On T. Elgindi's paper,

Finite-Time Singularity Formation for $C^{1,\alpha}$ Solutions to the Incompressible Euler Equations on $\mathbb{R}^3$

Sung-Jin Oh (UC Berkeley)

Apr. 29th, 2020

Introduction

The subject of this paper is the incompressible Euler equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1}$$

To eliminate the pressure, one introduces the *vorticity* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, which obeys

$O(\alpha)$

$\omega = \Omega(\varphi^\alpha)$

$$\begin{cases} \partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \times \mathbf{u} = \boldsymbol{\omega}. \end{cases} \tag{2}$$

Under reasonable assumptions on the decay of $\mathbf{u}$ at infinity, we have $\mathbf{u} = (-\Delta)^{-1}(\nabla \times \boldsymbol{\omega})$.

It is known classically that the IVP for (1) is locally well-posed for $\mathbf{u}(t = 0) = \mathbf{u}_0 \in C^{1,\alpha}$ (equivalently, $\boldsymbol{\omega} \in C^{0,\alpha}$) for any $0 < \alpha < 1$ (Lichnerowicz, Günther; recall Mohandas's talk).

Introduction

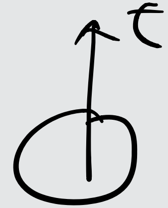
The main result of this paper is the existence of a self-similar, finite time blow-up solution in a locally well-posed class:

Main Theorem

$\omega_0 = \nabla \times u_0$

There exists $0 < \alpha \ll 1$ and a divergence-free initial data $\mathbf{u}_0 \in C^{\cancel{0}\,1,\alpha}(\mathbb{R}^3)$ with $|\cancel{\mathbf{u}}_0| \lesssim \frac{1}{1+|x|^\alpha}$ so that the vorticity $\boldsymbol{\omega}$ of the unique local solution $\mathbf{u}$ in $C^{1,\alpha}([0,1)_t \times \mathbb{R}^3_x)$ has the form

$$\boldsymbol{\omega}(t,x) = \underbrace{\frac{1}{1-t}}\,\overset{\omega_0}{\cancel{\mathbf{u}}_0}\left(\frac{x}{(1-t)^\lambda}\right)$$



$(1-t)^\lambda$

for some $\lambda > 0$. In particular,

$$\underbrace{\lim_{t\to 1-} \int_0^t \|\boldsymbol{\omega}(t')\|_{L^\infty}\, \mathrm{d}t' = +\infty.} \tag{3}$$

Note that (3) is the Beale–Kato–Majda blow-up criterion (cf. Ben's talk).

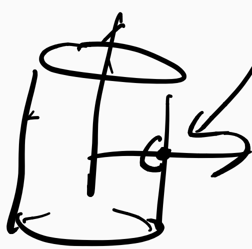
Introduction

- The decay at infinity is necessarily slow. Otherwise, there are rigidity theorems for self-similar solutions to the Euler equationis.
- The solution will be axi-symmetric without swirl (details will come soon). By the known global regularity results (cf. Ben's talk), $\alpha \leq \frac{1}{3}$ is necessary.

$\frac{\omega}{r} \in L^{3,1}(\mathbb{R}^3) \Rightarrow$ G.W.P

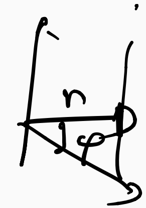
- Existence of a compactly supported initial data that develop finite time singularity, locally modeled by the above behavior, will be covered in Georgios's talk.
- A similar theorem (in particular, very rough initial data) was proved for the axi-symmetric Euler without swirl away from the axis and the Boussinesq equation by J. Chen–T. Hou.

requires a boundary



Set-up

As alluded earlier, one restricts to the case of axi-symmetric, vanishing swirl flow. Introduce the cylindrical coordinates (r, φ, x_3) by



$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, x_3 = x_3.$$

Assume that $(\mathbf{u}, p)$ is independent of φ. With respect to the orthonormal frame $(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_3) = (\partial_r, r^{-1}\partial_\varphi, \partial_3)$,

$$\begin{cases} \partial_t(ru^\varphi) + (u^r\partial_r + u^3\partial_r)(ru^\varphi) = 0, \\ r^{-1}\partial_r(ru^r) + \partial_3 u^3 = 0, \\ \partial_t\omega^\varphi + (u^r\partial_r + u^3\partial_r)\omega^\varphi = r^{-1}u^r\omega^\varphi. \end{cases}$$

