the $2-d$ Euler equation. Specifically, we have the following.

Theorem 4. Let u_0, s, T_*, u and p be as in Theorem 1. If $d=2$, then the time of existence T_* is infinite. The key will be the following conservation law

Lemma 5. For u_0, s, T_*, u, p as in Theorem 1 and $d=2$, we have
$$\|\omega_{12}(t)\|_{L^\infty(\mathbb{R}^d)} = \|\omega_{12}(0)\|_{L^\infty(\mathbb{R}^d)} \tag{31}$$
for every $0\le t<T_*$.

To see how Theorem 4 follows from the lemma, we see that
$$\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt \sim \int_0^{T_*} \|\omega_{12}(t)\|_{L^\infty} dt \tag{32}$$

must be finite unless $T_* = \infty$. Theorem 4 then follows from Theorem 3 and Theorem 1. $\Box$.

Now, let us prove Lemma 5. First, we make some useful simplifications.

First,it actually suffices to show that $\|\omega_{12}(t)\|_{L^q} = \|\omega_{12}(0)\|_{L^q}$ for any $2 \le q < \infty$, $0 \le t < T_*$ since we can just take the limit in q to get L^∞ conservation.

Secondly, in light of the well-posedness theory from Theorem 1, we might as well work with H^∞ solutions.

Finally, for technical reasons (that will soon become apparent), it suffices to prove (using an approximation argument)

$$\int_{\mathbb{R}^2} \varphi(\omega_{12}(t,x))dx = \int_{\mathbb{R}^2} \varphi(\omega_{12}(0,x))dx \tag{33}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth with compact support and vanishes in a neighborhood of 0. The reason for doing this is for purely technical reasons. The goal is to run the usual energy-like argument by showing that the time derivative of (33) is zero. To justify differentiating under the integral, we would like to somehow use dominated convergence. However, thanks to the fact that ω_{12} and its derivatives (essentially by assumption) belong to $L^\infty_{t,loc}H^\infty_x$ on $[0,T_*) \times \mathbb{R}^2$, it follows that for $0 < T < T_*$, ω_{12} is Lipschitz on $[0,T] \times \mathbb{R}^2$. In particular, for every $\epsilon > 0$, the set $\{(t,x) \in [0,T] \times \mathbb{R}^2 : |\omega_{12}| \ge \epsilon\}$ is compact. And so, $\varphi(\omega_{12})$ has compact support, and we may apply dominated convergence to obtain

$$\partial_t \int_{\mathbb{R}^2} \varphi(\omega_{12}(t,x))dx = \int_{\mathbb{R}^2} \varphi'(\omega_{12})\partial_t\omega_{12}dx \tag{34}$$

Now, we use the fact that $D_t\omega_{12} = 0$ in 2 dimensions, and then use incompressibility to obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi'(\omega_{12})\partial_t\omega_{12}dx &= -\int_{\mathbb{R}^2} \varphi'(\omega_{12})\partial_i\omega_{12}u_i dx \\ &= -\int_{\mathbb{R}^2} \partial_i\varphi(\omega_{12})u_i dx \\ &= \int_{\mathbb{R}^2} \varphi(\omega_{12})\partial_i u_i dx = 0 \end{aligned} \tag{35}$$

and so,

$$\int_{\mathbb{R}^2} \varphi(\omega_{12}(t,x))dx \tag{36}$$

is constant, as desired. $\Box$.

One might for a moment wonder if a conservation law similar to Lemma 5 holds in three dimensions. This turns out to be false (which hopefully comes as no surprise). On the other hand, with certain simplifications, we get a useful analogue. Hence, let us turn to our second application of the BKM theorem. Let us consider $d = 3$ and prove that axi-symmetric solutions without swirl (**for appropriate initial data**) exist for all time and are regular. Let us first introduce some definitions.

A flow v is axi-symmetric if

$$v = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3 \tag{37}$$

where e_r, e_θ and e_3 define the cylindrical coordinate system. That is,

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right),$$

$$e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right), \tag{38}$$

$$e_3 = (0, 0, 1)$$

The component v^θ is known as the swirl velocity.

One can also compute the vorticity in these coordinates,

$$\omega = -v^\theta_{x_3} e_r + (v^r_{x_3} - v^3_r)e_\theta + \frac{1}{r}(rv^\theta)_r e_3 \tag{39}$$

Now, we will prove a regularity criterion involving axisymmetric flows with zero swirl. As mentioned above, one of the key reasons for this simplification, is that there is a conserved quantity available that is similar (but not identical) to that of Lemma 5 to work with. Specifically, we prove the following from [3].