$\& \ \omega^\varphi = \partial_r u^3 - \partial_3 u^r$

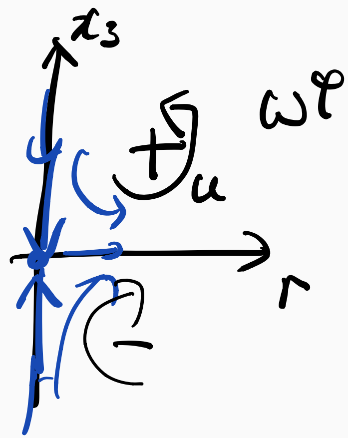
$\to (u^r, u^3)$ B.S.

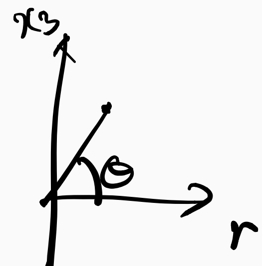
By the first equation, if u^φ (swirl) is zero initially, then it remains so (by uniqueness). We assume this condition for the remainder of the talk.

Set-up

Assume furthermore that ω^φ is *odd* with respect to x_3. Then $\mathbf{u}$ must have a hyperbolic stagnation point at the origin $(r, x_3) = (0, 0)$ (cf. Mitchell's talk, where a similar configuration arose).

ω^φ





$\theta = 0 \longleftrightarrow x_3 = 0$

$\theta = \pi/2 \longleftrightarrow \{r = 0\}$.

Let us introduce the *spherical polar coordinates* (ρ, φ, θ):

$$r = \rho\cos\theta, \quad ,\varphi = \varphi, \quad x^3 = \rho\sin\theta.$$

We will work with profiles that are nice functions of ρ^α for small α (cf. Thibault's talk). To this end, also define $R = \rho^\alpha$.

Key ideas

- Look for a *well-designed* profile (at the level of $\boldsymbol{\omega}$) that is merely $C^{0,\alpha}$ at the origin.
- Use α as a smallness parameter to approximate the Euler equations by a simpler model (*fundamental model*). Smallness of α will enter in two flavors:
    - To break the criticality and make the transport term $\mathbf{u} \cdot \nabla \boldsymbol{\omega}$ weaker than the votex stretching term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ (cf. Thibault's talk);
    - Approximate the non-local Biot–Savart law $\boldsymbol{\omega} \mapsto \mathbf{u}$ by a simpler operator (cf. Mitchell's talk).
- Identity a sufficiently rich class of self-similar solutions to the simpler model problem. Then come back to the Euler equations by a perturbation (i.e., implicit-function-theorem-like) argument.

Fundamental lemma

Fundamental lemma

One has

$$u_r(x) = \frac{1}{2} r L_{12}(\omega) + O(\alpha), \quad u_3(x) = -x_3 L_{12}(\omega) + O(\alpha),$$

where for $\omega = \omega^\varphi(\rho, \theta)$ (spherical coordinates),

$$L_{12}(\omega)(x) = 3 \int_{|x|}^{\infty} \int_0^{\frac{\pi}{2}} \frac{\omega(\rho', \theta') \sin\theta' \cos^2\theta'}{\rho'} \, \mathrm{d}\rho' \mathrm{d}\theta'.$$

We'll follow Elgindi. See T. Tao's blog (Elgindi's approximation of the Biot–Savart law) for an alternative derivation.

Fundamental lemma

To derive Biot–Savart's law, one introduces the *stream function* $\psi = \psi(r, x_3)$ solving,

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r - \partial_3^2 + r^{-2}\right)\psi = \omega^\varphi.$$

with $\psi(0, x_3) = 0$ and $\psi(r, 0) = 0$ (oddness w.r.t. x^3). Then

$$u_r = \partial_3 \psi, \quad u_3 = -\frac{1}{r}\psi - \partial_r \psi.$$

To make use of small α, pass to the spherical coordinates (ρ, φ, θ) and introduce the variable $R = \rho^\alpha$. We write $\omega^\varphi(r, x_3) = \Omega(\rho, \theta)$ and $\psi(r, x_3) = \rho^2 \Psi(\rho, \theta)$. Then

$$\begin{aligned}
&-\alpha^2 R^2 \partial_R^2 \Psi - \alpha(5+\alpha) R \partial_R \Psi - \partial_\theta^2 \Psi + \partial_\theta(\tan\theta \Psi) - 6\Psi = \Omega, \\
&u_r = \rho\left(2\sin\theta \Psi + \cos\theta \partial_\theta \Psi + \alpha \sin\theta R \partial_R \Psi\right), \\
&u_3 = \rho\left(-\frac{1}{\cos\theta}\Psi - 2\cos\theta\Psi + \sin\theta \partial_\theta \Psi - \alpha\cos\theta R\partial_R\Psi\right).
\end{aligned}$$

Fundamental lemma

Recap:

$$\mathcal{L}_\alpha \Psi := -\alpha^2 R^2 \partial_R^2 \Psi - \alpha(5+\alpha) R\partial_R \Psi - \partial_\theta^2 \Psi + \partial_\theta(\tan\theta\Psi) - 6\Psi = \Omega.$$

The idea is to formally take $\alpha = 0$; then only the angular part of $\mathcal{L}_\alpha$ remains:

$$\mathcal{L}_0(\Psi) = -\partial_\theta^2 \Psi + \partial_\theta(\tan\theta\Psi) - 6\Psi,$$

for which $\sin 2\theta$ is in the kernel, and $\sin\theta\cos^2\theta$ is in the kernel of the formal L^2_θ-adjoint. Hence, we expect a uniform-in-α (angular derivative!) estimates for $\mathcal{L}_\alpha\Psi = F$ if $\int_0^{\frac{\pi}{2}} F(R,\theta)\sin\theta\cos^2\theta\,\mathrm{d}\theta = 0$ for every R. In general, one writes $\hat{\Psi} = \Psi + G(\rho)\sin(2\theta)$ and observes that

$$\mathcal{L}_\alpha(\hat{\Psi}) = F + (\alpha^2 R^2 \partial_R^2 G + \alpha(5+\alpha)R\partial_R G)\sin(2\theta).$$

Requiring the angular orthogonality condition for the RHS at every R leads to an ODE for G, which we can solve~~.~~ explicitly. Then we arrive at

$$G = -\frac{1}{4\alpha} L_{12}(F) + O(\underset{1}{\cancel{\alpha}}).$$

when $F \in L^2_{\rho,\theta}$

Fundamental model

Recap:

$$\partial_t \omega^\varphi + (\mathbf{u} \cdot \nabla)\omega^\varphi = r^{-1} u_r \omega^\varphi.$$

By the fundamental lemma,

$$r^{-1} u_r = \frac{1}{2} L_{12}(\omega) + O(\alpha)$$

Plugging this in, and ignoring the transport term (cf. Thibault's talk), we arrive at the *fundamental model*:

$$\partial_t f(\rho, \theta, t) = f(\rho, \theta, t) L_{12}(f)(\rho, t),$$

where $f \approx \omega^\varphi$.

Fundamental model

Now let us focus on the model

$$\partial_t f(\rho, \theta, t) = f(\rho, \theta, t) L_{12}(f)(\rho, t),$$

where

$$L_{12}(f) = \int_{\rho}^{\infty} \int_{0}^{2\pi} \frac{\sin\theta' \cos^2\theta' f(\rho', \theta', t)}{\rho'} \, \mathrm{d}\theta' \mathrm{d}\rho'.$$

Amusingly, this equation is *explicitly solvable*! Change $(\rho, \theta) \to (\rho'', \theta'')$, mutiply by $\frac{\sin\theta'' \cos^2\theta''}{\rho''}$ and integrate on $(\rho'', \theta'') \in [\rho, \infty) \times [0, 2\pi]$. Then

$$\partial_t L_{12} f(\rho, t) = \frac{1}{2} L_{12} f(\rho, t)^2,$$

which is explicitly solvable. Moreover,

$$f(\rho, \theta, t) = \underbrace{f_0(\rho, \theta, \cancel{t}) \exp\left(\int_0^t L_{12} f(\rho', \theta, t) \, \mathrm{d}\rho' \right)}_{},$$

so

$$f(\rho, \theta, t) = \frac{f_0}{(1 - \frac{1}{2} t L_{12} f_0)^2}.$$

Fundamental model

Recap:

$$\partial_t f(\rho,\theta,t) = f(\rho,\theta,t) L_{12}(f)(\rho,t),$$
$$L_{12}(f) = \int_\rho^\infty \int_0^{2\pi} \frac{\sin\theta' \cos^2\theta' f(\rho',\theta',t)}{\rho'} \,\mathrm{d}\theta'\mathrm{d}\rho',$$