Theorem 6. Let $v \in C^1([0, T); C^{1,\gamma}(\mathbb{R}^3))$, $0 < \gamma < 1$ be a 3-dimensional axisymmetric flow without swirl, where the vorticity is given by

$$\omega = (v^r_{x_3} - v^3_r)e_\theta := \omega^\theta \tag{40}$$

If the initial vorticity has compact support, vanishes at $r = 0$ and satisfies

$$|\omega^\theta_0(r, x_3)| \leq cr \tag{41}$$

then v exists for all time. That is, $v \in C^1([0, \infty); C^{1,\gamma}(\mathbb{R}^3))$.

In the proof, I will omit some details since it will rely on a few technical lemmas, but see [3] for more details. Indeed, one can use the vorticity equation to show that

$$\frac{\overline{D}}{Dt}\xi = 0 \tag{42}$$

where $\xi = \frac{\omega^\theta}{r}$ and

$$\frac{\overline{D}}{Dt}\xi = \partial_t \xi + v^r \partial_r \xi + v^3 \partial_3 \xi \tag{43}$$

In particular ξ is conserved along particle trajectories $X(\alpha, t) = (X_1, X_2, X_3) := (X', X_3)$. Here, a particle trajectory is $X(\alpha, t)$ is defined by the ODE

$$\frac{d}{dt}X(\alpha, t) = v(X(\alpha, t), t), \qquad X(\alpha, 0) = \alpha \tag{44}$$

That is, $X(\alpha, t)$ represents the position of a fluid particle at time t with initial position α (with respect to the flow v). Again, see [3] for more details on this.

Now, we have

$$\frac{\omega^\theta(X(\alpha,t),t)}{|X'(\alpha,t)|} = \frac{\omega_0^\theta(\alpha',\alpha_3)}{|\alpha'|} \tag{45}$$

Now, from (41), we have

$$|\omega^\theta(X(\alpha,t),t)| \leq c|X'(\alpha,t)| \tag{46}$$

Thanks to the above two identities, if we can show that $|X'(\alpha,t)|$ is bounded (in α and t) on the support of ω_0^θ, then we are done thanks to the Theorem 3. Hence, we would like to control the quantity

$$R(t) := \sup_{\alpha \in supp(\omega_0^\theta)} |X(\alpha,t)| \tag{47}$$

We will use a Gronwall type argument. First, we invoke Lemma 4.5 in [3], so that if $K(t)^3$ is the measure of the support of $\omega(t)$, then we have

$$|v(X(\alpha,t),t))| \leq K(t)\|\omega(t)\|_{L^\infty(\mathbb{R}^3)} \tag{48}$$

As a quick remark, this estimate is basically due to the Biot-Savart law. It is sort of similar to the estimate in Lemma 2 to control $\|\nabla v\|_{L^\infty}$, except this time for the velocity. Again, see [3] for the technical details.

One can also show that K is constant using something akin to the Reynold's transport theorem and incompressibility of the flow (I think. Feel free to make comments if you want to make this more rigorous. I haven't had time to work out the details yet)

Taking this for granted, we get

$$|v(X(\alpha,t),t)| \leq cK(0) \sup_{\alpha \in supp(\omega_0)} |X'(\alpha,t)| \tag{49}$$

Hence,

$$\begin{aligned} \frac{d}{dt}R(t) &\leq \sup_{\alpha \in supp(\omega_0)} |v(X(\alpha,t),t)| \\ &\leq cK(0)R(t) \end{aligned} \tag{50}$$

and so, R is bounded on any compact time interval by Gronwall. Hence, Theorem 3 gives us global existence. $\Box$.

For some final remarks, it would be good to compare this result to the one we will soon be studying in [2], since Elgindi works in the regime of axial symmetry without swirl. One of the key issues, for instance appears to be the condition (41). Theorem 6 requires some kind of Lipschitz regularity on the vorticity near $r = 0$. There have been some attempts to relax this condition in the literature. One of the most recent results in this direction is a paper by Danchin [1], which proves global regularity under the assumption that $\frac{\omega_0}{r}$ belongs to the Lorentz space $L^{3,1}(\mathbb{R}^3)$. See also [4] and [5] for some results of the same flavor.

One common theme is that the mentioned results don't appear to cover the situation where one assumes that the initial veloctiy is in $C^{1,\alpha}$ for $0 < \alpha < \frac{1}{3}$. Elgindi's construction is actually performed in this range.

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