For $f(\rho,\theta,0) = f_0(\rho,\theta)$,

$$f(\rho,\theta,t) = \frac{f_0}{(1-\frac{1}{2}tL_{12}f_0)^2}.$$

From this explicit formula, we see that the fundamental model admits the following family of self-similar solutions (Γ is s.t. $c_\Gamma \neq 0$):

$$f(\rho,\theta,t) = 2\frac{\Gamma(\theta)}{c_\Gamma}\frac{1}{1-t}F_{*,rad}\left(\frac{\rho^\alpha}{1-t}\right), \quad F_{*,rad}(z) = \frac{z}{(1+z)^2},$$
$$c_\Gamma = \int_0^{2\pi} \sin\theta\cos^2\theta\Gamma(\theta)\,\mathrm{d}\theta.$$

Set-up of the perturbative scheme, naive formulation

Now the aim is to view the Euler equations in self-similar coordinates as a perturbation of the fundamental model.

Recap: $\omega(r,x_3) = \Omega(R,\theta)$, $\psi(r,x_3) = \rho^2\Psi(R,\theta)$. The vorticity tranport equation becomes

$$\partial_t\Omega + (-3\Psi - \alpha R\partial_R\Psi)\partial_\theta\Omega + (\partial_\theta\Psi - \tan\theta\Psi)\alpha R\partial_R\Omega$$
$$= \frac{1}{\cos\theta}\left(2\sin\theta\Psi + \cos\theta\partial_\theta\partial_\theta\Psi + \alpha\sin\theta R\partial_R\Psi\right)\Omega.$$

Recall:

$$\underbrace{\Psi(R,\Theta)} = \underbrace{-\frac{1}{4\alpha}\sin 2\theta L_{12}(F)} + O(\alpha).$$

Thus,

$$\partial_t\Omega - \frac{3}{2\alpha}\sin(2\theta)L_{12}(\Omega)\partial_\theta\Omega + L_{12}(\Omega)(\cos(2\theta) - \sin^2\theta)R\partial_R\Omega$$
$$= \frac{1}{\alpha}L_{12}(\Omega)\Omega + \cdots$$

Set-up of the perturbative scheme, naive formulation

Now introduce the ansatz

$$\Omega = \frac{1}{1-t} F\left(\frac{R}{1-t}, \theta\right) \Rightarrow \Psi = \frac{1}{1-t} \Phi\left(\frac{R}{1-t}, \theta\right).$$

and introduce the self-similar variable $z = \frac{R}{1-t}$. Then

$$F + z\partial_z F - \frac{1}{\alpha} L_{12}(F)F - \frac{3}{2\alpha}\sin(2\theta)L_{12}(F)\partial_\theta F + (\cos(2\theta) - \sin^2\theta)L_{12}(F)z\partial_z F$$
$$= \mathcal{N}.$$

Set-up of the perturbative scheme, naive formulation

Introduce the ansatz

$$F = F_* + g, \quad F_* = 2\alpha \frac{\Gamma(\theta)}{c_\Gamma} \frac{z}{(1+z)^2}.$$

Then, since $F_* + z\partial_z F_* - \frac{1}{\alpha} L_{12}(F_*) F_* = 0$,

$$g + z\partial_z g - 2\frac{g}{1+z} - \frac{2z\Gamma(\theta)}{c_\Gamma(1+z)^2} L_{12}(g) - \frac{3}{2\alpha} \sin(2\theta) L_{12}(F_*)\partial_\theta g$$
$$= \frac{3}{2}\sin(2\theta) L_{12}(F_*)\partial_\theta F_* - (\cos(2\theta) - \sin^2\theta) L_{12}(F_*) z\partial_z F_* + \mathcal{N}'.$$

Now choosing

$$\Gamma(\theta) = \sin^\alpha\theta \cos^{2\alpha}\theta$$

makes the transport terms (first two terms on the RHS) small.

Coercivity of the linearized operator & elliptic estimates

We wish to study the linear operator

$$\mathcal{L}_\Gamma^T g = g + z\partial_z g - 2\frac{g}{1+z} - \frac{2z\Gamma(\theta)}{c_\Gamma(1+z)^2} L_{12}(g) - \frac{3}{2\alpha}\mathbb{P}(\sin(2\theta)L_{12}(F_*)\partial_\theta g) = F.$$

where $\mathbb{P}$ is a suitable projection (to be detailed later).

- Ignoring the second two terms, looking at

$w\mathcal{L}_T(g) = (1+z\partial_z)wg$

$$\mathcal{L}_\Gamma := 1 + z\partial_z - \frac{2}{1+z},$$

$\int \mathcal{L}_T g \cdot g \, dz$

$= \int g^2$

$-\frac{1}{2}\int g^2$

$\geq \frac{1}{2}\int g^2.$

we are motivated to remove $\frac{2}{1+z}$ by conjugating with the weight $w = \frac{(1+z)^2}{z^2}$. Thus, we need $g(0) = g'(0) = 0$ and correspondingly $F(0) = F'(0) = 0$.

- Taking the second to last term into account, we also need to impose

$$(L_{12}g)(0) = 0, (L_{12}F)(0) = 0.$$

- Finally, the last term comes with $\mathbb{P}$ to ensure the preceding vanishing conditions (i.e., $L_{12}g(0) = 0 \Rightarrow L_{12}(\mathcal{L}_\Gamma^T g)(0) = 0$).

Coercivity of the linearized operator & elliptic estimates

- One commutes with $D_R = R\partial_R$ and $D_\theta = \sin(2\theta)\partial_\theta$ and proves an estimate with respect to the norm:

$(R \leftrightarrow z)$

$$\|f\|_{\mathcal{H}^2} = \sum_{k=0}^{2} \left| D_R^k f \frac{w}{\sqrt{\sin^{1-}(2\theta)}} \right|_{L^2_{R,\theta}} + \sum_{(k,j)=(1,1),(k,j)=(2,0)} \left| D_\theta^k D_R^j f \frac{w}{\sqrt{\sin^{1+10^{-1}\alpha}(2\theta)}} \right|_{L^2_{R,\theta}}$$

- We need the singular weights so that $\mathcal{H}^2 \hookrightarrow L^\infty$, which in turn allows us to prove algebra estimates for treating nonlinear terms. Note that $\Gamma(\theta) = \sin^{\alpha}\theta\cos^{2\alpha}\theta$. Small α is used for commuting with $D_\theta = \sin(2\theta)\partial_\theta$.
- Need to also upgrade the fundamental lemma (i.e., approximation of Biot–Savart) so that one has good elliptic bounds in $\mathcal{H}^2$ for the remainder.

Set-up of the perturbative scheme, modulation

Now we come back to the equation for g; note that we need to ensure the vanishing conditions.

Introduce modulation parameters μ, λ:

$$\Omega = \frac{1}{1-(1+\mu)t} F\left(\frac{R}{(1-(1+\mu)t)^{1+\lambda}}, \theta\right).$$

Correspondingly, the self-similar variable takes the form

$$z = \frac{R}{(1-(1+\mu)t)^{1+\lambda}},$$

and $\Phi(z,\theta) = (1-(1+\mu)t)\Psi(z,\theta)$.

Then

$$\mathcal{L}_\Gamma^T g = -\mu F_* - (\mu+\lambda+\mu\lambda) z\partial_z F_* + \mathcal{N}''$$

Set-up of the perturbative scheme, modulation

Recap:

$$\mathcal{L}_\Gamma^T g = -\mu F_* - (\mu + \lambda + \mu\lambda) z\partial_z F_* + \mathcal{N}''$$

The set-up gives $\mathcal{L}_\Gamma^T g(0) = 0$ for free; one sets λ, μ so that $(\mathcal{L}_\Gamma^T g)'(0) = L_{12}(\mathcal{L}_\Gamma^T g)(0) = 0.$

Setting $\Gamma(\theta) = \sin^\alpha \theta \cos^{2\alpha} \theta$ kills the main transport terms (cf. Thibault's talk).

The rest are easy to handle. As a result, one obtains an a-priori bound of the form

$$\|g\|_{\mathcal{H}^2} \lesssim \alpha^2.$$

$\lambda, \mu = O(\alpha)$

The a-priori bound can be upgraded to existence via soft argument.